Quasi-Homogeneous Linear Systems on $\mathbb{P}^2$ with Base Points of Multiplicity 5

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Abstract. In this paper we consider linear systems of $\mathbb{P}^2$ with all but one of the base points of multiplicity 5. We give an explicit way to evaluate the dimensions of such systems.

Introduction

We consider r points $p_1, \ldots, p_r$, in general position on $\mathbb{P}^2$ and to each p_i we associate a natural number m_i called the *multiplicity* of the point. Let r_j be the number of points of multiplicity m_j and let $\mathcal{L}(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k})$ be the linear system of curves of degree d with r_j general base points of multiplicity at least m_j for $j = 1 \ldots k$. A linear system is called *homogeneous* if all the multiplicities are equal and *quasi-homogeneous* if all but one of the multiplicities are equal.

Definitions 0.1 The *effective dimension* of the system is defined to be

$$l(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k}) = \dim \mathcal{L}(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k}),$$

the *virtual dimension* to be

$$v(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k}) = \frac{d(d+3)}{2} - \sum r_i \frac{m_i(m_i+1)}{2},$$

and the *expected dimension*

$$e = \max\{v, -1\}.$$

It follows immediately that for a given system

$$(0.1) \qquad\qquad v \le e \le l,$$

and the second inequality may be strict since the conditions imposed by the points may fail to be independent. In this case we say that the system is *special*.

Let Z be the 0-dimensional scheme defined by the multiple points p_i and consider the exact sequence of sheaves:

$$0 \to \mathcal{I}(Z) \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$$

where $\mathcal{I}(Z)$ is the ideal sheaf of Z. Tensoring with $L = \mathcal{O}_{\mathbb{P}^2}(d)$ and taking cohomology we obtain:

$$0 \to H^0\big(L \otimes \mathcal{I}(Z)\big) \to H^0(L) \to H^0(L_Z) \to H^1\big(L \otimes \mathcal{I}(Z)\big) \to 0.$$

In this way we see that

(0.2) $\qquad h^1\big(L \otimes \mathcal{I}(Z)\big) = l(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k}) - v(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k}).$

Consider the blow up S of $\mathbb{P}^2$ at $p_1, \ldots, p_r$ and denote by $\mathcal{L}$ the strict transform of the system $\mathcal{L}(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k})$. Define the virtual and the expected dimension of $\mathcal{L}$ as those of $\mathcal{L}(d, m_1{}^{r_1}, \ldots, m_k{}^{r_k})$. By Riemann-Roch, the virtual dimension of $\mathcal{L}$ may be given in the following way:

$$v(\mathcal{L}) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K_S}{2},$$

where K_S denotes the canonical bundle of S. We recall that E is a (-1)-curve on S if E is irreducible and $E^2 = E \cdot K_S = -1$. If $\mathcal{L} \cdot E = -t < 0$, then $\mathrm{Bs}\,|\mathcal{L}|$ contains E with multiplicity t. Let $\mathcal{M} = \mathcal{L} - tE$ be the residual system, then:

$$v(\mathcal{L}) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K_S}{2} = \frac{(\mathcal{M} + tE)^2 - (\mathcal{M} + tE) \cdot K_S}{2} = v(\mathcal{M}) + \frac{t - t^2}{2}.$$

So if $t \geq 2$ and $\mathcal{L}$, or equivalently $\mathcal{M}$, is not empty, then $l(\mathcal{L}) = l(\mathcal{M}) \geq v(\mathcal{M}) > v(\mathcal{L})$.

This allows us to recall the following definition given in [CM98]:

Definition 0.2 *A linear system $\mathcal{L}$ is (-1)-special if there are (-1)-curves $E_1, \ldots, E_r$ such that $\mathcal{L} \cdot E_j = -n_j$, with $n_j \geq 1$ for every j, $n_j \geq 2$ for some j, and the residual system $\mathcal{M} = \mathcal{L} - \sum_j n_j E_j$ has non-negative virtual dimension $v(\mathcal{M}) \geq 0$.*

The following conjecture was formulated for the first time in [Hir89]:

Conjecture 0.3 *A linear system $\mathcal{L}$ on $\mathbb{P}^2$ is special if and only if it is (-1)-special.*

The conjecture is known to be true in the following cases:

- The number of points $r \leq 9$ [Har98].
- The system is homogeneous of multiplicity ≤ 12 [CM].
- All the multiplicities are ≤ 4 [Mig00].
- The system is quasi-homogeneous with points of multiplicity ≤ 3 [CM98].
- The system is quasi-homogeneous with points of multiplicity 4 [Laf99], [Sei].
- The number of points is a power of 4 and the system is homogeneous [Lau99].

In this paper we prove the conjecture in the case of quasi-homogeneous systems of multiplicity 5.

The paper is organized as follows: in Section 1 we find a complete list of (-1)-curves that may produce quasi-homogeneous (-1)-special systems of multiplicity 5. Section 2 is devoted to the classification of such systems. In Section 3 we recall the degeneration of $\mathbb{P}^2$ presented in [CM98]. In Section 4 we prove the main theorem under the assumption of some technical lemmas, whose proof is given in Section 5.

1 (-1)-**Curves**

In this section we classify all the (-1)-curves on the blow up of $\mathbb{P}^2$ that may have negative (≤ -2) intersection with the elements of a quasi-homogeneous linear system. With abuse of language we call (-1)-curves also the curves on $\mathbb{P}^2$ whose proper transforms are (-1)-curves. In order to find such curves we first prove that they must be quasi-homogeneous. Then we restrict ourselves to the case of curves having points of multiplicity at most 2 (since otherwise they would have intersection ≥ -1 with an element of a quasi-homogeneous linear system of multiplicity 5), and we give a complete classification with the aid of numerical properties.

Proposition 1.1 ([CM98]) *Let $\mathcal{L}$ be a linear system of $\mathbb{P}^2$ and let E_1, E_2 be two distinct (-1)-curves, such that $E_i \cdot \mathcal{L} < 0$ for $i \in \{1, 2\}$. Then $E_1 \cdot E_2 = 0$.*

Proof By hypothesis $E_1, E_2 \in \mathrm{Bs}\,|\mathcal{L}|$. Suppose that $E_1 \cdot E_2 \geq 1$. From the exact sequence

$$0 \to H^0(S, E_2) \to H^0(S, E_1 + E_2) \to H^0\left(E_1, \mathcal{O}_{E_1}(E_1 \cdot E_2 - 1)\right) \to 0,$$

it follows that $h^0(S, E_1 + E_2) > 1$. This means that $E_1 + E_2$ moves in a linear system, but this is impossible since $E_1 + E_2 \in \mathrm{Bs}\,|\mathcal{L}|$. ∎

Now, consider a curve $E = L(d, m_0, m_1, \ldots, m_r)$, with $m_i \geq 0$. Suppose that E is a (-1)-curve such that $E \in \mathrm{Bs}\,|\mathcal{L}|$. Let $\sigma \in S_r$ be a permutation of the set $\{1, \ldots, r\}$ and consider the curve $E_\sigma = L(d, m_0, m_{\sigma(1)}, \ldots, m_{\sigma(r)})$. Clearly $E_\sigma \cdot E_\sigma = E \cdot E = -1$ and $E_\sigma \cdot K_{\mathbb{P}^2} = E \cdot K_{\mathbb{P}^2} = -1$, hence also E_σ is a (-1)-curve. Observe that since $\mathcal{L}$ is quasi-homogeneous, then if $E \in \mathrm{Bs}\,|\mathcal{L}|$, by symmetry also $E_\sigma \in \mathrm{Bs}\,|\mathcal{L}|$. Now consider the permutation σ_{ij} which switches i with j and leaves fixed all the other numbers. From Lemma 1.1 it follows that $E \cdot E_{\sigma_{ij}} = 0$. Hence we have:

$$E^2 = -1, \quad E \cdot E_{\sigma_{ij}} = 0.$$

These equations translate into the system:

$$\begin{cases} d^2 - m_0^2 - \sum_{k=1}^r m_k^2 = -1 \\ d^2 - m_0^2 - \sum_{\substack{k=1 \\ k \neq i,j}}^r m_k^2 - 2m_i m_j = 0. \end{cases}$$

From these, it follows that $(m_i - m_j)^2 = 1$ which gives $m_i = m_j \pm 1$. Hence, for each i, j, m_i and m_j are equal or they differ only by 1. This means that the (-1)-curve E may be rewritten as $L(d, m_0, m^s, (m+1)^{r-s})$. There are $\binom{r}{s}$ distinct and linearly independent curves of this kind. But $\dim \mathrm{Pic}(S) = r + 1$, so there are only four possibilities: $s = 0$, $s = 1$, $s = r - 1$, $s = r$. If $s = 0$ or $s = r$ then the (-1)-curve E belongs to a quasi-homogeneous system. We call the curve $E_{\mathrm{tot}} = \sum_\sigma E_\sigma$ *compound*.

Hence we have proved the following:

Proposition 1.2   *Let $\mathcal{L}$ be a quasi-homogeneous linear system on $\mathbb{P}^2$ and let E be a (-1)-curve such that $E \cdot \mathcal{L} < 0$. Then E is necessarily of the form:*

$$E = L(d, m_0, m^r),$$

$$E = L(d, m_0, m - 1, m^{r-1}), \quad E_{\text{tot}} = L\left(rd, rm_0, (rm - 1)^r\right),$$

$$E = L(d, m_0, m + 1, m^{r-1}), \quad E_{\text{tot}} = L\left(rd, rm_0, (rm + 1)^r\right).$$

In order to classify (-1)-curves belonging to quasi-homogeneous systems, we begin by classifying quasi-homogeneous curves of multiplicity $m \le 2$.

Proposition 1.3 ([CM98])   *For each $m > 1$, there are only a finite number of quasi-homogeneous (-1)-curves of multiplicity m. For $m \le 2$, the quasi-homogeneous (-1)-curves are:*

$$L(1, 1, 1), \quad L(2, 0, 1^5), \quad L(e, e - 1, 1^{2e}), \quad L(6, 3, 2^7).$$

Proof   Let $E = L(d, m_0, m^r)$ be a (-1)-curve. Then E must satisfy the following equations:

(1.1)
$$\begin{cases} d^2 - m_0^2 - rm^2 = -1 \\ -3d + m_0 + rm = -1. \end{cases}$$

Eliminating r from the two equations we obtain:

(1.2)
$$m - 3dm + mm_0 + d^2 - m_0^2 + 1 = 0.$$

If we put $x = 2d - 3m$ and $y = 2m_0 - m$, equation (1.2) becomes

(1.3)
$$x^2 - y^2 = 4(2m^2 - m - 1).$$

This is the equation of an irreducible conic for $m \ge 2$ and it has a finite number of solutions. In fact, the numbers $x - y$ and $x + y$ must be chosen among the divisors of $4(2m^2 - m - 1)$ and there are only a finite number of choices. In particular, for $m = 2$ we obtain $x^2 - y^2 = 20$, or equivalently $(d - 3)^2 - (m_0 - 1)^2 = 5$. Hence we must have $d - m_0 - 2 = 1$ and $d + m_0 - 4 = 5$ which gives $d = 6$, $m_0 = 3$ and the system $\mathcal{L}(6, 3, 2^7)$.

Finally, for $m = 1$, (1.2) becomes: $(d + m_0 - 2)(d - m_0 - 1) = 0$. This equation has the following solutions: $d = 1$, $m_0 = 1$, which gives the system $\mathcal{L}(1, 1, 1)$; $d = 2$, $m_0 = 0$, which gives the system $\mathcal{L}(2, 0, 1^5)$; $d = m_0 + 1$, corresponding to $\mathcal{L}(d, d - 1, 1^{2d})$. ∎

Proposition 1.4   *The (-1)-curves of multiplicity $m \le 2$ are listed in Table 1.1.*

(-1)-curves E	compound curve E_{tot}
$L(2, 0, 1^5)$	
$L(e, e-1, 1^{2e})$	
$L(6, 3, 2^7)$	
$L(1, 1, 1)$	$L(e, e, 1^e)$
$L(1, 0, 1^2)$	$L(3, 0, 2^3)$

Table 1.1: (-1)-curves of multiplicity ≤ 2.

Proof If $E = L(d, m_0, m-1, m^{r-1})$ then $E_{\text{tot}} = L\big(rd, rm_0, (rm-1)^r\big)$ and it must be $rm - 1 \leq 2$. Since $m \geq 1$ (otherwise $m - 1 < 0$) and $r \geq 2$ (otherwise the system is not compound), the only possibilities are $m = 1$ and $r = 2, 3$. In the first case, we obtain the (-1)-curve $E = L(1, 1, 1)$ which leads to the system $E_{\text{tot}} = L(r, r, 1^r)$. In the second case, $E = L(1, 0, 1^2)$ and $E_{\text{tot}} = L(3, 0, 2^3)$.

If $E = L(d, m_0, m+1, m^{r-1})$, then $E_{\text{tot}} = L\big(rd, rm_0, (rm+1)^r\big)$ and it must be $rm + 1 \leq 2$. If $m = 0$ then we obtain the (-1)-curve $E = L(1, 1, 1)$, which gives the compound curve: $E_{\text{tot}} = L(r, r, 1^r)$. If $m = 1$, then $r \leq 1$ and there are not compound systems with only two points. ∎

2   Quasi-Homogeneous Systems of Multiplicity 5

In this section we classify all the (-1)-special quasi-homogeneous linear systems $\mathcal{L}$ of multiplicity 5.

Proposition 2.1 *All the (-1)-special quasi-homogeneous systems of multiplicity 5 are listed in Table 2.1.*

Proof The procedure for finding these (-1)-special systems may be described as follows:

First step: find all the (-1)-curves E_i such that

$$(2.1) \qquad\qquad E_i \cdot \mathcal{L} = \mu_i \leq -2.$$

Second step: let $\mathcal{M} = \mathcal{L} - \sum \mu_i E_i$ be the residual system. We need to verify that

$$(2.2) \qquad\qquad v(\mathcal{M}) \geq 0.$$

Observe that to complete the first step, once we have found a (-1)-curve E_i that satisfies (2.1), we have to see if there is a (-1)-curve E_j such that $E_j \cdot (\mathcal{L} - \mu_i E_i) \leq -2$ and so on. We may speed the procedure by making an induction on the multiplicity of the system $\mathcal{L}$. Since the system $\mathcal{L} - \mu_i E_i$ has multiplicity less than 4, we need only to know if this system is again (-1)-special or not. In the affirmative case we search for the next (-1)-curve E_j, otherwise we proceed with step two. So we need a complete list of quasi-homogeneous (-1)-special systems with $m \leq 3$. The following table may be found in [CM98].

System		virtual dim.	effective dim.
$\mathcal{L}(d, d, 5^r)$		$d - 15r$	$d - 5r$
$\mathcal{L}(d, d - 1, 5^r)$		$2d - 15r$	$2d - 9r$
$\mathcal{L}(d, d - 2, 5^r)$	$(d, r) \neq (8e, 2e)$	$3d - 15r - 1$	$3d - 12r - 1$
$\mathcal{L}(8e, 8e - 2, 5^{2e})$		$-6e - 1$	0
$\mathcal{L}(d, d - 3, 5^r)$	$(d, r) \neq (7e, 2e)$	$4d - 15r - 3$	$4d - 14r - 3$
	$(7e + 1, 2e)$		
$\mathcal{L}(7e, 7e - 3, 5^{2e})$		$-2e - 3$	0
$\mathcal{L}(7e + 1, 7e - 2, 5^{2e})$		$-2e + 1$	2
$\mathcal{L}(8, 0, 5^2)$		14	15
$\mathcal{L}(10, 2, 5^4)$		2	3
$\mathcal{L}(3e + 5, 3e - 2, 5^{2e})$	$1 \leq e \leq 3$	$19 - 6e$	$20 - 6e$
$\mathcal{L}(3e + 4, 3e - 3, 5^{2e})$	$1 \leq e \leq 2$	$11 - 6e$	$14 - 6e$
$\mathcal{L}(4e + 4, 4e - 2, 5^{2e})$	$1 \leq e \leq 7$	$13 - 2e$	$14 - 2e$
$\mathcal{L}(4e + 3, 4e - 3, 5^{2e})$	$1 \leq e \leq 4$	$6 - 2e$	$9 - 2e$
$\mathcal{L}(4e + 2, 4e - 4, 5^{2e})$	$1 \leq e \leq 2$	$-2e - 1$	$5 - 2e$
$\mathcal{L}(5e + 3, 5e - 2, 5^{2e})$		8	9
$\mathcal{L}(5e + 2, 5e - 3, 5^{2e})$		2	5
$\mathcal{L}(5e + 1, 5e - 4, 5^{2e})$		-4	2
$\mathcal{L}(5e, 5e - 5, 5^{2e})$		-10	0
$\mathcal{L}(6e + 2, 6e - 2, 5^{2e})$		4	5
$\mathcal{L}(6e + 1, 6e - 3, 5^{2e})$		-1	2
$\mathcal{L}(6e, 6e - 4, 5^{2e})$		-6	0
$\mathcal{L}(8, 0, 5^3)$		-1	2
$\mathcal{L}(8, 1, 5^3)$		-2	1
$\mathcal{L}(11, 0, 5^5)$		2	5
$\mathcal{L}(11, 1, 5^5)$		1	4
$\mathcal{L}(11, 2, 5^5)$		-1	2

Table 2.1: (-1)-curves of multiplicity ≤ 2.

In order to satisfy (2.1) for a system $\mathcal{L} = \mathcal{L}(d, m_0, 4^r)$, the (-1)-curves E_i may have multiplicity at most 2. Hence the E_i are those listed in Table 1.1. So there are five possibilities to analyze.

- $\mathcal{L}(d, m_0, 5^r) \cdot L(1, 1, 1) = -\mu$.

  In this case $m_0 = d - 5 + \mu$ and the residual system is

  $$\mathcal{M} = \mathcal{L}\left(d - \mu r, d - \mu r - 5 + \mu, (5 - \mu)^r\right).$$

If $\mu = 5$ then the system is $\mathcal{L}(d, d, 5^r)$. The residual system $\mathcal{M}$ is non-special of dimension $d - 5r$.

If $\mu = 4$ then the system is $\mathcal{L}(d, d - 1, 5^r)$. The residual system $\mathcal{M}$ is non-special of dimension $2d - 9r$.

System		virtual dim.	effective dim.
$\mathcal{L}(4,0,2^5)$		$v=-1$	$l=0$
$\mathcal{L}(2e,2e-2,2^{2e})$	$e \geq 1$	$v=-1$	$l=0$
$\mathcal{L}(d,d,2^e)$	$d \geq 2e \geq 2$	$v=d-3e$	$l=d-2e$
$\mathcal{L}(4,0,3^2)$		$v=2$	$l=3$
$\mathcal{L}(6,0,3^5)$		$v=-3$	$l=0$
$\mathcal{L}(6,2,3^4)$		$v=0$	$l=1$
$\mathcal{L}(3e,3e-3,3^{2e})$	$e \geq 1$	$v=-3$	$l=0$
$\mathcal{L}(3e+1,3e-2,3^{2e})$	$e \geq 1$	$v=1$	$l=2$
$\mathcal{L}(4e,4e-2,3^{2e})$	$e \geq 1$	$v=-1$	$l=0$
$\mathcal{L}(d,d-1,3^e)$	$2d \geq 5e \geq 5$	$v=2d-6e$	$l=2d-5e$
$\mathcal{L}(d,d,3^e)$	$d \geq 3e \geq 3$	$v=d-6e$	$l=d-3e.$

Table 2.2: (-1)-curves of multiplicity ≤ 2.

If $\mu = 3$ then the system is $\mathcal{L}(d,d-2,5^r)$. The residual system $\mathcal{M}$ may be (-1)-special only if $d-3r = 2e$ and $r = 2e$. This implies that the system is $\mathcal{L}(8e,8e-2,5^{2e})$. In this case, $\mathcal{M} = \mathcal{L}(2e,2e-2,2^{2e})$ is (-1)-special of dimension 0. If $\mathcal{M}$ is non-special, then it has dimension $3d-12r-1$.

If $\mu = 2$ then the system is $\mathcal{L}(d,d-3,5^r)$. The residual system $\mathcal{M}$ may be (-1)-special only if $r = 2e$ and $d-2r = 3e$, or $d-2r = 3e+1$. In the first case one obtains the system $\mathcal{L}(7e,7e-3,5^{2e})$ of effective dimension 0. In the second case one obtains the system $\mathcal{L}(7e+1,7e-2,5^{2e})$ of effective dimension 2. If $\mathcal{M}$ is non-special, then it has dimension $4d-14r-3$.

- $\mathcal{L}(d,m_0,5^{2e}) \cdot L(e,e-1,1^{2e}) = -\mu$.

This gives:

$$(2.3) \qquad e(d-m_0-10) + m_0 + \mu = 0.$$

The residual system is $\mathcal{M} = \mathcal{L}\big(d-\mu e, m_0 - \mu(e-1), (5-\mu)^{2e}\big)$. This implies that $m_0 - \mu(e-1) \leq d-\mu e$, which gives $\mu + m_0 \leq d$. On the other hand, from equation (2.3) one deduces that $d-m_0-10 < 0$, since otherwise the sum cannot be 0. So we have the inequality $\mu \leq d-m_0 \leq 9$. Let $k = d-m_0$, then $\mu \leq k \leq 9$. Substituting in (2.3) we obtain:

$$(2.4) \qquad m_0 = (10-k)e-\mu, \quad d = (10-k)e-\mu+k.$$

The residual system is:

$$\mathcal{M} = \mathcal{L}\big((10-k-\mu)e-\mu+k, (10-k-\mu)e, (5-\mu)^{2e}\big).$$

If $k = 9$ then $10-k-\mu < 0$, so there are no systems.

If $k = 8$ then $10-k-\mu \geq 0$ only if $\mu = 2$. In this case $\mathcal{M} = \mathcal{L}(6,0,3^{2e})$ is non-special of dimension $27-12e$. This gives the systems $\mathcal{L}(8,0,5^2)$ and $\mathcal{L}(10,2,5^4)$.

If $k = 7$ then $\mu \leq 3$. If $\mu = 2$, the system $\mathcal{L}(3e + 5, 3e - 2, 5^{2e})$ has virtual dimension $19 - 6e$. The residual system $\mathcal{M} = \mathcal{L}(e + 5, e, 3^{2e})$ is non-special of dimension $20 - 6e$. If $\mu = 3$ the system $\mathcal{L}(3e + 4, 3e - 3, 5^{2e})$ has virtual dimension $11 - 6e$. The residual system $\mathcal{M} = \mathcal{L}(4, 0, 2^{2e})$ has dimension $14 - 6e$.

If $k = 6$ then $\mu \leq 4$. If $\mu = 2$ the system $\mathcal{L}(4e+4, 4e-2, 5^{2e})$ has virtual dimension $13 - 2e$ and the residual system $\mathcal{M} = \mathcal{L}(2e + 4, 2e, 3^{2e})$ is non-special of dimension $14 - 2e$. If $\mu = 3$ then the system $\mathcal{L}(4e + 3, 4e - 3, 5^{2e})$ has virtual dimension $6 - 2e$ and the residual system $\mathcal{M} = \mathcal{L}(e + 3, e, 2^{2e})$ is non-special of dimension $9 - 2e$. If $\mu = 4$ then the system $\mathcal{L}(4e + 2, 4e - 4, 5^{2e})$ has virtual dimension $-2e - 1$ and the residual system $\mathcal{M} = \mathcal{L}(2, 0, 1^{2e})$ is non special of dimension $5 - 2e$.

If $k = 5$ then if $\mu = 2$ the system $\mathcal{L}(5e + 3, 5e - 2, 5^{2e})$ has virtual dimension 8 and the residual system $\mathcal{M} = \mathcal{L}(3e + 3, 3e, 3^{2e})$ is non-special of dimension 9. If $\mu = 3$ the system $\mathcal{L}(5e + 2, 5e - 3, 5^{2e})$ has virtual dimension 2 and the residual system $\mathcal{M} = \mathcal{L}(2e + 2, 2e, 2^{2e})$ is non-special of dimension 5. If $\mu = 4$ the system $\mathcal{L}(5e + 1, 5e - 4, 5^{2e})$ has virtual dimension -4 and the residual system $\mathcal{M} = \mathcal{L}(e + 1, e, 1^{2e})$ is non-special of dimension 2. If $\mu = 5$ the system $\mathcal{L}(5e, 5e - 5, 5^{2e})$ has virtual dimension -10 and the residual system $\mathcal{M} = \mathcal{L}(0, 0, 0)$ is non-special of dimension 0.

If $k = 4$ then if $\mu = 2$ the system $\mathcal{L}(6e+2, 6e-2, 5^{2e})$ has virtual dimension 4 and the residual system $\mathcal{M} = \mathcal{L}(4e+2, 4e, 3^{2e})$ is non-special of dimension 5. If $\mu = 3$ the system $\mathcal{L}(6e + 1, 6e - 3, 5^{2e})$ has virtual dimension -1 and the residual system $\mathcal{M} = \mathcal{L}(3e+1, 3e, 2^{2e})$ is non-special of dimension 2. If $\mu = 4$ the system $\mathcal{L}(6e, 6e - 4, 5^{2e})$ has virtual dimension -6 and the residual system $\mathcal{M} = \mathcal{L}(2e, 2e, 1^{2e})$ is non-special of dimension 0. If $\mu = 5$ the system $\mathcal{L}(6e - 1, 6e - 5, 5^{2e})$ has virtual dimension -11 and the residual system is empty.

- $\mathcal{L}(d, m_0, 5^7) \cdot L(6, 3, 2^7) = -2$.

So $6d - 3m_0 - 68 = 0$. There are no integer values of d and m_0 that satisfy this equation.

- $\mathcal{L}(d, m_0, 5^5) \cdot L(2, 0, 1^5) = -\mu$

with $\mu \in \{2, 3, 4\}$. This equation gives $2d - 25 = -\mu$, so μ must be odd. If $\mu = 3$ then $d = 11$ and the residual system $\mathcal{M} = \mathcal{L}(5, m_0, 2^5)$ is not (-1)-special. Moreover $v(\mathcal{M}) = 5 - m_0(m_0 + 1)/2$, so $v(\mathcal{M}) \geq 0$ only if $m_0 \leq 2$. Hence the systems are: $\mathcal{L}(11, m_0, 5^5)$, $m_0 \in \{0, 1, 2\}$.

- $\mathcal{L}(d, m_0, 5^3) \cdot L(1, 0, 1^2) = -2$,

so $d = 8$ and the residual system $\mathcal{M} = \mathcal{L}(2, m_0, 1^3)$ has virtual dimension $2 - m_0(m_0 + 1)/2$. Hence it must be $m_0 \leq 1$ and the systems are: $\mathcal{L}(8, 0, 5^3)$ and $\mathcal{L}(8, 1, 5^3)$. ∎

3   Degeneration of Linear Systems on $\mathbb{P}^2$

In this section we recall the degeneration technique needed to specialize linear systems on $\mathbb{P}^2$. For a reference see [CM98, Sections 2 and 3].

The idea is to take a flat family X on a complex disk Δ, such that the fiber X_t over a point $t \neq 0$ is a plane, while the central fiber X_0 is the union of a plane $\mathbb{P}$ and a Hirzebruch surface $\mathbb{F} = \mathbb{F}_1$. These two surfaces are joined transversely along a curve R which is a line L in $\mathbb{P}$ and the exceptional divisor E on $\mathbb{F}$.

To give a linear system on X_0 is equivalent to giving two linear systems, on $\mathbb{P}$ and on $\mathbb{F}$, which agree on the curve R.

Fix a positive integer r and another non-negative integer $b \leq r$. Let us consider $r - b + 1$ general points $p_0, p_1, \ldots, p_{r-b}$ in $\mathbb{P}$ and b general points $p_{r-b+1}, \ldots, p_r$ in $\mathbb{F}$. These points are limits of $r + 1$ general points $p_{0,t}, p_{1,t}, \ldots, p_{r,t}$ in X_t. Let us call $\mathcal{L}_t$ the linear system $\mathcal{L}(d, m_0, m^r)$ in $X_t \cong \mathbb{P}^2$ based at the points $p_{0,t}, p_{1,t}, \ldots, p_{r,t}$.

For any integer k there exists a linear system $\mathcal{L}_0$ on X_0 that restricts to $\mathbb{P}$ as a system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d - k, m_0, m^{r-b})$ and to $\mathbb{F}$ as a system $\mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d - k, m^b)$. One can prove that $\mathcal{L}_0$ (for any k and b) can be obtained as a flat limit on X_0 of the system $\mathcal{L} = \mathcal{L}(d, m_0, m^r)$.

We say that $\mathcal{L}_0$ is obtained from $\mathcal{L}$ by a (k, b)-*degeneration*.

We denote by l_0 the dimension of the linear system $\mathcal{L}_0$ on X_0. By semi-continuity, l_0 is not smaller than the dimension of the linear system on the general fiber, *i.e.*,

$$l_0 = \dim(\mathcal{L}_0) \geq l = \dim \mathcal{L}(d, m_0, m^r).$$

Therefore we have the following:

Lemma 3.1 ([CM98, Lemma 2.1]) *If l_0 is equal to the expected dimension e of $\mathcal{L}$, then the system $\mathcal{L}$ is non-special.*

Let $\hat{\mathcal{L}}_{\mathbb{F}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ be the kernels of the restriction of the systems $\mathcal{L}_{\mathbb{F}}$ and $\mathcal{L}_{\mathbb{P}}$ to R, while $\mathcal{R}_{\mathbb{F}}$ and $\mathcal{R}_{\mathbb{P}}$ are the restricted systems. The dimension l_0 is obtained in terms of the dimensions of the systems $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$, and the dimensions of the subsystems $\hat{\mathcal{L}}_{\mathbb{P}} \subset \mathcal{L}_{\mathbb{P}}$ and $\hat{\mathcal{L}}_{\mathbb{F}} \subset \mathcal{L}_{\mathbb{F}}$ consisting of divisors containing the double curve R. Notice that by slightly abusing notation we have

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d - k, m_0, m^{r-b}), \quad \hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d - k - 1, m_0, m^{r-b}),$$

$$\mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d - k, m^b), \quad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d - k + 1, m^b).$$

We recall the following notations:

$v = v(d, m_0, m^r)$	the virtual dimension of $\mathcal{L}$,
$v_{\mathbb{P}} = v(d - k, m_0, m^{r-b})$	the virtual dimension of $\mathcal{L}_{\mathbb{P}}$,
$v_{\mathbb{F}} = v(d, d - k, m^b)$	the virtual dimension of $\mathcal{L}_{\mathbb{F}}$,
$\hat{v}_{\mathbb{P}} = v(d - k - 1, m_0, m^{r-b})$	the virtual dimension of $\hat{\mathcal{L}}_{\mathbb{P}}$,
$\hat{v}_{\mathbb{F}} = v(d, d - k + 1, m^b)$	the virtual dimension of $\hat{\mathcal{L}}_{\mathbb{F}}$,
$l = l(d, m_0, m^r)$	the dimension of $\mathcal{L}$,
$l_{\mathbb{P}} = l(d - k, m_0, m^{r-b})$	the dimension of $\mathcal{L}_{\mathbb{P}}$,
$l_{\mathbb{F}} = l(d, d - k, m^b)$	the dimension of $\mathcal{L}_{\mathbb{F}}$,
$\hat{l}_{\mathbb{P}} = l(d - k - 1, m_0, m^{r-b})$	the dimension of $\hat{\mathcal{L}}_{\mathbb{P}}$,
$\hat{l}_{\mathbb{F}} = l(d, d - k + 1, m^b)$	the dimension of $\hat{\mathcal{L}}_{\mathbb{F}}$,
$r_{\mathbb{P}} = l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1$	the dimension of the restricted system $\mathcal{R}_{\mathbb{P}}$,
$r_{\mathbb{F}} = l_{\mathbb{F}} - \hat{l}_{\mathbb{F}} - 1$	the dimension of the restricted system $\mathcal{R}_{\mathbb{F}}$.

Then, with the help of a transversality lemma of linear systems on R, one can prove the following proposition on the dimension l_0 that we are going to use in the next section.

Proposition 3.2 ([CM98, Proposition 3.3])   *The following identities hold:*

(a) *If $r_\mathbb{P} + r_\mathbb{F} \leq d - k - 1$, then $l_0 = \hat{l}_\mathbb{P} + \hat{l}_\mathbb{F} + 1$.*
(b) *If $r_\mathbb{P} + r_\mathbb{F} \geq d - k - 1$, then $l_0 = l_\mathbb{P} + l_\mathbb{F} - d + k$.*
(c) $v = v_\mathbb{P} + v_\mathbb{F} - d + k = v_\mathbb{F} + \hat{v}_\mathbb{P} + 1 = v_\mathbb{P} + \hat{v}_\mathbb{F} + 1$

4   Main Theorem

In this section we prove the main theorem on quasi-homogeneous linear systems of multiplicity 5, using the degeneration techniques and induction on the number of points.

Main Theorem $\mathcal{L}(d, m_0, 5^r)$ *is special if and only if it is (-1)-special.*

Before proving the main theorem, we state some technical lemmas that simplify the proof of the theorem. The proof of these lemmas is given in the next section.

Lemma 4.1   *A linear system with at most three base points is special if and only if it is (-1)-special.*

Lemma 4.2   *If $m_0 \geq d - 7$, then the system $\mathcal{L}(d, m_0, 5^r)$ is special if and only if it is (-1)-special.*

Lemma 4.3   *If $d \leq 150$, then the system $\mathcal{L}(d, m_0, 5^r)$ is special if and only if it is (-1)-special.*

Proof of Main Theorem   By Lemma 4.2, we may suppose that $d \geq m_0 + 8$. We proceed by induction on r.

Perform a $(4, b)$-degeneration obtaining the systems:

$$\mathcal{L}_\mathbb{P} = \mathcal{L}(d - 4, m_0, 5^{r-b}), \quad \mathcal{L}_\mathbb{F} = \mathcal{L}(d, d - 4, 5^b),$$
$$\hat{\mathcal{L}}_\mathbb{P} = \mathcal{L}(d - 5, m_0, 5^{r-b}), \quad \hat{\mathcal{L}}_\mathbb{F} = \mathcal{L}(d, d - 3, 5^b).$$

Step I $v\big(\mathcal{L}(d, m_0, 5^r)\big) \leq -1.$

Claim 1   *If we can find a $(4, b)$-degeneration such that*

(4.1) $\hat{l}_\mathbb{F} = -1, \quad \hat{v}_\mathbb{P} \leq v, \quad \hat{l}_\mathbb{P} = \hat{v}_\mathbb{P}, \quad v_\mathbb{F} = l_\mathbb{F}, \quad v_\mathbb{P} = l_\mathbb{P},$

then the system $\mathcal{L}(d, m_0, 5^r)$ is empty.

In fact we have that $\hat{l}_{\mathbb{F}} = -1$ and $\hat{l}_{\mathbb{P}} = \hat{v}_{\mathbb{P}} \leq v \leq -1$, and hence $\hat{\mathcal{L}}_{\mathbb{F}} = \hat{\mathcal{L}}_{\mathbb{P}} = \varnothing$. Moreover

$$
\begin{aligned}
r_{\mathbb{P}} + r_{\mathbb{F}} &= l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1 + l_{\mathbb{F}} - \hat{l}_{\mathbb{F}} - 1 \\
&= l_{\mathbb{P}} + l_{\mathbb{F}} \\
&= v_{\mathbb{P}} + v_{\mathbb{F}} \\
&= v + d - 4 \leq d - 5.
\end{aligned}
$$

By Proposition 3.2 and Lemma 3.1 we have that the system is empty and hence the claim follows.

We find conditions on the integer b necessary to guarantee (4.1).

First of all, $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d - 3, 5^b)$ is a (-1)-special system, and it is empty if $4d - 14b - 3 \leq -1$, *i.e.*, $b \geq \frac{2d-1}{7}$.

Moreover, $\hat{v}_{\mathbb{P}} - v = 15b - 5d + 5 \leq 0$ if and only if $b \leq \frac{d-1}{3}$.

Looking at Table 2.1 we have that the system $\mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d - 4, 5^b)$ can be special only if it is one of the following systems

$$
\mathcal{L}(6e + 2, 6e - 2, 5^{2e}),
$$
$$
\mathcal{L}(6e + 1, 6e - 3, 5^{2e}),
$$
$$
\mathcal{L}(6e, 6e - 4, 5^{2e}).
$$

But if b_0 is an integer such that $\mathcal{L}(d, d - 4, 5^{b_0}) = \mathcal{L}(6e + \epsilon, 6e + \epsilon - 4, 5^{2e})$, then taking $b_0 + \lambda$, with $\lambda \geq 1$, we obtain a non-special system.

Concerning the system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d - 4, m_0, 5^{r-b})$, since we are supposing that $m_0 \leq d - 8$ (and then the difference between the degree and m_0 is at least 4), it can not be special if $d \geq 12$ and $r - b$ is odd.

For the same reason we can say that $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d - 5, m_0, 5^{r-b})$ is non-special if $d \geq 12$ and $r - b$ is odd. The only remaining case is $m_0 = d - 8$, because $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d - 5, d - 8, 5^{r-b})$ is (-1)-special, and its dimension is $\hat{l}_{\mathbb{P}} = 4(d - 5) - 14(r - b) - 3$. In this case, instead of proving that $\hat{l}_{\mathbb{P}} = \hat{v}_{\mathbb{P}}$, we look for a b such that $\hat{l}_{\mathbb{P}} = 4(d - 5) - 14(r - b) - 3 < 0$. This is equivalent to saying that $b < r + \frac{23}{14} - \frac{2d}{7}$. Moreover we know that the linear system we started with was $\mathcal{L}(d, d - 8, 5^r)$, which has negative dimension only if $r > \frac{3d}{5} - \frac{28}{15}$. Then, if $b < \frac{3d}{5} - \frac{28}{15} + \frac{23}{14} - \frac{2d}{7} = \frac{66d-47}{210}$, the linear system $\hat{\mathcal{L}}_{\mathbb{P}}$ is empty.

Summarizing, if the difference

$$
\frac{66d - 47}{210} - \frac{2d - 1}{7}
$$

is at least 4, *i.e.*, $d \geq 143$, we can choose two values $b = b_0$ or $b_0 + 2$ between $\frac{2d-1}{7}$ and $\frac{66d-47}{210}$, and such that $r - b$ is odd. Clearly for one of them the system $\mathcal{L}(d, d - 4, 5^b)$ is non-special.

Hence all the conditions of (4.1) hold, and the theorem is true in the case $v \leq -1$.

Step II $v\big(\mathcal{L}(d, m_0, 5^r)\big) \geq 0$.

In this situation the following holds.

Claim 2 *If $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ are non-special and $v - 1 \geq \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}}$, then $v = l_0$.*

In fact, we have that

$$
\begin{aligned}
(4.2) \qquad r_{\mathbb{P}} + r_{\mathbb{F}} - (d - k - 1) &= l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1 + l_{\mathbb{F}} - \hat{l}_{\mathbb{F}} - 1 - (d - k - 1) \\
&= v_{\mathbb{P}} + v_{\mathbb{F}} - d + k - 1 - (\hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}}) \\
&= v - 1 - (\hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}}) \geq 0,
\end{aligned}
$$

which means that $r_{\mathbb{P}} + r_{\mathbb{F}} \geq d - k - 1$ and, by Proposition 3.2, $l_0 = l_{\mathbb{P}} + l_{\mathbb{F}} - d + k$. But since we are supposing that $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ are non-special, $l_0 = v_{\mathbb{P}} + v_{\mathbb{F}} - d + k = v$, and hence the claim.

As before, if $r - b$ is odd and $d \geq 12$, then $\mathcal{L}_{\mathbb{P}}$ is non-special. Moreover, if b can be choosen in a sufficiently large interval, also $\mathcal{L}_{\mathbb{F}}$ is non-special.

The system $\hat{\mathcal{L}}_{\mathbb{F}}$ is (-1)-special and $\hat{l}_{\mathbb{F}} - \hat{v}_{\mathbb{F}} = b$. If $m_0 < d - 8$, $r - b$ is odd and $d \geq 12$, then $\hat{\mathcal{L}}_{\mathbb{P}}$ is non-special. This implies that $\hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} = \hat{v}_{\mathbb{P}} + \hat{v}_{\mathbb{F}} + b$. We have to guarantee that $\hat{v}_{\mathbb{P}} + \hat{v}_{\mathbb{F}} + b \leq v - 1 = v_{\mathbb{F}} + \hat{v}_{\mathbb{P}}$, which is equivalent to $b \leq v_{\mathbb{F}} - \hat{v}_{\mathbb{F}} = d - 3$.

In the case $m_0 = d - 8$, $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d - 5, d - 8, 5^{r-b})$ is (-1)-special and $\hat{l}_{\mathbb{P}} = \hat{v}_{\mathbb{P}} + r - b$, which implies $\hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} = \hat{v}_{\mathbb{P}} + \hat{v}_{\mathbb{F}} + r$. If we want this sum to be smaller than or equal to $v - 1 = \hat{v}_{\mathbb{P}} + v_{\mathbb{F}}$, it must be $r \leq v_{\mathbb{F}} - \hat{v}_{\mathbb{F}} = d - 3$. But since the starting system was $\mathcal{L} = \mathcal{L}(d, d - 8, 5^r)$ and we are supposing that $v(\mathcal{L}) \geq 0$, we have that $r \leq \frac{3d}{5} - \frac{28}{15}$. In particular, if $d \geq \frac{17}{6}$, then $r \leq d - 3$, and we are in the hypothesis of Claim 2.

Summarizing, if $d \geq 12$, we can find a b such that the hypothesis of Claim 2 holds, and hence the theorem in the case $v(\mathcal{L}) \geq 0$. ∎

5 Proofs of Lemmas

Proof of Lemma 4.1 In order to prove this, it is sufficient to consider the three points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and then evaluate the condition imposed by these points on the monomials $x_0^{a_0} x_1^{a_1} x_2^{a_2}$ where $a_0 + a_1 + a_2 = $ degree. ∎

Proof of Lemma 4.2 The case $m_0 \geq d - m - 1$, for quasi-homogeneous systems of multiplicity m, has been proved in [CM98]. Hence we have only to prove the case $m_0 = d - 7$.

Performing recursively k Cremona transformations, we obtain the system $\mathcal{L}(d - 3k, d - 3k - 7, 5^{r-2k}, 2^{2k})$. Let us put $d - 7 = 3t + \epsilon$, with $\epsilon = 0, 1, 2$, and $r = 2q + \eta$, with $\eta = 0, 1$.

- $t \leq q$.

We perform t transformations to obtain the system $\mathcal{L}(7 + \epsilon, \epsilon, 5^{r-2t}, 2^{2t})$.

1. $\epsilon = 0$

   The system is $\mathcal{L}(7, 5^{r-2t}, 2^{2t})$. If $r - 2t \geq 2$, then the line passing through two of the points with multiplicity 5 is a (-1)-curve contained three times in the system. Then either $\mathcal{L}(7, 5^{r-2t}, 2^{2t})$ is (-1)-special, or it is empty.

   If $r - 2t = 0, 1$, then the linear system is homogeneous or quasi-homogeneous of multiplicity 2, and hence the conjecture holds.

2. $\epsilon = 1$

   As before, if $r - 2t \geq 2$, the linear system $\mathcal{L}(8, 1, 5^{r-2t}, 2^{2t})$ is (-1)-special or empty.

   If $r - 2t = 0, 1$, the linear system is equivalent to a homogeneous or a quasi-homogeneous linear system of multiplicity 2, and the conjecture holds.

3. $\epsilon = 2$

   We obtain the linear system $\mathcal{L}(9, 5^{r-2t}, 2^{2t+1})$.

   If $r - 2t \geq 4$, the system is (-1)-special or empty.

   If $r - 2t = 3$, if we cut away the three lines passing through the points of multiplicity 5, we obtain the equivalent system $\mathcal{L}(6, 3^3, 2^{2t})$, and then, after another Cremona transformation, $\mathcal{L}(3, 2^{2t})$, for which the conjecture holds.

   If $r - 2t = 2$, we cut away the line passing through the points of multiplicity 5, and then we perform a Cremona transformation to obtain $\mathcal{L}(6, 2^{2t+1})$. The conjecture still holds.

- $t > q$.

   We can perform q Cremona transformations and get $\mathcal{L}(d-3q, d-7-3q, 5^{\eta}, 2^{2q})$.

   If $\eta = 0$, the linear system is quasi-homogeneous of multiplicity 2, and the conjecture holds.

   If $\eta = 1$ the linear system is $\mathcal{L}(\delta, \delta - 7, 5, 2^{2q})$, with $\delta = d - 3q$. Perform a $(1, b)$-degeneration obtaining the systems:

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta - 1, \delta - 7, 5, 2^{2q-b}), \quad \mathcal{L}_{\mathbb{F}} = \mathcal{L}(\delta, \delta - 1, 2^b),$$

$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta - 2, \delta - 7, 5, 2^{2q-b}), \quad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(\delta, \delta, 2^b).$$

We proceed exactly as we did in the proof of Main Theorem.

Step I $v\big(\mathcal{L}(d, d - 7, 5^{2q+1})\big) \leq -1$.

We want to find a b such that

(5.1) $\hat{l}_{\mathbb{F}} = -1, \quad \hat{v}_{\mathbb{P}} \leq v, \quad \hat{l}_{\mathbb{P}} = \hat{v}_{\mathbb{P}}, \quad v_{\mathbb{F}} = l_{\mathbb{F}}, \quad v_{\mathbb{P}} = l_{\mathbb{P}}.$

Let us find out the conditions on the integer b necessary to guarantee (5.1).

Looking at Table 2.1 we have that the system $\mathcal{L}_{\mathbb{F}}$ is non-special (because it is not (-1)-special), while $\hat{\mathcal{L}}_{\mathbb{F}}$ is (-1)-special, of dimension $\hat{l}_{\mathbb{F}} = \delta - 2b$. In particular, if $b \geq \frac{\delta+1}{2}$, $\hat{\mathcal{L}}_{\mathbb{F}}$ is empty.

Concerning $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta - 1, \delta - 7, 5, 2^{2q-b})$ and $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta - 2, \delta - 7, 5, 2^{2q-b})$, we can perform some Cremona transformations and obtain that they are not special. Moreover, $\hat{v}_{\mathbb{P}} - v = 6b - 4\delta - 2$ is not bigger than zero if we choose $b \leq \frac{2\delta+1}{3}$.

Summarizing, if we can choose an integer b such that

$$\frac{\delta+1}{2} \le b \le \frac{2\delta+1}{3},$$

i.e., if the difference $\frac{2\delta+1}{3} - \frac{\delta+1}{2}$ is at least 1, then we can guarantee all the hypothesis of (5.1). But this is equivalent to saying that $\delta \ge 7$, or $d - 7 \ge 3q$, which is true since $d - 7 = 3t + \epsilon$, and we are supposing $t > q$.

Step II $\quad \nu\big(\mathcal{L}(d, d-7, 5^{2q+1})\big) \ge 0$.

In this case, Claim 2 still holds, and hence, since we have already seen that $\mathcal{L}_\mathbb{P}$ and $\mathcal{L}_\mathbb{F}$ are non-special, to end the proof of the lemma we only have to see that

$$(5.2) \qquad\qquad\qquad \nu - 1 \ge \hat{l}_\mathbb{P} + \hat{l}_\mathbb{F}.$$

The system $\hat{\mathcal{L}}_\mathbb{F}$ is (-1)-special and $\hat{l}_\mathbb{F} - \hat{v}_\mathbb{F} = b$, while $\hat{\mathcal{L}}_\mathbb{P}$ is non-special. Summing up, $\hat{l}_\mathbb{P} + \hat{l}_\mathbb{F} = \hat{v}_\mathbb{P} + \hat{v}_\mathbb{F} + b$, while $\nu - 1 = \hat{v}_\mathbb{P} + v_\mathbb{F}$ and hence (5.2) is equivalent to $b \le v_\mathbb{F} - \hat{v}_\mathbb{F} = \delta = d - 3q$.

Therefore, if we take an integer $b \le d - 3q$, and perform a $(1, b)$-degeneration, we are in the hypothesis of the claim. ∎

Proof of Lemma 4.3 We prove this lemma by writing an algorithm that performs degeneration on quasi-homogeneous linear systems. First we define a function that evaluates the effective dimension of a linear system $\mathcal{L}$ according to the conjecture. We do this in a recursive way: if $\mathcal{L}$ has negative product with some (-1)-curves E (*i.e.*, $\mathcal{L} \cdot E = -t < 0$), then we redefine $\mathcal{L}$ as $\mathcal{L} - tE$ and we restart from the beginning. Otherwise we define the effective dimension of $\mathcal{L}$ as its virtual dimension. If $\mathcal{L}$ is empty, it may happen that the procedure never ends, since at each step there is a (-1)-curve E such that $\mathcal{L} \cdot E < 0$. In this case after a finite number of steps the system $\mathcal{L}$ has a negative degree. This may happen only if the system is empty, in which case we set the effective dimension of $\mathcal{L}$ to be -1.

Then we define a test function, that tries all the possible degenerations of a linear system $\mathcal{L}$ and returns the value 0 if there exists a degeneration such that the dimension of $\mathcal{L}_0$ is equal to the virtual dimension of $\mathcal{L}$. The evaluation of $\dim \mathcal{L}_0$ is performed with the aid of Proposition 3.2. Due to the induction hypothesis the dimension of $\mathcal{L}_\mathbb{P}, \mathcal{L}_\mathbb{F}, \hat{\mathcal{L}}_\mathbb{P}, \hat{\mathcal{L}}_\mathbb{F}$ may be evaluated with the recursive function already defined.

In the final step we consider all the quasi-homogeneous linear systems $\mathcal{L}(d, m_0, 5^r)$ such that $8 \le d \le 150$ and $m_0 \le d - 8$. For each one of them all the possible degenerations are performed with the test function.

This procedure was written in `Maple` and may be found at
http://www.mat.unimi.it/~laface.

We give here a list of the systems for which the degeneration does not work:

Almost all these systems can be studied with quadratic transformations. But for some of them it has been necessary to evaluate the Hilbert function of the corresponding ideal. We did this evaluation with the aid of `Singular` [GPS01]. The

Degree	System			
12	$\mathcal{L}(12, \alpha, 5^5)$,	$\mathcal{L}(12, \beta, 5^6)$;	$1 \leq \alpha \leq 4$	$0 \leq \beta \leq 4$
13	$\mathcal{L}(13, \alpha, 5^6)$,	$\mathcal{L}(13, \beta, 5^7)$;	$\alpha = 4, 5$	$\beta = 0, 1$
14	$\mathcal{L}(14, \alpha, 5^7)$,	$\mathcal{L}(14, \beta, 5^8)$;	$\alpha = 4, 5, 6$	$\beta = 0, 1$
	$\mathcal{L}(14, 6, 5^6)$,			
16	$\mathcal{L}(16, \alpha, 5^9)$,	$\mathcal{L}(16, \beta, 5^{10})$;	$\alpha = 5, 6$	$\beta = 0, 1, 2$
17	$\mathcal{L}(17, 8, 5^9)$;			
18	$\mathcal{L}(18, 10, 5^9)$;			
19	$\mathcal{L}(19, 9, 5^{11})$,	$\mathcal{L}(19, 11, 5^{10})$;		
20	$\mathcal{L}(20, 11, 5^{10})$,	$\mathcal{L}(20, 12, 5^{10})$;		
21	$\mathcal{L}(21, 13, 5^{11})$;			
23	$\mathcal{L}(23, 14, 5^{13})$;			
26	$\mathcal{L}(26, 18, 5^{14})$;			
28	$\mathcal{L}(28, 20, 5^{15})$;			
33	$\mathcal{L}(33, 25, 5^{18})$.			

Table 5.1: The exceptional cases.

source of our program can be downloaded at the following url:
http://www.mat.unimi.it/~laface.

Even in these cases the conjecture was true. ∎

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