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HOPF BIFURCATION FOR IMPLICIT NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. An analog of the Hopf bifurcation theorem is proved for implicit neutral functional differential equations of the form $F(x_t, D'(x_t, \alpha), \alpha) = 0$. The proof is based on the method of S^1-degree of convex-valued mappings. Examples illustrating the theorem are provided.

1. **Introduction.** This paper is a continuation of the work in [17, 30]. We show how the S^1-degree developed in [6, 7, 10, 20, 30] can be applied to investigate the bifurcation problem in order to provide an analog, for implicit neutral functional differential equations (INFDEs), of a local Hopf bifurcation theorem in [17], for delay differential equations.

We consider the following INFDE

$$F\big(x_t, D'(x_t, \alpha), \alpha\big) = 0, \quad \alpha \in \mathbb{R}, \tag{1.1}$$

where "$'$" denotes the derivative with respect to $t \in \mathbb{R}$, $D(\phi, \alpha) = \phi(0) - b(\phi, \alpha)$, $\phi \in C := C([-r, 0], \mathbb{R}^n)$, $b\colon C \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $F\colon C \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ are continuous, $r > 0$ is a constant. Since the equation (1.1) is usually not solvable for $D'(x_t, \alpha)$, it cannot be reduced to a neutral functional differential equation (NFDE) of the type

$$\frac{d}{dt} D(x_t, \alpha) = f(x_t, \alpha), \quad \alpha \in \mathbb{R}, \tag{1.2}$$

for which a global as well as a local Hopf bifurcation theorem is available. We refer to [4, 7, 13, 14, 21, 22, 26, 29] for the detailed results on Hopf bifurcation for retarded functional differential equations and [20, 23, 28] for neutral functional differential equations. The idea of our approach is to embed the INFDE (1.1) into a family of differential inclusions parametrized by a constant $0 < k \leq 1$ such that when $k = 1$ we get the original equation (1.1) and when $k < 1$ the embedded equations are of simpler form for which the known results in [6, 7, 20] apply.

The functional differential inclusions of neutral type has been considered by many authors (*cf.* [11, 12, 18, 19, 24, 30] and the references therein). We extend their studies by investigating the Hopf bifurcation phenomenon for INFDEs, which exhibits the dynamics of solutions as well as the periodic solutions for implicit differential equations.

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This paper is organized as follows. In Section 2, we deal with a general case where the embedded differential inclusions for $k < 1$ are not necessarily differential equations. Section 3 is an application of result in Section 2 and a (local) Hopf bifurcation theorem for INFDE (1.1) is proved. Finally, in Section 4, we give two examples showing how our bifurcation theorem can be applied to obtain periodic solutions for INFDE (1.1).

2. **Hopf bifurcation for neutral functional differential inclusions.** Let $r \geq 0$ be given. We denote by C the Banach space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| := \sup_{r \leq \theta \leq 0} |\phi(\theta)|$, $\phi \in C$, $|x| = \max(|x_1|, \ldots, |x_n|)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Given any $\rho > 0$, $C_\rho := \{\phi \in C : \|\phi\| \leq \rho\}$. For any $x \in C\big([-r, \infty); \mathbb{R}^n\big)$, $x_t \in C$, $t \geq 0$, where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. By $K(E)$ we denote the family of all non-empty compact convex subsets of a normed space E.

We are concerned with the following family of neutral functional differential inclusions (NFDIs) with one parameter

$$\frac{d}{dt}[x(t) - b(x_t, \alpha)] \in G(x_t, \alpha, k), \quad \alpha \in R, \quad k \in (0, 1], \tag{2.1}$$

where G: $C \times \mathbb{R} \times (0, 1] \longrightarrow K(\mathbb{R}^n)$ is an upper semicontinuous mapping (u.s.c. mapping) and $0 \in G(0, \alpha, k)$ for all $(\alpha, k) \in R \times (0, 1]$, b: $C \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is a C^1-mapping. We assume that there is a constant $0 \leq L < 1$ such that

$$|b(\phi, \alpha) - b(\psi, \alpha)| \leq L\|\phi - \psi\| \tag{2.2}$$

for all $\phi, \psi \in C$ and $\alpha \in \mathbb{R}$. Moreover, there exists $\varrho > 0$ such that, for every $k \in (0, 1)$,

$$G(\phi, \alpha, k) = B^{-1}(\alpha, k)A(\alpha, k)\phi + R(\phi, \alpha k) \tag{2.3}$$

for all $(\phi, \alpha) \in C_\varrho \times \mathbb{R}$, where B: $\mathbb{R} \times (0, 1] \longrightarrow M_{n\times n}$ ($n \times n$ matrix space) is continuous and $B(\alpha, k)$ is invertible for each $(\alpha, k) \in \mathbb{R} \times (0, 1)$, $B^{-1}(\alpha, k)$ is the inverse of $B(\alpha, k)$, A: $\mathbb{R} \times (0, 1] \longrightarrow \mathrm{BL}(C_\varrho; \mathbb{R}^n)$ (bounded linear operator space) is continuous, R: $C_\varrho \times \mathbb{R} \times (0, 1) \longrightarrow K(\mathbb{R}^n)$ is an u.s.c. mapping satisfying

$$|R(\phi, \alpha, k)| := \sup\{|y| : y \in R(\phi, \alpha, k)\} \leq g(\alpha)H(\|\phi\|) \tag{2.4}$$

with $\lim_{x \to 0^+} H(x)/x = 0$, and g: $\mathbb{R} \longrightarrow \mathbb{R}^+$ locally bounded.

The assumption (2.3) indicates that for every $k \in (0, 1)$, G admits a linear approximation at $\phi = 0$ and this is not assumed for $k = 1$. Our purpose is to study the Hopf bifurcation of (2.1) when $k = 1$, by examining these inclusions of (2.1) when $k < 1$ which are of simpler form.

Since $0 \in G(0, \alpha, k)$ for $(\alpha, k) \in \mathbb{R} \times (0, 1]$, $x(t) \equiv 0$ is a solution of (2.1) for all $\alpha \in R$. We call $\alpha_0 \in \mathbb{R}$ *nonsingular* if the matrix $A(\alpha_0, k)|_{\mathbb{R}^n}$ is invertible for every $k \in (0, 1]$.

Given a nonsingular $\alpha_0 \in \mathbb{R}$, we associate with it a *characteristic equation*

$$\det \Delta_{\alpha_0, k}(\lambda) = 0 \tag{2.5}$$

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where, for any complex number $\lambda \in \mathbb{C}$, $\Delta_{\alpha_0,k}(\lambda)$ is an $n \times n$ complex matrix defined by

$$\Delta_{\alpha_0,k}(\lambda) := \lambda B(\alpha_0, k)[I - D_\phi b(0, \alpha_0)(e^{\lambda \cdot}\cdot)] - A(\alpha_0, k)(e^{\lambda \cdot}\cdot) \tag{2.6}$$

in which $(e^{\lambda\cdot}\cdot)(\theta, v) = e^{\lambda\theta}v$, $\theta \in [-r, 0]$, $v \in \mathbb{C}^n$.

A solution λ_0 to the characteristic equation (2.5) is called a *characteristic value* of (α_0, k). The assumption that α_0 is nonsingular implies that $\lambda = 0$ is not a characteristic value of (α_0, k), $0 < k \leq 1$.

We call a nonsingular point α_0 a *center* of equation (2.1) with $k = 1$, if $\det \Delta_{\alpha_0,1}(\lambda) = 0$ has a purely imaginary characteristic value. It is called an *isolated center* if there is a neighbourhood of α_0 in $\mathbb{R}$ such that (2.1) with $k = 1$ has no centers other than α_0.

As a standard assumption in Hopf bifurcation theory, we assume in the rest of this section that $\alpha_0 \in \mathbb{R}$ is an isolated center of (2.1) with $k = 1$. It then follows that there exist $\beta_0 > 0$, $\delta_0 > 0$, $\tau > 0$ and $b = b(\alpha_0, \beta_0) > 0$, $c = c(\alpha_0, \beta_0) > 0$ such that

(i) $\det \Delta_{\alpha_0,1}(i\beta_0) = 0$;
(ii) if $0 < |\alpha - \alpha_0| < \delta_0$, then $i\mathbb{R} \bigcap \{\lambda \in \mathbb{C} : \det \Delta_{\alpha,1}(\lambda) = 0\} = \emptyset$;
(iii) for every $k \in [\tau, 1)$, $\det \Delta_{\alpha_0 \pm \delta_0,k}(\lambda) \neq 0$ on $\partial\Omega$, where $\Omega := (0, b) \times (\beta_0 - c, \beta_0 + c) \subset \mathbb{R}^2$.

This leads to the *crossing number of* (α_0, β_0, k) for each $k \in [\tau, 1]$ defined by

$$\gamma(\alpha_0, \beta_0, k) = \deg_B\big(\det \Delta_{\alpha_0-\delta_0,k}(\cdot), \Omega\big) - \deg_B\big(\det \Delta_{\alpha_0+\delta_0,k}(\cdot), \Omega\big) \tag{2.7}$$

where $\deg_B$ denotes the classical Brouwer degree. By the property (iii) above and the continuity of $\det \Delta_{\alpha,k}(\lambda)$ (given by (2.6)) with respect to k, the following proposition follows from the homotopy invariance of degree.

PROPOSITION 2.1. *The number $\gamma(\alpha_0, \beta_0, k)$ defined in (2.7) is independent of $k \in [\tau, 1]$.*

We then write $\gamma(\alpha_0, \beta_0) := \gamma(\alpha_0, \beta_0, k)$, $k \in [\tau, 1]$.

We may now formulate the main result:

THEOREM 2.2 (LOCAL HOPF BIFURCATION THEOREM). *Suppose that b and G satisfy (2.2)–(2.4). If $\gamma(\alpha_0, \beta_0) \neq 0$, then (α_0, β_0) is a bifurcation point of (2.1) for $k = 1$. In particular, there exists a sequence $\big(x_n(t), \alpha_n, \beta_n\big)$ such that $\alpha_n \longrightarrow \alpha_0$, $\beta_n \longrightarrow \beta_0$, $x_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $x_n(t) \in L^2_{loc}(\mathbb{R}, \mathbb{R}^n)$ is a nonconstant $2\pi/\beta_n$-periodic solution for the following inclusion:*

$$\frac{d}{dt}\big(x(t) - b(x_t, \alpha_n)\big) \in G(x_t, \alpha_n, 1).$$

PROOF. First we normalize the period. Let $x(t) = z(\beta t)$. Then (2.1) is transformed into

$$\frac{d}{dt}[z(t) - b(z_t, \beta, \alpha)] \in \frac{1}{\beta}G(z_{t,\beta}, \alpha, k) \tag{2.8}$$

where $z_{t,\beta}(\theta) := z(t+\beta\theta)$ for $\theta \in [-r, 0]$, and $z(t)$ is a 2π-periodic solution of (2.8) if and only if $x(t) := z(\beta t)$ is a $2\pi/\beta$-periodic solution of (2.1).

Let $S^1 := \mathbb{R}/2\pi\mathbb{Z}$, $V := C(S^1; \mathbb{R}^n)$ and $W := L^2(S^1; \mathbb{R}^n)$, and $\mathcal{D}(\alpha_0, \beta_0) = (\alpha_0, \delta_0, \alpha_0 + \delta_0) \times (\beta_0 - c, \beta_0 + c)$. We define the following maps.

$$
(2.9) \qquad \begin{cases} L_0\colon \mathrm{Dom}(L_0) \subset V \longrightarrow W, L_0 z(t) = z'(t),\ z \in \mathrm{Dom}(L_0) = H^1(S^1; \mathbb{R}^n), \\ B_0\colon V \times \mathbb{R}^2 \longrightarrow V,\ B_0(z, \alpha, \beta)(t) := b(z_t, \beta, \alpha), \\ N_0\colon V \times \mathbb{R}^2 \times [\tau, 1] \longrightarrow K(W),\ N_{0k}(\cdot, \cdot, \cdot) = N_0(\cdot, \cdot, \cdot, k), \\ N_0(z, \alpha, \beta, k) := \{f \in W : f(t) \in \frac{1}{\beta} G(z_t, \beta, \alpha),\ \text{a.e. on } S^1\}, \\ (z, \alpha, \beta) \in V \times \mathcal{D}(\alpha_0, \beta_0), t \in S^1. \end{cases}
$$

We define the circle group $S^1 := \{e^{i\theta} \in \mathbb{C} : 0 \le \theta \le 2\pi\}$ and the actions of S^1 on V and W by the argument shift. It follows that V and W are both isometric Banach representations of S^1. By the assumption (2.3)–(2.4) on G, N_{0k} is a well-defined upper semicontinuous mapping with nonempty weakly compact convex values (see [25]). Under the action of $S^1, L_0\colon \mathrm{Dom}(L_0) \subset V \longrightarrow W$ is an S^1- equivariant closed Fredholm operator of index zero defined in (2.9) and $\mathrm{Ker}\, L_0 = \mathbb{R}^n \subset V$ is the invariant subspace of constant functions. Define $K_0\colon V \longrightarrow W$ by

$$
K_0 z := \frac{1}{2\pi} \int_0^{2\pi} z(t)\, dt, \quad z \in V.
$$

Then K_0 is the composition of the inclusion $V \subset W$ with the orthogonal projection of W onto the subspace $\mathbb{R}^n \subset W$. Therefore, K_0 is an equivariant finite dimensional resolvent of the operator L_0. Moreover, by the Sobolev inequality, the operator $R_{K_0} = i(L + K_0)^{-1}\colon W \longrightarrow V$, where i is the inclusion operator, is completely continuous, which renders commutative the following diagram:

$$
\begin{tikzcd}
H^1(S^1; R^n) \arrow[r, "i"] & C^1(S^1; R^n) =: V \\
W := L^2(S^1; R^n) \arrow[u, "(L_0+K)^{-1}"] \arrow[ur, "R_{K_0}"] & H^1(S^1; R^n) \arrow[l, "(L_0+K_0)"'] \arrow[u, "i"']
\end{tikzcd}
$$

Note that the composition of a weakly compact upper semicontinuous mapping with a compact linear operator is compact ([2]). $R_{K_0} N_{0k}$ is therefore compact for every $k \in [\tau, 1]$. By definition (2.9), N_{0k} and B_0 are equivariant. Moreover, the assumption (2.2) implies that B_0 is condensing.

Now, finding periodic solutions of inclusions (2.1) is reduced to finding solutions of the following composite coincidence problem.

$$
(2.10) \qquad \begin{cases} \text{Find } (z, \alpha, \beta) \in V \times \mathcal{D}(\alpha_0, \beta_0) \\ \text{such that } z - B_0(z, \alpha, \beta) \in \mathrm{Dom}(L_0) \\ \text{and } L_0[z - B_0(z, \alpha, \beta)] \in N_{0k}(z, \alpha, \beta) \end{cases}
$$

which is next equivalent to the fixed point problem

$$z \in \Theta_{K_0}(B_0, N_{0k})(z, \alpha, \beta), \quad (z, \alpha, \beta) \in V \times \mathcal{D}(\alpha_0, \beta_0),$$

where $\Theta_{K_0}(B_0, N_{0k})\colon V \times \mathcal{D}(\alpha_0, \beta_0) \longrightarrow K(V)$ is a condensing map given by

$$\Theta_{K_0}(B_0, N_{0k}) := B_0 + R_{K_0}[N_{0k} + K_0(\pi_0 - B_0)]$$

and $\pi_0\colon V \times \mathbb{R}^2 \longrightarrow V$ is the natural projection. Note that π_0 is equivariant. As the composition of equivariant maps, $\Theta_{K_0}(B_0, N_{0k})$ is equivariant for all $k \in [\tau, 1]$.

Let $M = \{0\} \times \mathcal{D}(\alpha_0, \beta_0) \subset V^{S^1} \times \mathbb{R}^2$, where $V^{S^1} := \{z \in V : gz = z \text{ for all } g \in S_1\}$. Then M are the trivial solutions of (2.10). To prove the theorem, we assume that (α_0, β_0) is not a bifurcation point of (2.1) for $k = 1$. Then we find an invariant neighbourhood U of $(0, \alpha_0, \beta_0)$ in $V \times \mathbb{R}^2$ such that U contains only trivial solutions. Note that the center α_0 is isolated. We can construct, by Gleason-Tietze G-Extention lemma (see [3]), an invariant function, called *a complementing function*, with the property that $\varphi(z, \alpha, \beta) \neq 0$ if $(z, \alpha, \beta) \in \overline{U} \cap M$ (see [7, 10, 15, 20]). Let us define now a convex-valued mapping $\Gamma\colon \overline{U} \times [\tau, 1] \longrightarrow K(V \times \mathbb{R}^2)$ by

$$\Gamma(z, \alpha, \beta, k) = \big(z - \Theta_{K_0}(B_0, N_{0k})(z, \alpha, \beta), \varphi(z, \alpha, \beta)\big).$$

It follows that $\Gamma_k := \Gamma(\cdot, \cdot, \cdot, k)$ is an equivariant homotopy between convex-valued condensing fields Γ_τ and Γ_1. Since $\varphi(z, \alpha, \beta) \neq 0$ on $\overline{U} \cap M$ and there is no other solution $(z, \alpha, \beta) \in U$ (nontrivial solutions) such that $z \in \Theta_{K_0}(B_0, N_{01})(z, \alpha, \beta)$, by the contradiction hypothesis. This gives that $z \in \Gamma_1(z, \alpha, \beta)$ for all $(z, \alpha, \beta) \in \overline{U}$ and the existence property of S^1-degree for multivalued mappings (see [30]) implies that S^1-$\mathrm{Deg}(\Gamma_1, U) = 0$. By the homotopy invariance, there exists $k_0 \in [\tau, 1)$ such that S^1-$\mathrm{Deg}(\Gamma_{k_0}, U) = S^1$-$\mathrm{Deg}(\Gamma_1, U) = 0$. However, if $k = k_0$, the equation (2.1) is equivalent to

$$\frac{d}{dt}[x(t) - b(x_t, \alpha)] \in B^{-1}(\alpha, k_0)A(\alpha, k_0)x_t + R(x_t, \alpha, k_0),$$

and by [30], S^1-$\mathrm{Deg}(\Gamma_{k_0}, U) = \sigma\gamma(\alpha_0, \beta_0, k_0) = \sigma\gamma(\alpha_0, \beta_0) \neq 0$, where $\sigma = \pm 1 \neq 0$. This is a contradiction and the proof is completed. ∎

3. **Hopf bifurcation for implicit neutral equations.** In this section, we study the Hopf bifurcation for the following implicit neutral functional differential equation (INFDE).

(3.1) $$F\big(x_t, D'(x_t, \alpha), \alpha\big) = 0$$

where "$'$" denotes the derivative with respect to $t \in \mathbb{R}$, $D(\phi, \alpha) = \phi(0) - b(\phi, \alpha)$, $\phi \in C$, $b\colon C \times \mathbb{R} \longrightarrow \mathbb{R}^n$ satisfies the assumption (2.2) and $F\colon C \times \mathbb{R}^n \times R \longrightarrow \mathbb{R}^n$, is a completely continuous C^1-map. By $D_\varphi F$ and $D_y F$ we denote the partial derivatives of $F = F(\phi, y, \alpha)$.

Let $\alpha_0 \in \mathbb{R}$ be given. Throughout this section, we assume the following:

(i) $F(0, 0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$ and $D_\varphi F(0, 0, \alpha_0)|_{\mathbb{R}^n}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible;

(ii) there exists $h > 0$ and $\delta > 0$ such that all eigenvalues of $D_yF(\varphi, y, \alpha)$ are in the disc $|1 - z| \leq 1$, for all $(\varphi, y, \alpha) \in C_h \times \mathbb{R}^n \times (\alpha_0 - \delta, \alpha_0 + \delta)$;

(iii) for any constant $K > 0$, there exist $M > 0$ and $0 < a < 1$ such that

$$\|y - F(\phi, y, \alpha)\| \leq M + a\|y\| \text{ if } \|\phi\| + |\alpha - \alpha_0| \leq K, \quad y \in \mathbb{R}^n.$$

The assumption (i) implies that $x(t) \equiv 0$ is a solution of (3.1) for all $\alpha \in \mathbb{R}$. We call it the trivial solution. Our aim is to find a sufficient condition under which, near α_0, (3.1) admits non-constant periodic solutions.

In order to apply the theorem in §2, we imbed (3.1) into a family of equations parametrized by a constant $0 < k \leq 1$

$$F\big(x_t, kD'(x_t, \alpha), \alpha\big) + (1 - k)D'(x_t, \alpha) = 0. \tag{3.2}$$

The following proposition is essential in applying Theorem 2.2.

PROPOSITION 3.1. *Under the conditions (ii) and (iii), the equations (3.2) are equivalent to the neutral functional differential inclusions (*NFDIs*)*

$$\frac{d}{dt}[x(t) - b(x_t, \alpha)] \in G(x_t, \alpha, k) \tag{3.3}$$

where $G\colon C_h \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (0, 1] \longrightarrow K(\mathbb{R}^n)$ satisfies (2.3) and (2.4) in §2 with $A(\alpha, k) = -kD_\varphi F(0, 0, \alpha), k \in (0, 1]$ and $B(\alpha, k) = (1-k)I + kD_yF(0, 0, \alpha)$ for $k \in (0, 1)$.

PROOF. (3.2) is equivalent to (3.3) with G defined by

$$G(\varphi, \alpha, k) = \{y \in \mathbb{R}^n : (1 - k)y + F(\varphi, ky, \alpha) = 0\}.$$

Let $f\colon C \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be defined by $f(\varphi, y, \alpha) = y - F(\varphi, y, \alpha)$. Then $G(\varphi, \alpha, k) = \{y \in \mathbb{R}^n : y = f(\varphi, ky, \alpha)\}$. The condition (ii) implies that all eigenvalues of D_yf are in the disc $|z| \leq 1$, therefore f is Lipshitzian with the constant 1. By (iii) and by the fixed point theorem for nonexpansive mappings (see *e.g.* [5]), $G(\varphi, \alpha, k)$ is a nonempty compact convex set. By the closed graph property, $G\colon C_h \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (0, 1] \longrightarrow K(\mathbb{R}^n)$ is upper semicontinuous. If $k < 1$, the mapping $y \longrightarrow f(\phi, ky, \alpha)$ is a contraction and G is a single-valued map. Since, in that case, $I - kD_yF$ is nonsingular, the implicit function theorem implies that

$$D_\varphi G = [I - kD_yf]^{-1}D_\varphi f = -[(1 - k) + kD_yF]^{-1}D_\varphi F,$$

so the formulas for A and B are verified. ∎

By definition in §2 and Proposition 3.1, we see that $\{0\} \times (\alpha_0 - \delta, \alpha_0 + \delta)$ are trivial solutions of (3.3) and α_0 is a nonsingular point. The characteristic equation at α_0 is given by

$$\det \Delta_{\alpha_0,k}(\lambda) = 0, \quad k \in [0, 1]$$

where $\Delta_{\alpha_0,k}(\lambda) = \lambda\big[(1 - k)I + k\big(D_yF(0, 0, \alpha_0)\big)\big][I - D_\phi b(0, \alpha_0)e^{\lambda\cdot}\cdot] + D_\varphi F(0, 0, \alpha_0)e^{\lambda\cdot}\cdot$ is a complex $n \times n$ matrix-valued function of $\lambda \in \mathbb{C}$.

We need one more assumption.

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(iv) α is an isolated center for (3.3) with $k = 1$.

By (iv), there exist $\beta_0 > 0$ such that $\det \Delta_{\alpha_0,1}(i\beta) = 0$ and constants $b, c > 0$ as in § 2. We define the *crossing number* at (α_0, β_0) by

$$\gamma(\alpha_0, \beta_0) = \deg_B\Big(\deg_B(\det \Delta_{\alpha_0-\delta,1}(\cdot), \Omega\Big) - \deg_B\Big(\det \Delta_{\alpha_0-\delta,1}(\cdot), \Omega\Big)$$

where $\Omega := (0, b) \times (\beta_0 - b, \beta_0 + b)$. We now reach the following result:

THEOREM 3.2 (HOPF BIFURCATION THEOREM). *Suppose that (i)–(iv) are satisfied. If $\gamma(\alpha_0, \beta_0) \neq 0$, then (α_0, β_0) is a bifurcation point of (3.1), i.e. there exists a sequence $\{(x_n(t), \alpha_n, \beta_n)\}$ such that $x_n(t) \longrightarrow 0, \alpha_n \longrightarrow \alpha_0, \beta_n \longrightarrow \beta_0$ as $n \longrightarrow \infty$ and $x_n(t)$ is a non constant periodic solution of (3.1) with $\alpha = \alpha_n$ and period $2\pi/\beta_n$.*

PROOF. By Proposition 3.1, this is a direct consequence of Theorem 2.2. ∎

Before concluding this section, we remark that the equation (3.1) is a special type of equation considered in [27]. We also note that when the neutral term $b = 0$, then (3.1) becomes a retarded functional differential equation of implicit type

$$F\big(x_t, \dot{x}(t), \alpha\big) = 0. \tag{3.4}$$

We will discuss some examples in the next section.

4. Examples.

We give two examples to illustrate the applications of Theorem 3.2.

EXAMPLE 1. Consider the following scalar INFDE

$$D'(x_t, \alpha) = \alpha x(t) + cx(t-r) + \big(1 - x^2(t)\big)\frac{D'(x_t, \alpha)}{1 + [D'(x_t, \alpha)]^2} + g(x_t) \tag{4.1}$$

where $r > 0$ and $c \neq 0$ are constants, $\alpha \in \mathbb{R}$ is a parameter, $D(x_t, \alpha) = x(t) - b(x_t, \alpha)$ with b satisfying (2.2), and $g\colon C \longrightarrow \mathbb{R}$ is such that $g(0) = 0$ and $g'_\phi(0) = 0$. It is easily seen that the map F is of the form

$$F(\phi, y, \alpha) = y - \alpha\phi(0) - c\phi(-r) - \big(1 - \phi^2(0)\big)\frac{y}{y^2} - g(\phi).$$

We choose $h = 1$ and $\delta > 0$ arbitrary. An elementry calculation shows that the condition (i) is satisfied. Similarly, condition (iii) holds for $a = 1/2$. Note that we have $f_y(0, 0, \alpha) = 1$. The equation (4.1) is not solvable for $D'(x_t, \alpha)$ about the origin $(0, 0) \in C \times \mathbb{R}^n$ and, therefore, the results in [20, 23, 28] cannot be applied. However, we are able to use Theorem 3.2 to find the periodic solution around $0 \in C$ for some parameter $\alpha \in \mathbb{R}$.

The characteristic equation takes the form

$$\alpha + be^{-\lambda r} = 0. \tag{4.2}$$

In order to locate the isolated centers, we let $\lambda = i\beta_0$ in (4.2) and solve for $\beta_0 > 0$. By a direct computation, it follows that $\alpha_0 = c$ is a nonsingular point and $\beta_0 = (2n + 1)\pi/r$ where n is an arbitrarily nonnegative integer.

To compute the crossing number at each (α_0, β_0), we let $\lambda = x(\alpha) + iy(\alpha)$ in (4.2) and we obtain the following equivalent equations

$$(4.3) \qquad \begin{cases} \alpha e^{x(\alpha)r} \cos y(\alpha)r = b \\ \alpha e^{x(\alpha)r} \sin y(\alpha)r = 0. \end{cases}$$

Note that $x(\alpha_0) = 0$ and $y(\alpha_0) = \beta$. It follows from (4.3) that $x'(\alpha_0) = -\frac{1}{\alpha_0 r} < 0$. Therefore, $\gamma(\alpha_0, \beta_0) = 1$ for all $(\alpha_0, \beta_0) = \big(c, (2n+1)\pi/r\big)$. By Theorem 3.2, $\big(c, (2n+1)\pi/r\big)$ are all bifurcation points, and nonconstant periodic solutions exist for equation (4.1) when α is near c.

It is very interesting to note that the above conclusion is independent of the neutral term $b(\phi, \alpha)$ in $D(\phi, \alpha)$. And therefore, by choosing $b = 0$, the following retarded implicit functional differential equation (RIFDE)

$$x'(t) = \alpha x(t) + bx(t-r) + \big(1 - x^2(t)\big)\frac{x'(t)}{1 + [x'(t)]^2} + g(x_t)$$

has bifurcation points $(\alpha_0, \beta_0) = \big(c, (2n+1)\phi/r\big)$ where $n \geq 0$ is an integer.

EXAMPLE 2. Let us consider the following 2-dimensional RIFDE

$$(4.4) \qquad \begin{cases} x_1'(t) = ax_1(t) + \alpha bx_1(t-r) + g(x_t) \\ x_2'(t) = cx_2(t-r) + u\big(x_2'(t)\big) \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $0 < a < b$, $r > 0$ and $c \neq 0$ are constants, $\alpha \in \mathbb{R}$ is a parameter, $g\colon C \longrightarrow \mathbb{R}$ is of class C^1 with $g(0) = 0$ and $g'_\phi(0) = 0$, $u\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a differential functional with the property that $\lim_{x\to\infty} |u(x)/x| < 1$, $u(x) = x$ in a neighbourhood of zero and $|u'(x)| \leq 1$ for all $x \in \mathbb{R}$. One of these u can be given by

$$u(x) = \begin{cases} x, & \text{if } |x| \leq d \\ \operatorname{sgn} x(d + \ln|\frac{x}{d}|), & \text{if } |x| > d, \end{cases}$$

where $d > 0$ is arbitrarily given. It follows that f in (4.4) is of the form

$$F(\phi, y, \alpha) = \begin{bmatrix} y_1 - a\phi_1(0) - \alpha b\phi_1(-r) - g(\phi) \\ y_2 - c\phi_2(-r) - u(y) \end{bmatrix}, \quad \phi = (\phi_1, \phi_2) \in C([-r, 0]; \mathbb{R}^2),$$

and f satisfies the conditions (i)–(iii). To check the condition (iv), we look at the characteristic equation at α which reads

$$(4.5) \qquad \lambda - ae^{\lambda} - \alpha be^{\lambda r} = 0.$$

Let $\lambda = i\beta$ in (4.5). We obtain two equations which are equivalent to (4.5)

$$(4.6) \qquad \begin{cases} \tan\beta(r) = \beta/a \\ \alpha^2 = (a^2 + \beta^2)/b^2. \end{cases}$$

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It follows that (4.6) has infinite many solutions for $\beta > 0$ and α, and we denote those solutions by $(\alpha_{\pm n}, \beta_n)$, where

$$\alpha_{\pm n} = \pm\sqrt{a^2 + \beta_n^2}/b, \quad n = 1, 2, 3, \ldots$$

so that (4.4) has isolated centers $(\alpha_{\pm}n, \beta_n)$, $n = 1, 2, \ldots$.

We now compute their crossing numbers. Let $\lambda = x(\alpha) + iy(\alpha)$ in (4.5). As in Example 1, one can show that for each $n = 1, 2, \ldots$

$$x'(\alpha_{\pm n}) = -\frac{ab\alpha_{\pm n}(1 + r\alpha_{\pm}nb)}{A_{\pm n}^2 + B_n^2}$$

where $A_{\pm n} = \alpha_{\pm n}b - r\beta_n^2 + a^2 r$, $B_n = 2a\gamma\beta_n$. Therefore if $r \geq 1/a$, then $x'(\alpha_{\pm n}) \neq 0$ and $\gamma(\alpha_{\pm n}, \beta_n) = \pm 1 \neq 0$. By Theorem 3.2, all $(\alpha_{\pm n}, \beta_n)$ are bifurcation points if $r \geq 1/a$.

Note that since we have $u(x) = x$ if x is small, (4.4) is unsolvable for $x_2'(t)$. Moreover the above analysis is independent of $d > 0$. We choose d sufficiently large so that $u\big(x_2'(t)\big) = x_2'$ since $x_2'(t)$ is bounded. Therefore the second equation in (4.4) gives that $x_2(t) = 0$ and (4.4) is reduced to

$$\begin{cases} x_1'(t) = ax_1(t) + \alpha bx_1(t - r) + \bar{g}(x_{1t}) \\ x_2'(t) = 0 \end{cases} \tag{4.7}$$

where $\bar{g}(x_{1t}) = g(\tilde{x}_t), \tilde{x}_t = (x_{1t}, 0)$. Thus, as discussed above, (4.7) has bifurcation points $(\alpha_{\pm n}, \beta_n)$ if $r \geq 1/a$. However, since (4.7) is singular at the right hand side, the known result (*cf.* [4, 7, 14, 21]) is not directly applicable, even though the system (4.7) is explicit for $\big(x_1(t), x_2(t)\big)$.

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