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EXTENDED CESÀRO OPERATOR BETWEEN SOME HOLOMORPHIC FUNCTION SPACES

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We characterize the boundedness and compactness of the extended Cesàro operator T_g from H^∞ to the mixed norm space and Bloch-type space (or little Bloch-type space), where g is a given holomorphic function in the unit ball of $\mathbf{C}^n$ and T_g is defined by

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)(dt/t).$$

1. INTRODUCTION

Let $\mathbf{B} = \{z \in \mathbf{C}^n; |z| < 1\}$ be the unit ball of $\mathbf{C}^n$, and let $H(\mathbf{B})$ be the family of all holomorphic functions on $\mathbf{B}$. We denote by H^∞ the space of all bounded functions in $H(\mathbf{B})$. H^∞ is a Banach space under the norm

$$\|f\|_\infty = \sup\Big\{|f(z)|; z \in \mathbf{B}\Big\}.$$

A positive continuous function φ on $[0,1)$ is called normal if there are three constants $0 \leqslant \delta < 1$ and $0 < a < b$ such that

$$(P_1) \qquad \frac{\varphi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1^-} \frac{\varphi(r)}{(1-r)^a} = 0;$$

$$(P_2) \qquad \frac{\varphi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1^-} \frac{\varphi(r)}{(1-r)^b} = \infty.$$

We extend it to $\mathbf{B}$ by $\varphi(z) = \varphi(|z|)$. For $f \in H(\mathbf{B})$ we set

$$\|f\|_{p,q,\varphi} = \left\{\int_0^1 M_q^p(f,r)\frac{\varphi^p(r)}{1-r}dr\right\}^{1/p}, \qquad 0 < p < \infty,$$

and

$$\|f\|_{\infty,q,\varphi} = \sup_{0<r<1} M_q(f,r)\varphi(r).$$

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Here

$$M_q(f,r) = \left\{\int_{\partial\mathbf{B}} |f(r\zeta)|^q d\sigma(\zeta)\right\}^{1/q}, \qquad 0 < q < \infty;$$
$$M_\infty(f,r) = \sup_{\zeta\in\partial B} |f(r\zeta)|.$$

The mixed norm space $H_{p,q}(\varphi)$, $0 < p, q \leqslant \infty$, is the space of all functions $f \in H(\mathbf{B})$ for which $\|f\|_{p,q,\varphi} < \infty$. When $0 < p = q < \infty$, $H_{p,q}(\varphi)$ is just the weighted Bergman space

$$A_a^p(\varphi) = \left\{f \in H(\mathbf{B}) : \|f\|_{A_a^p} = \left\{\int_{\mathbf{B}} |f(z)|^p \frac{\varphi^p(z)}{1-|z|} dv(z)\right\}^{1/p} < \infty\right\}.$$

A function $f \in H(\mathbf{B})$ is said to belong to the Bloch-type space $\mathcal{B}_\varphi$ if

$$\|f\|_{\mathcal{B}_\varphi} = \sup_{z\in\mathbf{B}} \varphi(z)|\nabla f(z)| < \infty;$$

and it is said to belong to the little Bloch-type space $\mathcal{B}_{\varphi,0}$ if

$$\lim_{|z|\to 1} \varphi(z)|\nabla f(z)| = 0.$$

Here

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$$

is the complex gradient of f. It is easy to check that both $\mathcal{B}_\varphi$ and $\mathcal{B}_{\varphi,0}$ are Banach spaces under the norm $\|f\|_\varphi = |f(0)| + \|f\|_{\mathcal{B}_\varphi}$, and $\mathcal{B}_{\varphi,0}$ is a closed subspace of $\mathcal{B}_\varphi$. When $\varphi(r) = 1 - r^2$ and $\varphi(r) = (1-r^2)^{1-\alpha}$ with $\alpha \in (0,1)$, two typical normal weights, the induced spaces $\mathcal{B}_\varphi$ are the Bloch space and Lipschitz type space, respectively.

Let $\mathbf{D}$ denote the open unit disc in the complex plane $\mathbf{C}$. For a holomorphic function $f(z)$ on $\mathbf{D}$ with Taylor expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the Cesàro operator acting on f is

$$C[f](z) = \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j} a_k\right) z^j.$$

The behaviour of the operator $C[\cdot]$ have been studied extensively on various spaces of holomorphic functions (see [**5, 6, 7, 8, 11, 12**]). A little calculation shows

$$C[f](z) = \frac{1}{z}\int_0^z f(t)\big(\log(1/1-t)\big)' dt.$$

Hence, on most holomorphic function spaces, $C[\cdot]$ is bounded if and only if the integral operator

$$f \longmapsto \int_0^z f(t)\big(\log(1/1-t)\big)' dt$$

is bounded. From this point of view it is natural to consider the extended Cesàro operator T_g on $H(\mathbf{D})$ with holomorphic symbol g,

$$T_g f(z) = \int_0^z f(t) g'(t) dt. \tag{1.1}$$

The boundedness and compactness of this operator on Hardy spaces, Bergman spaces, Bloch-type spaces and Lipschitz spaces have been studied in [**1**, **2**, **10**].

For $f \in H(\mathbf{B})$, the radial derivative of f is

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j}.$$

Given $g \in H(\mathbf{B})$, the operator T_g on $H(\mathbf{B})$ is defined by

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(\mathbf{B}), \quad z \in \mathbf{B}. \tag{1.2}$$

It is trivial that (1.2) is just (1.1) when $n = 1$. In the unit ball, Hu [**3**] got the characterisation on g for which the induced extended Cesàro operator is bounded or compact on the Bergman space $L^p_{a,\omega}$, Zhang [**13**] studied the same problems between $\mathcal{B}_{(1-r^2)^p}$ and $\mathcal{B}_{(1-r^2)^q}$ for $0 < p, q < \infty$. And also, Hu discussed the boundedness and compactness of T_g on the mixed norm space $H_{p,q}(\varphi)$, where $0 < p, q \leqslant \infty$ (see [**4**]). The purpose of this work is to obtain the sufficient and necessary conditions on $g \in H(\mathbf{B})$, such that the operator $T_g : H^\infty \to H_{p,q}(\varphi)$ (respectively, $H^\infty \to \mathcal{B}_\varphi$, $H^\infty \to \mathcal{B}_{\varphi,0}$) is bounded or compact.

In what follows, C will stand for positive constants whose value may change from line to line but not depend on the functions in $H(\mathbf{B})$. The expression $A \simeq B$ means $C^{-1}A \leqslant B \leqslant CA$.

2. Some preliminary results

Lemma 2.1. ([4]) *Let $0 < p, q \leqslant \infty$ and φ be normal. Then for any $f \in H(\mathbf{B})$,*

$$\|f\|_{p,q,\varphi} \simeq |f(0)| + \left\{ \int_0^1 M_q^p(\Re f, r)(1-r^2)^p \frac{\varphi^p(r)}{1-r} dr \right\}^{1/p}.$$

Lemma 2.2. ([9]) *Let φ be normal and $f \in H(\mathbf{B})$. Then*

(A) *$f \in \mathcal{B}_\varphi$ if and only if $\sup_{z \in \mathbf{B}} \varphi(z)|\Re f(z)| < \infty$. Moreover,*

$$\|f\|_\varphi \simeq |f(0)| + \sup_{z \in \mathbf{B}} \varphi(z)|\Re f(z)|.$$

(B) *$f \in \mathcal{B}_{\varphi,0}$ if and only if $\lim_{|z| \to 1} \varphi(z)|\Re f(z)| = 0$.*

Lemma 2.3. *Let φ be normal, $0 < p, q \leqslant \infty$ and $g \in H(\mathbf{B})$. Then $T_g : H^\infty \to H_{p,q}(\varphi)$ (or $H^\infty \to \mathcal{B}_\varphi$) is compact if and only if for any bounded sequence $\{f_j\} \subseteq H^\infty$ which converges to 0 uniformly on any compact subset of $\mathbf{B}$, we have $\lim_{j\to\infty} \|T_g f_j\|_{p,q,\varphi} = 0$ (or $\lim_{j\to\infty} \|T_g f_j\|_\varphi = 0$).*

Proof. It can be proved by Montel's Theorem and the definition of compact operator. The details are omitted here. □

3. Main results

Theorem 3.1. *Let φ be normal, $0 < p < \infty$, $0 < q \leqslant \infty$ and $g \in H(\mathbf{B})$. Then the following statements are equivalent:*

(A) $T_g : H^\infty \to H_{p,q}(\varphi)$ *is bounded;*

(B) $T_g : H^\infty \to H_{p,q}(\varphi)$ *is compact;*

(C) $g \in H_{p,q}(\varphi)$.

In this case, $\|T_g\| \simeq \big\|g - g(0)\big\|_{p,q,\varphi}$.

Proof: The implication (B)⇒(A) is trivial.

(A)⇒(C). Suppose $T_g : H^\infty \to H_{p,q}(\varphi)$ is bounded, by the fact that $g(z) = g(0) + T_g(1)(z)$ we know $g \in H_{p,q}(\varphi)$. Moreover,

$$\big\|g - g(0)\big\|_{p,q,\varphi} = \big\|T_g(1)\big\|_{p,q,\varphi} \leqslant C\|T_g\|. \tag{3.1}$$

(C)⇒(B). First, for $f, g \in H(\mathbf{B})$, by direct calculation we see

$$\Re(T_g f)(z) = f(z)\Re g(z).$$

Let $g \in H_{p,q}(\varphi)$, $\{f_j\} \subseteq H^\infty$ satisfying $\|f_j\|_\infty \leqslant 1$. By Montel's Theorem, there exists some subsequence of $\{f_j\}$ converging to f uniformly on any compact subset of $\mathbf{B}$. Without loss of generality, we suppose the subsequence is $\{f_j\}$ itself. Then $f \in H(\mathbf{B})$ and $\|f\|_\infty \leqslant 1$. Hence

$$M_q^p\big((f_j - f)\Re g, r\big) \leqslant 2^p M_q^p(\Re g, r).$$

By $g \in H_{p,q}(\varphi)$, Lemma 2.1 and the dominated convergence theorem we obtain

$$\int_0^1 M_q^p\big((f_j - f)\Re g, r\big)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} dr \to 0 \quad (j \to \infty).$$

Lemma 2.1 implies, as $j \to \infty$,

$$\begin{aligned} \|T_g f_j - T_g f\|_{p,q,\varphi}^p &\leqslant C \int_0^1 M_q^p(\Re(T_g f_j - T_g f), r)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} dr \\ &= C \int_0^1 M_q^p\big((f_j - f)\Re g, r\big)(1 - r^2)^p \frac{\varphi^p(r)}{1 - r} dr \\ &\to 0. \end{aligned}$$

Therefore, $T_g : H^\infty \to H_{p,q}(\varphi)$ is compact.

Furthermore, for any $f \in H^\infty$, Lemma 2.1 yields

$$\begin{aligned}\|T_g f\|^p_{p,q,\varphi} &\leqslant C\int_0^1 M_q^p(\Re g, r)\sup\Big\{|f(z)|^p; |z| = r\Big\}(1-r^2)^p\frac{\varphi^p(r)}{1-r}dr\\ &\leqslant C\big\|g-g(0)\big\|^p_{p,q,\varphi}\|f\|^p_\infty.\end{aligned}$$

This, together with (3.1), means $\|T_g\| \simeq \big\|g - g(0)\big\|_{p,q,\varphi}$. The proof is completed. ∎

REMARK. When $p = \infty$, the implication (C)⇒(B) does not hold for any $0 < q \leqslant \infty$ in general. For example, we let $n = 1$, and choose some g and φ satisfying (C) but $T_g : H^\infty \to H_{p,q}(\varphi)$ is not compact. In fact, set $g(z) = \big(z/(1-z)^{1+(1/q)}\big)$, where $z \in \mathbf{D}$ and $0 < q \leqslant \infty$, $\varphi(r) = 1 - r$. Then for $0 < q < \infty$,

$$M_q(g, r) = \left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{r^q d\theta}{|1 - re^{i\theta}|^{q+1}}\right\}^{1/q} \simeq \frac{1}{1-r},$$

as $r \to 1^-$. For $q = \infty$,

$$\sup_{0\leqslant\theta<2\pi}\frac{r}{|1-re^{i\theta}|} \simeq \frac{1}{1-r} \quad \text{as } r \to 1^-.$$

Hence, for $0 < q \leqslant \infty$ and $1/2 \leqslant r < 1$,

$$M_q(g,r)\varphi(r) \simeq \frac{1}{1-r}\varphi(r) = 1.$$

Write $f_j(z) = z^j$, $z \in \mathbf{D}$. Then $\|f_j\|_\infty \leqslant 1$ and $\{f_j\}$ converges to 0 uniformly on any compact subset of $\mathbf{D}$. However, for each j, Lemma 2.1 and $g(0) = 0$ yield

$$\begin{aligned}\|T_g f_j\|_{\infty,q,\varphi} &\simeq \sup_{0<r<1} M_q(\Re T_g(f_j), r)(1-r^2)\varphi(r)\\ &= \sup_{0<r<1} r^j M_q(\Re g, r)(1-r^2)\varphi(r)\\ &\geqslant C \sup_{(1/2)\leqslant r<1} r^j M_q(g,r)\varphi(r)\\ &\geqslant C\lim_{r\to1^-} r^j = C,\end{aligned}$$

where the constant C is independent of j.

THEOREM 3.2. *Let φ be normal and $g \in H(\mathbf{B})$. Then the following statements are equivalent:*

(A) $T_g(H^\infty) \subseteq \mathcal{B}_{\varphi,0}$;

(B) $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ *is bounded;*

(C) $T_g : H^\infty \to \mathcal{B}_\varphi$ *is compact;*

(D) $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ *is compact;*

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(E) $g \in \mathcal{B}_{\varphi,0}$.

In this case, $\|T_g\| \simeq \sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z)\big|$.

PROOF: The implications (B)$\Rightarrow$(A) and (D)$\Rightarrow$(C) are obvious.

(B)$\Rightarrow$(E). Suppose $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ is bounded, then $g = g(0) + T_g(1) \in \mathcal{B}_{\varphi,0}$. Furthermore,

$$\sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z)\big| \simeq \big\|g - g(0)\big\|_\varphi = \big\|T_g(1)\big\|_\varphi \leqslant C\|T_g\|. \tag{3.2}$$

(E)$\Rightarrow$(B). Let $g \in \mathcal{B}_{\varphi,0}$, then for any $f \in H^\infty$, $T_g f \in \mathcal{B}_{\varphi,0}$. Moreover,

$$\|T_g f\|_\varphi \simeq \sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z)\big|\big|f(z)\big| \leqslant \|f\|_\infty \sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z)\big| \leqslant C\|f\|_\infty.$$

This, together with (3.2), shows $\|T_g\| \simeq \sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z)\big|$.

(A)$\Rightarrow$(B). Suppose $\{f_j\} \subseteq H^\infty$, $f \in H^\infty$ and $h \in \mathcal{B}_{\varphi,0}$ satisfying $\lim_{j\to\infty} \|f_j - f\|_\infty = 0$ and $\lim_{j\to\infty} \|T_g f_j - h\|_\varphi = 0$. Then

$$f_j(z) \to f(z) \quad (j \to \infty), \qquad z \in \mathbf{B}. \tag{3.3}$$

And

$$\big|T_g f_j(0) - h(0)\big| + \sup_{z\in\mathbf{B}} \varphi(z)\big|\Re g(z) f_j(z) - \Re h(z)\big| \to 0 \quad \text{as } j \to \infty.$$

So $h(0) = 0$ and for every $z \in \mathbf{B}$

$$f_j(z)\Re g(z) \to \Re h(z) \quad (j \to \infty). \tag{3.4}$$

By (3.3), we have

$$\lim_{j\to\infty} f_j(z)\Re g(z) = f(z)\Re g(z), \qquad z \in \mathbf{B}. \tag{3.5}$$

Thus, (3.4) and (3.5) imply $f(z)\Re g(z) = \Re h(z)$. Therefore,

$$h(z) = \int_0^1 \Re h(tz)\frac{dt}{t} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} = (T_g f)(z).$$

Consequently, $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ is a closed operator. By the closed graph theorem, $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ is bounded.

(C)$\Rightarrow$(E). Suppose $g \notin \mathcal{B}_{\varphi,0}$. Then there would be some $\varepsilon_0 > 0$ and some sequence $\{z^j\} \subseteq \mathbf{B}$ satisfying $\lim_{j\to\infty} |z^j| = 1$, but for each j, $\varphi(z^j)\big|\Re g(z^j)\big| > \varepsilon_0$. Set

$$f_j(z) = \frac{1 - |z^j|^2}{1 - \langle z, z^j\rangle}, \qquad z \in \mathbf{B}.$$

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It is easy to check that $\{f_j\}$ is a bounded sequence in H^∞ and $f_j \to 0$ uniformly on any compact subset of $\mathbf{B}$ as $j \to \infty$. Since $T_g : H^\infty \to \mathcal{B}_\varphi$ is compact, by Lemma 2.3,

$$\|T_g f_j\|_\varphi \to 0 \quad (j \to \infty). \tag{3.6}$$

On the other hand,

$$\begin{aligned} \|T_g f_j\|_\varphi &\simeq |T_g f_j(0)| + \sup_{z\in\mathbf{B}} \varphi(z)|\Re g(z)||f_j(z)| \\ &\geqslant \varphi(z^j)|\Re g(z^j)||f_j(z^j)| \\ &\geqslant \varphi(z^j)|\Re g(z^j)| \\ &\geqslant \varepsilon_0. \end{aligned}$$

This is a contradiction to (3.6).

(E)$\Rightarrow$(D). Suppose $g \in \mathcal{B}_{\varphi,0}$, then for any $f \in H^\infty$, $T_g f \in \mathcal{B}_{\varphi,0}$. And also, for every $\varepsilon > 0$, there exists some $r > 0$ such that

$$\varphi(z)|\Re g(z)| < \varepsilon \qquad \text{whenever } |z| > r. \tag{3.7}$$

Let $\{f_j\}$ be any bounded sequence in H^∞, say $\|f_j\|_\infty \leqslant 1$ and $f_j \to 0$ uniformly on any compact subset of $\mathbf{B}$ as $j \to \infty$. Then for the above ε, there is a positive integer J such that for $|z| \leqslant r$ and $j > J$,

$$|f_j(z)| < \frac{\varepsilon}{\|g\|_\varphi + 1}. \tag{3.8}$$

Thus, combining (3.7) and (3.8), we have

$$\|T_g f_j\|_\varphi \simeq \sup_{z\in\mathbf{B}} \varphi(z)|\Re g(z)||f_j(z)| < \varepsilon \quad \text{if } j > J.$$

By Lemma 2.3, $T_g : H^\infty \to \mathcal{B}_{\varphi,0}$ is compact. The proof is completed. ∎

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