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GORENSTEIN GRADED ALGEBRAS AND THE EVALUATION MAP

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ABSTRACT. We consider graded connected Gorenstein algebras with respect to the evaluation map $\mathrm{ev}_G = \mathrm{Ext}_G(k,\varepsilon) :: \mathrm{Ext}_G(k,G) \longrightarrow \mathrm{Ext}_G(k,k)$. We prove that if $\mathrm{ev}_G \neq 0$, then the global dimension of G is finite.

This paper fits into the general program of Halperin *et al.* to identify and use new algebraic topological constructs derived from differential homological algebra to classify spaces ([3], [6], [7]).

A graded connected, finite type algebra G defined over a field k is called *Gorenstein* if the graded vector space, $\mathrm{Ext}_G(k,G)$ has dimension one. The global dimension of G, $\mathrm{gldim}\, G$, is the minimum integer n such that the residual field k admits a free resolution of length n. If k does not admit free resolutions of finite length, then the global dimension is infinite. The evaluation map

$$\mathrm{ev}_G \colon \mathrm{Ext}_G(k,G) \longrightarrow \mathrm{Ext}_G(k,k)$$

is the canonical map induced by the augmentation $\varepsilon \colon G \longrightarrow k$. Our first result is:

THEOREM 1. *Let G be a graded Gorenstein algebra such that $\mathrm{ev}_G \neq 0$, then* $\mathrm{gldim}\, G < \infty$, *and* $\dim \mathrm{Ext}_G(k,k) < \infty$.

Elliptic Hopf algebras are examples of Gorenstein algebras. For recall an elliptic Hopf algebra is a graded connected finite type Hopf algebra with finite depth and polynomial growth ([4]). Finite depth means that $\mathrm{Ext}_G(k,G) \neq 0$ and polynomial growth r means that there exists $A > 0$ and $B > 0$ such that for all n large enough, we have :

$$An^r \leq \sum_{i=0}^{n} \dim G_i \leq Bn^r.$$

An elliptic Hopf algebra G is Gorenstein and $\mathrm{Ext}^r_G(k,G) \neq 0$, with r equal to the polynomial growth of G. For instance a finitely generated nilpotent Hopf algebra is an elliptic Hopf algebra.

Gorenstein algebras are important in algebraic topology: for a finite complex X, $H^*(X;\mathbb{Z}/p)$ is Gorenstein precisely when it satisfies Poincaré duality, and this occurs precisely when the Spivak fibre F_X localizes to a sphere, $(F_X)_p \simeq S^k_p$ (see [5]).

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We define in a similar way Gorenstein differential graded algebras. A differential graded algebra (A, d) is called *Gorenstein* if the vector space, $\mathrm{Ext}_{(A,d)}\big(k, (A, d)\big)$ has dimension one. It happens that for a simply connected finite CW complex, the cochain algebra $C^*(X; k)$ is Gorenstein if and only if $H^*(X; k)$ is a Poincaré duality algebra ([5]).

Now let X be a simply connected finite type CW complex. According to ([6], [7]), for each p (prime or zero) there are exactly two possibilities: either

(1) there are constants $C > 0$ and $r \in \mathbb{N}$ such that

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbb{Z}/p) \leq Cn^r, \quad n \geq 1,$$

or else

(2) there are constants $K > 1$ and $N \in \mathbb{N}$ such that

$$\sum_{i=0}^{n} \dim H_i(\Omega X; \mathbb{Z}/p) \geq K^{\sqrt{n}}, \quad n \geq N.$$

In the first case, the space X is called *elliptic* (more precisely $\mathbb{Z}_{(p)}$-elliptic). In the second case the space is called *hyperbolic*. An elliptic space X is a Poincaré complex, its Euler-Poincaré characteristic is non-negative and its loop space homology with $\mathbb{Z}_{(p)}$ coefficients is a finitely generated left noetherian ring.

The loop space homology $H_*(\Omega X; \mathbb{Z}/p)$ of an hyperbolic space has a subexponential growth, is not noetherian as a left module over itself, and is not nilpotent as a Hopf algebra.

In ([12]) Murillo shows that the evaluation map $\mathrm{ev}_{H_*(\Omega X;\mathbb{Q})}$ detects finite dimensionality of $H^*(X; \mathbb{Q})$ when X has $\dim \pi_* X \otimes \mathbb{Q} < \infty$: $H^*(X; \mathbb{Q})$ is finite dimensional if $\mathrm{ev}_{H_*(\Omega X;\mathbb{Q})}$ is nonzero. Because spaces with $\dim(\pi_* X \otimes \mathbb{Q}) < \infty$ have $H_*(\Omega X; \mathbb{Q})$ Gorenstein, our second result is a generalization to arbitrary characteristic of Murillo's result.

Since elliptic spaces have a lot of interesting properties, it is very important to detect elliptic spaces. This is one of the roles of Theorem 2.

THEOREM 2. *Let X be a simply connected finite type* CW *complex. Suppose $G = H_*(\Omega X; k)$ is a Gorenstein Hopf algebra. If ev_G is nonzero, then $H^*(X; k)$ is finite dimensional.*

PROOF. By Theorem 1, $\mathrm{Ext}_G(k, k)$ is finite dimensional. Theorem 2 results then directly from the convergence of the Moore spectral sequence ([11], [5])

$$\mathrm{Ext}_G(k, k) \Rightarrow \mathrm{Ext}_{C_*(\Omega X;k)}(k, k) \cong H^*(X; k). \qquad \blacksquare$$

The converse is clearly not true. For instance the space $X = \mathbb{C}P^2$ is a finite CW complex. Its loop space homology $G = H_*(\Omega X; k)$ is a Gorenstein Hopf algebra, $G \cong k[x_4] \otimes \Lambda x_1$. However a simple inspection shows that $\mathrm{ev}_G = 0$.

There are other relations and results connecting Gorenstein algebras and algebraic topology. The relation with the Gottlieb groups is for instance described in [9].

We remark finally that Theorem 1 appears as a complement to Gammelin's result ([10]):

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THEOREM (H. GAMMELIN). *Let (A,d) be a simply-connected Gorenstein commutative differential graded algebra such that $H^*(A,d)$ is noetherian. If* $\mathrm{ev}_A \neq 0\colon \mathrm{Ext}_A(k,A) \longrightarrow \mathrm{Ext}_A(k,k)$*, then $H^*(A,d)$ is finite dimensional.*

PROOF OF THEOREM 1. Denote by $\cdots P_m \xrightarrow{d} P_{m-1} \xrightarrow{d} \cdots \longrightarrow k$ a free minimal G-resolution of k:

$$P = \bigoplus_{m\geq 0} P_m, \quad P_m = G \otimes X_m.$$

Each X_j is a graded vector space, $X_j = \bigoplus_{r\geq 0} X_{j,r}$ such that $\dim X_{j,r} < \infty$ for $r \geq 0$.

Suppose that $\mathrm{Ext}^n_G(k,G) \neq 0$. Since the evaluation is nonzero, there is an element $x \in X_n$ and a G-module map $f\colon P \longrightarrow G$ satisfying $f \circ d = 0$ and $f(x) = 1$. There clearly exists then a decomposition of P_n into the form

$$P_n = G \otimes (kx \oplus V_n),$$

with $f(V_n) = 0$. Moreover since f is a cocycle we have $d(P_{n+1}) \subset G \otimes V_n$.

We call the complex which computes $\mathrm{Ext}^*_G(k,G)$,

$$(Q,\delta) = \big(\mathrm{Hom}_G(P,G),\delta\big), \quad Q_{-r} \cong G \otimes \mathrm{Hom}(X_r,k).$$

$$0 \longrightarrow Q_0 \xrightarrow{\delta} Q_{-1} \xrightarrow{\delta} Q_{-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} Q_{-n} \xrightarrow{\delta} \cdots.$$

Since G is Gorenstein we have

$$Q_{-n} \cong (G \otimes kf) \oplus \big(G \otimes \mathrm{Hom}(V_n,k)\big)$$

$$\delta(f) = 0$$

$$H_{-r}(Q,d) = \mathrm{Ext}^r_G(k,G) = \begin{cases} 0 & \text{if } r \neq n \\ kf & \text{if } r = n. \end{cases}$$

We show by induction on j, $j = 1,\ldots,n$, that Q_{-n+j} admits a decomposition

$$Q_{-n+j} = G \otimes (W_j \oplus R_j)$$

with

$$\begin{cases} \delta(W_j) \subset G \otimes W_{j-1}, \quad \delta(R_j) \subset G \otimes R_{j-1}, \quad j = 1,\ldots,n, \\ W_0 = kf, \quad R_0 = \mathrm{Hom}(V_n,k). \end{cases}$$

Suppose this is true for $j-1$. We choose a G-module decomposition of Q_{-n+j},

$$Q_{-n+j} = G \otimes T_j,$$

with T_j a direct sum $T_j = E \oplus F \oplus S$ such that E and F are graded subvector spaces of maximal dimension with respect to the properties

$$\delta(E) \subset G \otimes W_{j-1}, \quad \delta(F) \subset G \otimes R_{j-1}.$$

Since T_j is a finite type graded vector space, a maximal such decomposition exists. We want to prove that $S = 0$, *i.e.*, $T_j = E \oplus F$.

We suppose $S \neq 0$ and we take a nonzero element $x \in S$. We have

$$\delta x = \delta_1 x + \delta_2 x, \quad \delta_1 x \in G \otimes W_{j-1}, \quad \delta_2 x \in G \otimes R_{j-1}.$$

By induction hypothesis, $\delta_1 x$ and $\delta_2 x$ are cocycles. Since G is Gorenstein, $\delta_1 x$ and $\delta_2 x$ are coboundaries,

$$\delta_1 x = \delta(\alpha_1), \quad \delta_2 x = \delta(\alpha_2).$$

For $\alpha \in G \otimes T$, we write $\alpha = \bar{\alpha} + \alpha'$ with $\bar{\alpha} \in k \otimes T$ and $\alpha' \in G_+ \otimes T$.

If $\bar{\alpha}_1 \in E \oplus F$, $x - \alpha_1$ can be taken as a basis element of a new G-basis of $G \otimes T_j$ in contradiction with the maximality condition of the chosen decomposition. The same contradiction appears when $\bar{\alpha}_2 \in E \oplus F$. Therefore $\bar{\alpha}_1$ and $\bar{\alpha}_2$ do not belong to $E \oplus F$. There are two cases: x either belongs to $k\bar{\alpha}_1 \oplus k\bar{\alpha}_2 \oplus E \oplus F$ or not. In the first case the element α_1 can be taken as a new basis element of a G-basis of $G \otimes T_j$, which is not possible. In the second case, the element $x - \alpha_1 - \alpha_2$ is a new basis element of a G-basis once again in contradiction with the maximality hypothesis.

It follows that the direct sum

$$A_* = \bigoplus_{j=0}^{n} G \otimes W_j$$

is a complex satisfying

$$\begin{cases} A_p = 0 \quad \text{for } p \notin \{0, 1, \ldots, n\} \\ H_p(A, \delta) = \begin{cases} kf & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} . \end{cases}$$

This shows that (A, δ) is a G-free resolution of k. Since the length of this resolution is n, we have $\operatorname{gldim} G \leq n$.

We can therefore suppose that k admits a minimal free G-resolution (P_*, d) of length n. Since every linear map $W_n \longrightarrow k$ extends to a cocycle that is not a coboundary, the evaluation map $\operatorname{Ext}^n_G(k, G) \longrightarrow \operatorname{Ext}^n_G(k, k)$ is surjective. Because G is Gorenstein, the graded vector space $\operatorname{Ext}^n_G(k, k)$ has dimension one and is concentrated in total degree r for some r.

Then every linear map $W_{n-1,>r} \longrightarrow k$ extends to a cocycle that is not a coboundary. This implies the surjectivity of the evaluation map

$$\operatorname{Ext}^{n-1}_G(k, G) \longrightarrow \operatorname{Ext}^{n-1}_G(k, k)$$

in total degree greater than r. Therefore, since G is Gorenstein, $\operatorname{Ext}^{n-1}_G(k, k)$ is zero in total degree greater than r. By the same argument $\operatorname{Ext}^p_G(k, k)$ can be shown to be zero in total degree $> r$ for any p. This implies the finiteness of the total dimension of $\operatorname{Ext}_G(k, k)$.

■

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