



Pure quotients and Morita's theorem for k_ω -spaces

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Abstract. A k_ω -space X is a Hausdorff quotient of a locally compact, σ -compact Hausdorff space. A theorem of Morita's describes the structure of X when the quotient map is closed, but in 2010 a question of Arkhangel'skii's highlighted the lack of a corresponding theorem for nonclosed quotient maps (even from subsets of \mathbb{R}^n). Arkhangel'skii's specific question had in fact been answered by Siwiec in 1976, but a general structure theorem for k_ω -spaces is still lacking. We introduce pure quotient maps, extend Morita's theorem to these, and use Fell's topology to show that every quotient map can be "purified" (and thus every k_ω -space is the image of a pure quotient map). This clarifies the structure of arbitrary k_ω -spaces and gives a fuller answer to Arkhangel'skii's question.

1 Introduction

A k_ω -space—or a hemicompact k -space—is a Hausdorff space which is the image of a locally compact, σ -compact Hausdorff space under a quotient map. The class of k_ω -spaces was extensively studied from the 1940s and the literature of the subject (see [10] for a useful summary) contains some famous names: R. Arens, M. Graev, E. Michael, J. Milnor, K. Morita, N. Steenrod, and others. Since then k_ω -spaces have become a standard tool (like locally compact Hausdorff spaces).

For general topologists, the class of k_ω -spaces has good closure properties (under quotients, closed subspaces, and finite products). In the study of spaces of continuous functions, the k_ω -spaces are those for which the space $C_k(X)$ of continuous real-valued functions on X , with the compact-open topology, is completely metrizable (see [16] for example). In C^* -algebras (the authors' interest) the Glimm spaces of σ -unital C^* -algebras belong to this class [14, Theorem 2.6]. In topological algebras, the spectrum of a Fréchet algebra is usually (but not always) a k_ω -space; and k_ω -spaces crop up, too, in the study of topological groups and semi-groups, and in mathematical economics, see [7] for example.

A k_ω -space is paracompact and normal [20, Lemma 5], but can be nowhere first countable, even when it is the quotient of a second countable space. The standard example is the countable space S_ω introduced by Arkhangel'skii and Franklin [6]. One well-known weakening of first countability is the *Fréchet-Urysohn* property: a topological space X is *Fréchet-Urysohn* at $x \in X$ if $A \subseteq X$ and $x \in \overline{A}$ implies the existence of a sequence $(x_n)_{n \geq 1} \subseteq A$ with $\lim_n x_n = x$. If X is Fréchet-Urysohn at each

Received by the editors October 14, 2020; accepted July 1, 2021.

Published online on Cambridge Core July 15, 2021.

AMS subject classification: 54B15, 54D45.

Keywords: k_ω -space, Morita's theorem, pure quotient.

point then X is a *Fréchet-Urysohn space*. The sequential fan (consisting of countably many convergent sequences with the non-isolated points identified to a single point) is a countable Fréchet-Urysohn k_ω -space with first countability failing at a single point. Attempts to build more complicated examples run into difficulties, however, and in 2010, in exhibiting “a countable Tychonoff Fréchet-Urysohn space which is nowhere first countable,” Arkhangel'skii raised the question of whether a quotient of a locally compact second countable metric space could be found with all these features [5]. In point of fact this question had already been answered negatively by Siwiec in 1976 [22], but the fact that the question was even asked is significant because it highlights a basic gap in the understanding of k_ω -spaces.

The general picture is as follows. Recall that a point x in a topological space X is a k -point if $E \subseteq X$ and $x \in \bar{E}$ implies that there is a compact set K such that $x \in \bar{E} \cap K$. Clearly every point with a compact neighborhood is a k -point. As temporary notation, for a k_ω -space X , let L denote the set of points with compact neighborhood, F the set of k -points without compact neighborhood, and N the set of non- k -points. Then L is obviously an open set, and Morita and Arkhangel'skii (Theorems 2.1 and 2.3) showed that if X is the image of a closed quotient map then L is dense, F is discrete, and N is empty. Furthermore, the points of L are characterized by the fact that their inverse images have compact boundary.

When a k_ω -space X is the image of a general (nonclosed) quotient map q , the set L can still be characterized by the biquotient property and $L \cup F$ by pseudo-openness (definitions below), but there is no explicit extension of Morita's theorem relating these properties to q being locally closed or attempting to describe the sets F and N .

In this paper, we introduce pure quotient maps—a partial generalization of closed quotient maps—and show that whenever X is the image of a quotient map q from a locally compact σ -compact Hausdorff space Y , there is a corresponding pure quotient map $q_* : Y_* \rightarrow X$ where Y_* is a locally compact σ -compact Hausdorff space derived from Y using Fell's topology on closed sets. Thus, every k_ω -space is the image of a pure quotient map.

For pure quotient maps, N is precisely the set of points at which the map is not locally closed, and we extend Morita's theorem by showing that F is contained in the set of P-points of $N \cup F$ and is thus a P-space with empty interior in X (Theorem 4.5). Thus a general k_ω -space X decomposes as the disjoint union $L \cup F \cup N$ where L is open and F has empty interior and is contained in the set of P-points of the closed set $F \cup N$. If X is countably tight (in particular if X is a quotient of a locally compact subset of \mathbb{R}^n) then the P-points of $F \cup N$ are isolated points of $F \cup N$, lying in the closure of L ; so N is closed, $L \cup F$ is open with L dense, and F is discrete (Corollary 4.6). For general X , however, F need not be discrete, nor N closed, nor F lie in the closure of L ; and we show by example that $F \cup N$ can be any compact Hausdorff space with F as its set of P-points (Example 4.7). A Fréchet-Urysohn space is countably tight with N empty, so the answer to Arkhangel'skii's question quickly follows from the description just given (see Corollary 5.3 and Theorem 5.4, where more general results are obtained).

The authors encountered pure quotient maps in work on Glimm spaces of C^* -algebras, where they occur naturally (indeed Theorem 4.5, in the second countable

case, was originally proved entirely using C^* -algebras). They look less obvious when translated into a topological context, but perhaps that is part of their wider interest.

2 Morita's theorem and locally closed maps

Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Then, we can write $Y = \bigcup_{i \geq 1} Y_i$ where each Y_i is compact and is contained in the interior of its successor. We will say that $\{Y_i : i \geq 1\}$ is a *compact decomposition* for Y , while the compact sets $\{q(Y_i) : i \geq 1\}$ are a k_ω -decomposition for X . The space X is *hemicompact* with regard to a k_ω -decomposition; that is, for any compact $K \subseteq X$, eventually $K \subseteq q(Y_i)$ for some i (see [10, p. 113]).

One of the central results on k_ω -spaces is due to Morita [19, Theorem 4].

Theorem 2.1 (Morita, 1956) *Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. If q is closed then X is locally compact except at a closed discrete set of exceptional points, namely those points x for which the fiber $q^{-1}(x)$ has noncompact boundary.*

As Morita observes, the discrete set of exceptional points is in fact contained in a larger discrete set of points with noncompact fiber. For all other points, the fiber is compact.

If q is not closed, the situation becomes more complicated, and to discuss this the following definition is useful. Let X and Y be topological spaces. A map $q : Y \rightarrow X$ is *locally closed* at a point $x \in X$ if for every closed set $W \subseteq Y$, $x \in q(W)$ implies $x \in q(W)$ [13, Section 13.XIV]. Clearly if q is surjective then q is closed if and only if q is locally closed at each point of x . One basic result is as follows.

Proposition 2.2 *Let Y be a locally compact Hausdorff space and $q : Y \rightarrow X$ a continuous surjective map with X Hausdorff. Let $x \in X$ and suppose that q is locally closed at x and that the boundary of $q^{-1}(x)$ is compact. Then x has a compact neighborhood in X .*

Proof If the boundary of $q^{-1}(x)$ is compact then it is contained in an open subset O of Y whose closure is compact. The set $U := q^{-1}(x) \cup O$ is open and the map q is locally closed at x . Thus $V := X \setminus \overline{q(Y \setminus U)}$ is an open neighborhood of x . We have $q^{-1}(x) \subseteq q^{-1}(V)$, hence $\overline{V} \subseteq \overline{q(U)} = \{x\} \cup \overline{q(O)} = \{x\} \cup q(\overline{O})$, and thus \overline{V} is compact. ■

In general, there seems to be little relation between the compactness of the fiber and “local closedness” of the quotient map on the one hand, and the local compactness of the image on the other. The following example is typical. Let $Y = \bigcup_{n \geq 0} L_n$ where L_n is the line $y = n$ in \mathbb{R}^2 , and let $q : Y \rightarrow X$ be the projection of Y onto the x -axis, which is a quotient map. Then X is locally compact and Hausdorff but q is nowhere locally closed and $q^{-1}(x)$ has noncompact boundary for every $x \in X$.

A rather similar definition to “locally closed” was introduced independently by various authors, the earliest of whom seems to have been McDougle [17]. Let X and Y be topological spaces. A surjective map $f : Y \rightarrow X$ is *pseudo-open at $x \in X$* if whenever U is an open subset of Y containing $f^{-1}(x)$, $f(U)$ is a neighborhood of x . If f is

pseudo-open at every point of X then f is said to be *pseudo-open*. A pseudo-open map is easily seen to be a quotient map, and if f is locally closed at x then f is pseudo-open at x . The example above, with q open but nowhere locally closed, shows that the converse is not true. We will see, however, that the two conditions are equivalent if the quotient map q is pure.

The next theorem was essentially proved in [2, Theorems 3.3 and 3.4] (see also [1]) but our statement of it is somewhat different, so we include the short proof here.

Theorem 2.3 (Arkhangel'skii, 1963) *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Let $x \in X$. Then x is a k -point if and only if q is pseudo-open at x .*

Proof Suppose first that q is pseudo-open at x (this does not use the σ -compactness of Y). Let $E \subseteq X$ with $x \in \bar{E}$, and set $F = q^{-1}(E)$. Then \bar{F} meets $q^{-1}(x)$, at y say, because q is pseudo-open (for otherwise there is an open set U containing $q^{-1}(x)$ and disjoint from F , and then $q(U)$ is a neighborhood of x disjoint from E). Let L be a compact neighborhood of y and set $K = q(L)$. Then $y \in \overline{F \cap L}$ and $x \in \overline{E \cap K}$.

Conversely, suppose that x is a k -point and let U be an open subset of Y containing $f^{-1}(x)$. Let $E = X \setminus q(U)$ and $F = q^{-1}(E)$. Then F does not meet U . If $x \in \bar{E}$ then there exists a compact set K such that $x \in \overline{E \cap K}$. By hemicompactness, there exists compact L in Y such $q(L) = K$. Then $\overline{F \cap L}$ is compact and $x \in q(\overline{F \cap L})$. Hence, \bar{F} meets $q^{-1}(x)$, contradicting the fact that F does not meet U . Thus $x \notin \bar{E}$, and $q(U)$ is a neighborhood of x . ■

For a quotient map $q : Y \rightarrow X$, let H_q be the set of points in X at which q is pseudo-open. Theorem 1.3 shows that if Y is a locally compact σ -compact Hausdorff space and X is Hausdorff then H_q is the set of k -points in X .

Among the k -points, the points x of local compactness in a k_ω -space (and more generally) can be characterized as those which satisfy the *biquotient* property: every open cover \mathcal{U} of $q^{-1}(x)$ has a finite subset $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$ such that $\{q(U_1), \dots, q(U_n)\}$ covers a neighborhood of x (see [18, 23]). While the characterizations given by the biquotient property and by pseudo-openness are elegant and versatile, they do not seem to lead to a structure theory for general k_ω -spaces along the lines of Morita's theorem. Something is needed to bring the ideas together.

3 Pure quotient maps

In this section, we introduce pure quotient maps. They are weaker than closed quotient maps in one important aspect, but with a compensatory strengthening in another direction; and while they retain some of the nice properties of closed quotient maps, they are much more general. We begin with a preparatory lemma.

Lemma 3.1 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff space. Then the following are equivalent:*

- (i) q is not closed;
- (ii) there are nets (y_α) (with the points y_α from distinct fibers) and (z_α) in Y with $y_\alpha \rightarrow \infty$ and $z_\alpha \not\rightarrow \infty$ such that $q(y_\alpha) = q(z_\alpha)$ for all α ; and

- (iii) *there are sequences (y_n) (with the points y_n from distinct fibers) and (z_n) in Y with $y_n \rightarrow \infty$ and (z_n) contained in a compact set such that $q(y_n) = q(z_n)$ for all n .*

Proof (i) \Rightarrow (ii). Let W be a closed subset of Y such that $q(W)$ is not closed. Then the saturation W_0 of W is not closed, so there is a net (z_α) in W_0 , which can be chosen from distinct fibers, such that $z_\alpha \rightarrow w \notin W_0$. Let (y_α) be a corresponding net in W with y_α and z_α from the same fiber for each α . Then if (y_α) had any convergent subnet (y_β) with limit $y \in W$, y and w would lie in the same fiber so w would belong to W_0 , a contradiction. Hence $y_\alpha \rightarrow \infty$.

(ii) \Rightarrow (iii). If (ii) holds, then (z_α) is frequently in some compact set K , so for each $n \geq 1$, we may choose z_n from the set $\{z_\alpha\}$ such the corresponding $y_n \in \{y_\alpha\}$ does not belong to K_n (where $Y = \bigcup_{n \geq 1} K_n$ is a compact decomposition for Y). Then $y_n \rightarrow \infty$ while (z_n) is contained in K .

(iii) \Rightarrow (i). If (iii) holds, let K be the compact set containing (z_n) . Then (z_n) has subnet converging to some $y \in K$, so $q(y)$ lies in the closure of the set $\{q(z_n) : n \geq 1\}$. But $\{q(z_n) : n \geq 1\} = q(\{y_n : n \geq 1\})$, the image of a closed set, so q is not a closed map. ■

Definition Let Y be a locally compact σ -compact space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. We say that q is *pure* if there is a subset D of Y such that (i) D is dense in Y and the restriction of q to D is injective, and (ii) for every net (d_α) in D , if $d_\alpha \rightarrow \infty$ (i.e., eventually escapes from every compact set in Y) then $e_\alpha \rightarrow \infty$ for any other net (e_α) for which $q(d_\alpha) = q(e_\alpha)$ for all α .

Condition (ii) is equivalent to requiring that if $d_\alpha \rightarrow \infty$ then the net $q^{-1}(q(d_\alpha))$ converges to infinity in the Fell topology. Lemma 3.1 shows that condition (ii) is automatically satisfied whenever q is a closed quotient map.

On the other hand, condition (i) implies that for every $x \in X$, the interior of $q^{-1}(x)$ is either empty or consists of an isolated point. In fact, condition (i) is equivalent to this when Y is second countable. To see this, let D_0 be the (countable) set of isolated points in Y and fix a countable base $\{U_i\}_{i \geq 1}$ for $Y \setminus \overline{D_0}$. Inductively chose a point from each U_i which does not belong to any fiber with nonempty interior or any fiber previously chosen. This is possible because U_i contains no isolated point and hence is an uncountable Baire space, and every fiber intersects U_i in a closed set with empty interior. Let D_1 be the set thus obtained, and set $D = D_0 \cup D_1$. Then D is dense in Y and q restricted to D is injective.

Here is an alternative description of pure quotient maps that will be useful later.

Lemma 3.2 *Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Then the following are equivalent:*

- (a) *q is pure and*
- (b) *there is a subset D in Y such that (i) D is dense and the restriction of q to D is injective, and (ii) whenever $Y = \bigcup_{i \geq 1} Y_i$ is a compact decomposition of Y , the sets $D_i := \{d \in D : q(d) \in q(Y_i)\}$ have compact closure in Y .*

Proof (a) \Rightarrow (b). Let W be the closure of D_i . If W is not compact, then there is a sequence (d_j) in D_i with $d_j \rightarrow \infty$ as $j \rightarrow \infty$. But for each j , there exists $y_j \in Y_i$ such that $q(y_j) = q(d_j)$. Since Y_i is compact, $y_j \not\rightarrow \infty$, contradicting condition (ii) for pure quotient maps.

(b) \Rightarrow (a). Let (d_α) be a net in D with $d_\alpha \rightarrow \infty$. Let (y_α) be any net in Y such that $q(y_\alpha) = q(d_\alpha)$ for each α . Then for any Y_i , if (y_α) were frequently in Y_i then (d_α) would frequently be in the compact closure of D_i , contradicting the convergence of (d_α) to infinity. Hence, (y_α) is eventually outside each compact set Y_i , so $y_\alpha \rightarrow \infty$. ■

Thus, if q is a pure quotient map and $Y = \bigcup_{i \geq 1} Y_i$ is a compact decomposition of Y , then for each Y_i we may find $Y_{d(i)}$ with $d(i) \geq i$ such that $D_i \subseteq Y_{d(i)}$.

The usefulness of pure quotient maps appears in the following converse to Proposition 2.2 (note that because a fiber of a pure quotient map has at most a singleton as its interior, the fiber is compact if and only if its boundary is compact).

Theorem 3.3 *Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff. If $x \in X$ has a compact neighborhood then q is locally closed at x and $q^{-1}(x)$ is compact.*

Proof If Y is compact then q is closed and $q^{-1}(x)$ is compact for all $x \in X$, so we may suppose that Y is noncompact. Let $Y = \bigcup_{i \geq 1} Y_i$ be a compact decomposition for Y . Suppose first that q is not locally closed at x . Let W be a closed subset of Y such that $x \in \overline{q(W)}$ but W does not meet $q^{-1}(x)$. Then for any neighborhood U of x , $q(W) \cap U$ is not closed in U , so $W \cap q^{-1}(U)$ is not contained in any of the compact sets Y_i . Thus for each i , there exists $w_i \in (W \cap q^{-1}(U)) \setminus Y_i$. By the density of D , there exists $d_i \in (D \cap q^{-1}(U)) \setminus Y_i$. Then the set $F = \bigcup_i q^{-1}(q(d_i))$ is a saturated subset of Y , and F is closed by condition (ii) for pure quotient maps. Hence $q(F)$ is an infinite closed discrete subset of U , so U is not countably compact.

Now suppose instead that $q^{-1}(x)$ is noncompact, and let U be a neighborhood of x . Then for each $i \geq 1$, there exists $y_i \in q^{-1}(x) \setminus Y_i$, so by the density of D , there exists $d_i \in q^{-1}(U) \setminus Y_i$. Then again the set $F = \bigcup_i q^{-1}(q(d_i))$ is a saturated subset of Y , and F is closed by condition (ii) for pure quotient maps. Hence $q(F)$ is an infinite closed discrete subset of U , so again U is not countably compact. ■

Pure quotient maps are not uncommon. Clearly, every locally compact σ -compact Hausdorff space X admits the trivial pure quotient, taking $Y = X$ and q as the identity map, and we now show that every k_ω -space X admits a pure quotient. Specifically, we show that if Y is a locally compact, σ -compact Hausdorff space, and $q : Y \rightarrow X$ a quotient map with X Hausdorff, then q can be “purified”: that is, a locally compact σ -compact space Y_* can be derived from Y such that the quotient map $q_* : Y_* \rightarrow X$ is pure (and induces the same topology on X as q).

For a topological space X , let $\text{Cl}(X)$ be the hyperspace of closed subsets of X with the Fell topology. A base for the Fell topology consists of the family of all sets

$$U(K, \Phi) = \{S \in \text{Cl}(X) : S \cap K = \emptyset, S \cap O \neq \emptyset, O \in \Phi\},$$

where K is a compact subset of X and Φ is a finite family of open subsets of X . Then $\text{Cl}(X)$ is a compact space [9, Lemma 1], and is Hausdorff if X is locally compact [9, Theorem 1]. The next lemma is presumably standard.

Lemma 3.4 *Let X be a locally compact σ -compact space. Then $\text{Cl}(X) \setminus \{\emptyset\}$ is σ -compact.*

Proof We may write $X = \bigcup_i X_i$ where each X_i is compact. For each i , set $K_i = \{K \in \text{Cl}(X) : K \cap X_i \neq \emptyset\}$. Let (K_α) be a net in K_i . Then by the compactness of $\text{Cl}(X)$, (K_α) has a convergent subnet (K_β) with limit W . But $W \cap X_i$ must be nonempty, since each K_β meets X_i and hence $W \in K_i$. Thus K_i is compact, and $\text{Cl}(X) \setminus \{\emptyset\} = \bigcup_i K_i$ is σ -compact. ■

Lemma 3.5 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Let $\text{Cl}(Y)$ be the hyperspace of closed subsets of Y with the Fell topology, and let Y_* be the closure of the set $D = \{q^{-1}(x) : x \in X\}$ in $\text{Cl}(Y) \setminus \{\emptyset\}$. Then for each $F \in Y_*$ there exists $F' \in D$ such that $F \subseteq F'$.*

Proof Let (F_α) be a net in D with limit $F \neq \emptyset$ in $\text{Cl}(Y)$. Let $y \in F$ and set $F' = q^{-1}(q(y))$. Then there is a net (y_α) with $y_\alpha \in F_\alpha$ such that $y_\alpha \rightarrow y$. Hence, $q(y_\alpha) \rightarrow q(y)$. Let $y' \in F$. Then there is another net (y'_α) with $y'_\alpha \in F_\alpha$ such that $y'_\alpha \rightarrow y'$. Hence, $q(y_\alpha) = q(y'_\alpha) \rightarrow q(y')$, so $y' \in F'$. Thus $F \subseteq F'$. ■

It follows from Lemma 3.5 that, in the context of the lemma, we may consistently define a map $q_* : Y_* \rightarrow X$ by $q_*(F) = q(y)(y \in F)$.

Theorem 3.6 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Let $\text{Cl}(Y)$ be the hyperspace of closed subsets of Y with the Fell topology, and let Y_* be the closure of the set $D = \{q^{-1}(x) : x \in X\}$ in $\text{Cl}(Y) \setminus \{\emptyset\}$. Then Y_* is a locally compact, σ -compact Hausdorff and the map $q_* : Y_* \rightarrow X$ is a pure quotient map inducing the same topology on X as q .*

Proof It follows from Lemma 3.4 that Y_* is σ -compact.

Let C be a subset of X . We must show that $q_*^{-1}(C)$ is closed if and only if $q^{-1}(C)$ is closed. Suppose first that $q^{-1}(C)$ is closed, and let $F \in Y_*$ with $q_*(F) = y \notin C$. Then the complement of $q^{-1}(C)$ defines an open subset of Y_* containing F , and thus no net in $q_*^{-1}(C)$ can converge to F . Thus $q_*^{-1}(C)$ is closed. Conversely, suppose that $q_*^{-1}(C)$ is closed. Let (y_α) be a net in $q^{-1}(C)$ with limit y . Set $F_\alpha = q_*^{-1}(q(y_\alpha))$. Then $F_\alpha \not\rightarrow \infty$, so by local compactness of Y_* there exists a convergent subnet of (F_α) with limit $F \in Y_*$, and $y \in F$. Then $F \in q_*^{-1}(C)$ by assumption, and $q_*(F) = q(y) \in C$. Thus $y \in q^{-1}(C)$ as required.

Finally, it is immediate that the restriction of q_* to D is injective, while condition (ii) of pure quotient maps follows from Lemma 3.5. ■

Note that $q_*^{-1}(x)$ is the set consisting of $q^{-1}(x)$ and a family of closed subsets of the boundary of $q^{-1}(x)$, obtained as limits of nets of fibers; see Lemma 3.5. (Thus in the

example after Proposition 2.2, D is homeomorphic to \mathbb{R} and is closed in $\text{Cl}(Y) \setminus \{\emptyset\}$. If the boundary of $q^{-1}(x)$ is compact then $q_*^{-1}(x)$ is also compact because Y_* is the closure of D in $\text{Cl}(Y) \setminus \{\emptyset\}$. On the other hand, if the boundary of $q^{-1}(x)$ is noncompact then a net of its closed subsets might converge to \emptyset in the Fell topology, so $q_*^{-1}(x)$ might be compact or noncompact.

Apart from the fact that its fibers are more likely to be compact (or to have compact boundary) than those of q , a further advantage that q_* has over q is that it is more likely to be locally closed.

Proposition 3.7 *Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Let q_* be the pure quotient constructed in Theorem 3.6, and let $x \in X$. If q is locally closed at x , then so is q_* .*

Proof Suppose that q is locally closed at x and let C be a closed subset of Y_* disjoint from $q_*^{-1}(x)$. Set $W = \bigcup_{K \in C} K$. Then W is disjoint from $q^{-1}(x)$. Let (y_α) be a net in W with limit $y \in Y$. Then there is a net (K_α) in C with $y_\alpha \in K_\alpha$ for each α , and $K_\alpha \not\rightarrow \infty$, so by the local compactness of Y_* there is a net (K_β) converging to some K , which belongs to C since C is closed. Since $y \in K$, it follows that $y \in W$, and hence that W is closed. Since $q(W) = q_*(C)$, it follows that $x \notin \overline{q_*(C)}$. ■

In particular, if q is closed then q_* is closed. Note, however, that in this case if the boundary of $q^{-1}(x)$ is noncompact then x does not have a compact neighborhood in X by Theorem 2.1, and hence $q_*^{-1}(x)$ is noncompact by Proposition 2.2 applied to q_* .

Proposition 3.7 raises the question of whether q_{**} might be locally closed at even more points of X than q_* is, but we will see in the next section that q_* already reaches the limit.

4 An extension of Morita's theorem

In this section, we use pure quotient maps to extend Morita's theorem to general k_ω -spaces.

Given a quotient map q from a locally compact σ -compact Hausdorff space Y onto a Hausdorff space X , we partition X into three sets as follows:

- (i) $L_q = \{x \in X : q \text{ is locally closed at } x \text{ and } q^{-1}(x) \text{ has compact boundary}\};$
- (ii) $F_q = \{x \in X : q \text{ is locally closed at } x \text{ and } q^{-1}(x) \text{ has noncompact boundary}\};$
- and (iii) $N_q = \{x \in X : q \text{ is not locally closed at } x\}.$

We saw in Proposition 2.2 that if $x \in L_q$ then x has a compact neighborhood in X , and in Theorem 3.3 that if q is a pure quotient map then L_q is precisely the set of points in X which have a compact neighborhood. We now show that if q is a pure quotient map, the sets F_q and N_q can similarly be characterized in terms of the topology of X .

Theorem 4.1 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff. Then q is locally closed at $x \in X$ if and only if x is a k -point.*

Proof We have seen that if q is locally closed at x then it is pseudo-open at x , and hence x is a k -point by Theorem 2.3.

Conversely, suppose that q is not locally closed at x , and let N be a closed subset of Y disjoint from $q^{-1}(x)$ with $x \in \overline{q(N)}$. Let W be a closed subset of Y disjoint from $q^{-1}(x)$ with N in its interior. Let D be the dense subset of Y from the pure quotient property, and set $M = W \cap D$ and $E = q(M)$. Then $\overline{M} \supseteq N$, so $x \in \overline{E} \setminus E$. Let $Y = \bigcup_i Y_i$ be a compact decomposition for Y and set $K_i = q(Y_i)$. Then for any i , $q^{-1}(K_i \cap E) = \{d \in M : q(d) \in K_i\}$ so $q^{-1}(K_i \cap E) \subseteq Y_{d(i)}$ (in the terminology introduced after Lemma 3.2). Thus

$$K_i \cap E \subseteq q(Y_{d(i)} \cap M) \subseteq q(Y_{d(i)} \cap \overline{M}) \subseteq q(Y_{d(i)} \cap W)$$

and this latter set is compact and does not contain x . Since every compact subset of X is contained in some K_i by hemicompactness, it follows that there does not exist compact K with $x \in \overline{E} \cap \overline{K}$. Hence x is not a k -point. ■

Thus if q is a pure quotient map, q is locally closed at x if and only if q is pseudo-open at x . The following corollary is immediate from Theorem 4.1.

Corollary 4.2 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff. Let $x \in X$. Then*

- (i) $x \in L_q$ if and only if x has a compact neighborhood in X ;
- (ii) $x \in F_q$ if and only if x is a k -point without a compact neighborhood in X ; and
- (iii) $x \in N_q$ if and only if x is not a k -point.

In particular, for any quotient map with domain a locally compact, σ -compact Hausdorff space and Hausdorff range, $N_{q^{**}} = N_{q^*}$, answering the question raised in Section 3. It follows from Corollary 4.2 that for a k_ω -space X , the sets L_{q^*} , F_{q^*} , and N_{q^*} are independent of the original quotient map q , and in future we may use this notation even when there is no particular original quotient map in mind.

The next corollary is also immediate from Theorem 4.1 (applied to q^*) and Theorem 2.3 (applied to q). Recall that H_q denotes the set of points of X at which q is pseudo-open.

Corollary 4.3 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. For $x \in X$, $x \in H_q$ if and only if q^* is locally closed at x . Hence, q is pseudo-open if and only if q^* is closed.*

We turn now to consider the topology of the sets L_q , F_q , and N_q when q is pure (in which case $H_q = L_q \cup F_q$ by Corollary 4.3). It follows from Corollary 4.2 that L_q is open, and hence that $F_q \cup N_q$ is closed in X .

Recall that a point x in a Tychonoff space X is a P-point if the intersection of any countable family of neighborhoods of x is a neighborhood of x . If every point is a P-point then X is a P-space. If U is an open subset of X and x is a P-point in U then x is P-point in X . Conversely, a P-point in X is a P-point in any subspace of X , and hence the set of P-points of X forms a P-space. If X is a compact P-space then X is

finite [11, 4K]. The next theorem shows that F_q is contained in the set of P-points of $F_q \cup N_q$. For this, we need the following observation.

Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff. Then q extends to a continuous map $\bar{q} : \beta Y \rightarrow \beta X$, and \bar{q} is continuous and surjective from a compact Hausdorff space, and is therefore also a quotient map. If $x \in X$ and q is not locally closed at x , then there is a closed set $N \subseteq Y$ disjoint from $q^{-1}(x)$ such that $x \in \overline{q(N)}$. Hence, there is a point $y \in \overline{N}^{\beta Y}$ such that $\bar{q}(y) = x$.

Lemma 4.4 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff. Let $x \in N_q$ and let $y \in \beta Y \setminus Y$ such that $\bar{q}(y) = x$. Let V be any neighborhood of x in X . Then Y has a zero set $Z \subseteq q^{-1}(V)$ with $y \in \overline{Z}^{\beta Y}$.*

Proof Let $U \subseteq V$ be a cozero set containing x with complement W in X . Then $q^{-1}(W)$ is a saturated zero subset in Y . If Z is a zero set in Y contained in $q^{-1}(W)$ then $q(Z)$ is contained in W , so $y \notin \overline{Z}^{\beta Y}$. Hence, by the z -ultrafilter property [11, 2.6 Theorem], there is a zero set Z in Y with $y \in \overline{Z}^{\beta Y}$ such that $Z \cap q^{-1}(W)$ is empty. ■

Theorem 4.5 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff.*

- (i) *If $x \in X$ is a P-point then q is locally closed at x .*
- (ii) *Let W be the set of P-points in $F_q \cup N_q$ and V the interior of $F_q \cup N_q$ in X . Then*

$$V \cap W \subseteq F_q \subseteq W.$$

- (iii) *F_q is a P-space and has empty interior in X .*

Proof (i) (This does not require q to be pure). Let C be a closed subset of Y disjoint from $q^{-1}(x)$. Then

$$C = \bigcup_{n \geq 1} C \cap K_n,$$

where $Y = \bigcup_{n \geq 1} K_n$ is a compact decomposition for Y , so $q(C) = \bigcup_{n \geq 1} q(C \cap K_n)$. Since $q(C \cap K_n)$ is compact, hence closed in X , the P-point x does not lie in $q(C)$. Thus q is locally closed at x .

(ii) It follows from (i) that $F_q \supseteq V \cap W$. Now suppose that $x_0 \in F_q \cup N_q$ is not a P-point of $F_q \cup N_q$. Let $\{E_i : i \geq 1\}$ be a countable family of closed sets in $F_q \cup N_q \setminus \{x_0\}$ with $x_0 \in \overline{\bigcup_i E_i}$. Fix i , and let U_i be a closed neighborhood of x_0 in X that does not meet E_i . For each $x \in E_i$ choose $y \in q^{-1}(x)$ with $y \notin Y_{i+1}$ if this is possible (as it is for each $x \in F_q$ since $q^{-1}(x)$ is noncompact). Otherwise, by the observation preceding Lemma 4.4, there exists $y \in \beta Y \setminus Y$ such that $\bar{q}(y) = x$. By Lemma 4.4, there is a zero set Z in Y with $y \in \overline{Z}^{\beta Y}$ and $Z \cap q^{-1}(U_i)$ empty, and we may also arrange that $Z \cap Y_{i+1}$ is empty.

Let N'_i be the union of the collection of elements y and zero sets Z thus obtained, and let N_i be the closure of N'_i in Y . Then N_i does not meet Y_i since Y_i is contained

in the interior of Y_{i+1} , and $\overline{q(N_i)} \supseteq E_i$, but $q(N_i) \cap U_i$ is empty (so in particular $x_0 \notin q(N_i)$). Set $N = \bigcup N_i$. Then N does not meet $q^{-1}(x_0)$, and N is closed in Y because, for any Y_j , only finitely many of the sets N_i meet Y_j , so $N \cap Y_j$ is closed. But $\overline{q(N)} \supseteq \bigcup_i E_i$, so $x_0 \in \overline{q(N)} \setminus q(N)$. Hence, $x_0 \in N_q$, so $F_q \subseteq W$.

(iii) F_q is a P-space by (ii). Let U be the interior of F_q in $F_q \cup N_q$. Then $O := q^{-1}(U)$ is open in Y , hence a locally compact Hausdorff space, and $q|_O$ is a quotient map. Thus U is a k -space, so a subset of U is closed if and only if its intersection with every compact subset of U is closed. But U is P-space, so compact subsets are finite [11, 4K]. Thus, every subset of U is closed, so U is discrete. Hence, each point in U lies in the closure of L_q (for otherwise it would be an isolated point of X and would belong to L_q itself). Thus, F_q has empty interior in X . ■

Theorem 4.5 is an extension of Morita's theorem (1.1) because if q is a closed quotient map then q_* is closed (Proposition 3.7), so N_q and N_{q_*} are both empty. Furthermore $F_q = F_{q_*}$ by the remark after Proposition 3.7, so F_q is closed, and thus σ -compact. Being a P-space by Theorem 4.5, F_q must be a countable discrete set in the closure of L_q , and thus we recover Morita's theorem.

In some cases, we can go further. Recall that a topological space X has *countable tightness* at $x \in X$ if for each $A \subseteq X$ with $x \in \overline{A}$, there is a countable subset $B \subseteq A$ with $x \in \overline{B}$; and X is *countably tight* if it has countable tightness at every point. Clearly every metric space is countably tight.

Corollary 4.6 *Let Y be a locally compact σ -compact Hausdorff space and $q : Y \rightarrow X$ a pure quotient map with X Hausdorff.*

(i) *If $x \in F_q$ is a point of countable tightness in X then x is isolated in $F_q \cup N_q$ and lies in the closure of L_q .*

(ii) *If X is countably tight then N_q is closed in X , and F_q is discrete and lies in the closure of L_q .*

Proof (i) follows immediately from the fact that x is a P-point in $F_q \cup N_q$, and (ii) follows from (i). ■

Corollary 4.6 seems to be new even for quotients of locally compact subsets of \mathbb{R}^n .

We have just seen that if X is countably tight then there are no points of F_q in the interior of $N_q \cup F_q$, but if X is not countably tight, such points may occur. For example, let X be the quotient of $\omega_1 + 1 \times S_\omega$ (where S_ω is the Arkhangel'skii–Franklin space [6] obtained by identifying $\{\omega_1\} \times S_\omega$ to a point x_0). Then L_q is empty and $F_q = \{x_0\}$.

Not every P-point of $F_q \cup N_q$ need belong to F_q . For example, let $Y = \mathbb{R}$ and for $n \geq 1$, identify the pairs of points n and $1/n$. Let X be the resulting quotient space and $q : Y \rightarrow X$ the quotient map, which is obviously pure. Then $F_q \cup N_q$ consists of the singleton $\{q(0)\}$, but F_q is empty. The following example, however, has the satisfying feature that F_q consists precisely of the P-points in $F_q \cup N_q$.

Example 4.7 *Let Z be a locally compact σ -compact Hausdorff space and V a closed subset with empty interior (so that $V \subseteq \overline{Z \setminus V}$). We shall find a locally compact σ -compact Hausdorff space Y and a pure quotient map $q : Y \rightarrow X$ with X Hausdorff*

such that V is homeomorphic to $F_q \cup N_q$, with F_q corresponding to the P-points in V .

Let $Y = \bigcup_i Z_i$ be the disjoint union of countably many copies of Z . Set $W = \bigcup_i V_i$, and for each i let ϕ_i be a homeomorphism from V to V_i . Let O be the complement of W in Y . Let \sim be the equivalence relation defined on points $y \in Y$ by $y \sim y'$ if and only if either (i) $y = y'$; or (ii) $y, y' \in W$ and there exists $v \in V$ such that $\phi_i(v) = y$ and $\phi_j(v) = y'$ for some i and j . Set $X = Y / \sim$ and let q be the quotient map. Then q is pure because V has empty interior in Z . Note that $q(W)$ is homeomorphic to V , and that $q|_{V_i}$ is a homeomorphism from V_i onto $q(W)$ for each i . For ease of notation, we will identify $q(W)$ with V , so that $\phi_i : V \rightarrow V_i$ is the inverse of $q|_{V_i}$.

Clearly $q|_O$ is a homeomorphism from O onto $q(O)$, so if $x \in q(O)$ then x is a point of local compactness. Now let $x \in V$ and suppose first that x is not a P-point in V . Let $\{N_i\}$ be a countable family of closed subsets of V such that $x \notin N_i$ for all i but $x \in \bigcup_i \overline{N_i}$. Set $N = \bigcup_i \phi_i(N_i)$. Then N is a closed subset of Y disjoint from $q^{-1}(x)$, but $x \in \overline{q(N)}$. Hence $x \in N_q$.

Now suppose that x is a P-point in V . Let N be a closed subset of Y not meeting $q^{-1}(x)$. Then the sets $q(N \cap V_i)$ are a countable family of closed set in V (since $q|_{V_i}$ is a homeomorphism), and they each miss x , so $M := \bigcup_i q(N \cap V_i)$ also misses x . Set $N' = N \cup \bigcup_i \phi_i(M)$. Then N' is a closed saturated set in Y containing N and missing $q^{-1}(x)$, so $x \notin \overline{q(N)}$. Hence $x \in F_q$.

Thus in Example 4.7, N_q is closed in X if and only if every P-point in V is an isolated point of V . If V has a nonisolated P-point, for example, if $V = \omega_1 + 1$ (and Z is the cone over V), then N_q is not closed. If $Z = \beta\mathbb{N}$ and $V = \beta\mathbb{N} \setminus \mathbb{N}$, then it depends on the set theoretic axioms adopted as to whether F_q is empty or dense in $F_q \cup N_q$.

5 Arkhangel'skii's question and Fréchet-Urysohn spaces

Finally, we turn to consider first countable and Fréchet-Urysohn spaces and Arkhangel'skii's question (for a vast amount of information on this whole area, see [21]).

As mentioned in the Introduction, a point x in a topological space X is a *Fréchet-Urysohn* point if whenever $E \subseteq X$ with $x \in \overline{E}$ there is a sequence (v_n) in E with $v_n \rightarrow x$. Clearly, every point of first countability is a Fréchet-Urysohn point. If every point is a Fréchet-Urysohn point then X is a *Fréchet-Urysohn* space. A Fréchet-Urysohn point in a k_ω -space is easily seen to be a k -point (Proposition 5.1(ii)) and thus belongs either to L_{q_*} or to F_{q_*} . Ordman showed that a point of first countability must belong to L_{q_*} (see [10, p. 113]) but a general Fréchet-Urysohn point may belong to F_{q_*} . Being a point of countable tightness, however, it must belong to the closure of L_{q_*} , and the set of such points must be countable (Corollary 4.6(i)).

The importance of Fréchet-Urysohn spaces is that every quotient of a compact first countable (or Fréchet-Urysohn) space is Fréchet-Urysohn (see Corollary 5.2). For example, let $Y = [0, 1] \times [0, 1]$ with the order topology of the lexicographic order and let $V = [0, 1] \times \{0, 1\}$. Let $q : Y \rightarrow X$ be the quotient map that takes V to a point x_0 . Then Y is compact, Hausdorff, and first countable, and V is closed but not

a zero-set, so X is compact, Hausdorff and Fréchet-Urysohn but not first countable at x_0 .

Recall that a topological space X is *sequential* if every sequentially closed subset of X is closed. Every Fréchet-Urysohn space is sequential, but there are compact Hausdorff sequential spaces which are not Fréchet-Urysohn. More generally, every sequential space has countable tightness; and the famous Moore-Mrowka problem—shown by Ostaszewski et al. to depend on set-theoretic axioms—was whether there exists a compact Hausdorff space of countable tightness which is not sequential.

If Y is a second countable locally compact Hausdorff then $\text{Cl}(Y)$ is also second countable [9, Remark 3, p. 474], but the same does not hold for first countability. Indeed, if Y is a locally compact Hausdorff space, then the following three conditions are equivalent: (a) $\text{Cl}(Y) \setminus \{\emptyset\}$ is first countable; (b) Y is both hereditarily Lindelof and hereditary separable (for a locally compact space, first countability is a consequence of being hereditarily Lindelof); and (c) $\text{Cl}(Y) \setminus \{\emptyset\}$ is countably tight [12], [8, Corollary 2.16]. The unit square with the lexicographic order (above) is neither separable nor hereditarily Lindelof. The passage from Y to Y_* in Theorem 3.6 thus preserves second countability but it is not clear that it will preserve first countability or the Fréchet-Urysohn property (although we do not have specific counter-examples). This problem, however, is easily circumvented.

Proposition 5.1 *Let Y be a locally compact, σ -compact Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff.*

(i) *If Y is a Fréchet-Urysohn space and q_* is locally closed at x then x is a Fréchet-Urysohn point.*

(ii) *Suppose that q is pure. If $x \in X$ is a Fréchet-Urysohn point then q is locally closed at x .*

Proof (i) Let V be a subset of X and suppose that $x \in \overline{V}$. With $E := q_*^{-1}(V)$, we have $\overline{E} \cap q_*^{-1}(x) \neq \emptyset$ because q_* is locally closed at x . Let (C_α) be a net of points in E converging to some $C \in q_*^{-1}(x)$. Let $y \in C$. Then there exists a net (y_α) , with $y_\alpha \in C_\alpha$ for each α , such that $\lim_\alpha y_\alpha = y$. Then the set $\{y_\alpha\} \subseteq q_*^{-1}(V)$, and hence $\overline{q_*^{-1}(V)} \cap q_*^{-1}(x) \supseteq \{y\}$. Since Y is a Fréchet-Urysohn space, there is a sequence (y_n) in $q_*^{-1}(V)$ converging to y . Thus $(q(y_n))$ gives the required sequence in V converging to x .

(ii) Let $x \in X$ be a Fréchet-Urysohn point and let $E \subseteq X$ with $x \in \overline{E}$. Then by assumption there is a sequence (x_n) in E such that $x_n \rightarrow x$. Let K be the compact set $\{x\} \cup \{x_i : i \geq 1\}$. Then $x \in \overline{E} \cap \overline{K}$, so x is a k -point, and hence q is locally closed at x by Theorem 4.1. ■

Corollary 5.2 *Let Y be a Fréchet-Urysohn locally compact, σ -compact, Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff.*

(i) *The open set $H_{q_*} = L_{q_*} \cup F_{q_*}$ is the set of Fréchet-Urysohn points of X .*

(ii) *X is a Fréchet-Urysohn space if and only if q_* is closed.*

(iii) *If X is locally compact then X is a Fréchet-Urysohn space.*

Proof (i) follows from Proposition 5.1 and Corollary 4.6; (ii) from Proposition 5.1; and (iii) from Proposition 5.1 and Theorem 3.3. ■

Corollary 5.3 *Let Y be a locally compact, σ -compact, Hausdorff space and $q : Y \rightarrow X$ a quotient map with X Hausdorff and a Fréchet-Urysohn space. Then*

- (i) q_* is closed and L_{q_*} is a dense open subset of X with discrete complement and
- (ii) if Y is hereditarily Lindelof then X is first countable at each point of L_{q_*} .

Proof (i) The first statement follows from Theorems 2.1 and 3.6 and Proposition 5.1(ii).

(ii) For second statement, note (as already mentioned) that a locally compact hereditarily Lindelof space is first countable. Furthermore, the property of being hereditarily Lindelof is equivalent, in the presence of local compactness, to every open subset being σ -compact. The set L_{q_*} is locally compact and open, and each open subset of L_{q_*} inherits the property of σ -compactness from its inverse image under q . Hence, L_{q_*} is first countable. ■

Arkhangel'skii's question. In [5, Problems 5.14 and 5.15], Arkhangel'skii asked for a Hausdorff quotient of a locally compact second countable Hausdorff space which is (countable and) Fréchet-Urysohn but nowhere first countable. A locally compact second countable Hausdorff space is perfectly normal (i.e., every open set is an F_σ), and the same is true in every Hausdorff quotient, so every point in a quotient is a G_δ , and it is well known that a G_δ point with a compact neighborhood is first countable. Thus, the required example had to have L_{q_*} empty and $F_{q_*} \cup N_{q_*}$ as a Fréchet-Urysohn space, along the lines of the space S_ω mentioned in Section 1.

In the absence of a structure theory for k_ω -spaces, Arkhangel'skii's question is very pertinent, but in the course of this paper, we have seen multiple reasons why such an example cannot exist: for instance, every Fréchet-Urysohn point in a k_ω -space lies in the closure of L_{q_*} (Corollary 4.6(ii)). As we have mentioned, the question had, in fact, already been answered negatively by Siwiec in 1976 [22] who showed (among other things) that if a k_ω -space is Fréchet-Urysohn and has a k_ω -decomposition of compact metric spaces then it is the closed image of a locally compact separable metric space (and hence Morita's theorem implies that L_{q_*} is dense).

Since a locally compact second countable space is hereditarily Lindelof, Corollary 5.3(i) and (ii) gives a more general negative answer to Arkhangel'skii's question. If the Continuum Hypothesis is assumed, a still more general answer can be given.

Theorem 5.4 (CH) *Let X be a sequential k_ω -space. Then X is first countable at a dense subset of points if and only if the set of non- k -points of X has empty interior.*

Proof The set L_{q_*} of points with compact neighborhoods is open and hence sequential. Since X is sequential it is countably tight, so it follows from Corollary 4.6(ii) that N_{q_*} has empty interior if and only if L_{q_*} is dense. If X is first countable at $x \in X$ then $x \in L_{q_*}$, see [10, p. 113]. Thus if X is first countable at a dense subset of points, N_{q_*} has empty interior. Conversely, if N_{q_*} has empty interior, then L_{q_*} is dense in X , and Arkhangelski showed in 1970 ([3]; see also [4, p. 381]) that under (CH)

every (locally) compact Hausdorff sequential space is first countable at a dense subset of points. ■

In particular, under (CH), every Fréchet-Urysohn k_ω -space is first countable at a dense subset of points. On the other hand, in 1987 Malyhin used the method of forcing to exhibit a compact Hausdorff Fréchet-Urysohn which is nowhere first countable [15]. Thus Corollary 5.3 may be about as far as one can go in ZFC in answer to Arkhangel'skii's question.

Acknowledgment We are grateful to Eric Wofsey and Henno Brandsma for help on Math Stack Exchange at early stages of this enquiry.

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