ON THE SEMIGROUPS OF FREDHOLM MAPPINGS

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Let E_1 and E_2 be real Banach spaces and let $\mathscr{L}(E_1)$ and $\mathscr{L}(E_2)$ be the Banach algebras of all continuous linear mappings on E_1 and E_2 respectively. It is a well-known result of M. Eidelheit [1] that $\mathscr{L}(E_1)$ that $\mathscr{L}(E_2)$ are isomorphic as rings if and only if E_1 and E_2 are topologically and algebraically isomorphic. It is easy to see that the essential part of his proof is the following fact.

LEMMA. *Let E be a real Banach space and $\mathscr{F}(E)$ be the ring of all continuous linear mappings of finite rank of E into itself. Then, for any ring automorphism ϕ of $\mathscr{F}(E)$ such that $\phi(\alpha u) = \alpha\phi(u)$ for every $u \in \mathscr{F}(E)$ and for every real number α, there exists a topological linear isomorphism h of E such that, for every $u \in \mathscr{F}(E)$,*

$$\phi(u) = huh^{-1}.$$

Now, let $\mathscr{A}$ be a subset of $\mathscr{L}(E)$ such that

(1) *if $f, g \in \mathscr{A}$, then $fg \in \mathscr{A}$, where $(fg)(x) = f(g(x))$ for $x \in E$;*

(2) *if $f \in \mathscr{A}$ and $u \in \mathscr{F}(E)$, then $f+u \in \mathscr{A}$;*

(3) *every topological linear isomorphism belongs to $\mathscr{A}$.*

It is known that the set $\mathscr{F}r(E)$ of all Fredholm linear mappings of E into itself satisfies these conditions. (For example, see [3].) Although it is a semigroup with respect to the composition of mappings, the conditions $f \in \mathscr{F}r(E)$ and $g \in \mathscr{F}r(E)$ do not always imply $f+g \in \mathscr{F}r(E)$.

Let ϕ be an automorphism of the semigroup $\mathscr{A}$ defined above; in other words, ϕ is assumed to be a bijection of $\mathscr{A}$ such that, for $f, g \in \mathscr{A}$,

$$\phi(fg) = \phi(f)\phi(g).$$

We define that an automorphism ϕ is said to be *additive* if $f_i \in A$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} f_i \in \mathscr{A}$ imply

$$\phi(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} \phi(f_i).$$

Then, we can prove the following theorem, the proof of which can easily be modified to show that *two real Banach spaces E_1 and E_2 are topologically and* *algebraically isomorphic if and only if the semigroups $\mathscr{F}r(E_1)$ and $\mathscr{F}r(E_2)$ are isomorphic by an additive isomorphism.*

THEOREM. *Let $\mathscr{A}$ be the semigroup defined above. Then, every additive automorphism ϕ is inner; i.e., there exists a topological linear isomorphism h of E such that, for every $f \in \mathscr{A}$,*

$$\phi(f) = hfh^{-1}. \tag{4}$$

PROOF. (i) $\phi(\alpha) = \alpha$ *for any non-zero real number* α when it is regarded as a topological linear isomorphism: $x \to \alpha x$. (The zero mapping is not assumed to be in $\mathscr{A}$.)

In fact, for any $u \in \mathscr{F}(E)$, since $1+u \in \mathscr{A}$, there exists $f \in \mathscr{A}$ such that $\phi(f) = 1+u$. Then,

$$\begin{aligned}\phi(\alpha)+\phi(\alpha)u &= \phi(\alpha)(1+u) = \phi(\alpha)\phi(f) = \phi(\alpha f)\\ &= \phi(f\alpha) = \phi(f)\phi(\alpha) = \phi(\alpha)+u\phi(\alpha).\end{aligned}$$

Hence, $\phi(\alpha)$ commutes with every element of $\mathscr{F}(E)$. Particularly, for the one-dimensional mapping $a \otimes \bar{a}$ defined by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a}\rangle a,$$

where $a \in E$, $\bar{a} \in \bar{E}$ (the conjugate space of E) and $\langle x, \bar{a}\rangle$ is the value of $\bar{a}$ at x, we have

$$\phi(\alpha)(a \otimes \bar{a}) = (a \otimes \bar{a})\phi(\alpha),$$

or, assuming that $\langle a, \bar{a}\rangle = 1$,

$$\phi(\alpha)(a) = \langle \phi(\alpha)(a), \bar{a}\rangle a$$

for any non-zero real number α. Put

$$\lambda(\alpha) = \langle \phi(\alpha)(a), \bar{a}\rangle.$$

Then, if $\alpha\beta \neq 0$, we have

$$\lambda(\alpha+\beta) = \lambda(\alpha)+\lambda(\beta) \text{ and } \lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta),$$

from which it follows that $\lambda(\alpha) = \alpha$ for any non-zero real number α. Since a could be any non-zero element, we have $\phi(\alpha) = \alpha$ for any non-zero real number α.

(ii) *If we put $\psi(u) = \phi(1+u)-1$ for $u \in \mathscr{F}(E)$, then $\psi(\alpha u) = \alpha\psi(u)$ for any $u \in \mathscr{F}(E)$ and any real number α.*

Since $\psi(0) = 0$, we can assume that $\alpha \neq 0$. Then,

$$\begin{aligned}\psi(\alpha u) &= \phi(1+\alpha u)-1 = \phi(\alpha(\alpha^{-1}+u))-1\\ &= \phi(\alpha)\phi(\alpha^{-1}+u)-\alpha\alpha^{-1} = \alpha[\phi(\alpha^{-1}+u)-\alpha^{-1}]\\ &= \alpha\psi(u),\end{aligned}$$

because from the additivity of ϕ it follows that, for any $f \in \mathscr{A}$,

$$\phi(f+u)-\phi(f) = \phi(1+u)-1.$$

(iii) *ψ is additive and multiplicative.*

This also follows from the corresponding properties of ϕ.

(iv) *If u is one-dimensional, then $\psi(u)$ is also one-dimensional.*

As Eidelheit [1, Lemma 2] has pointed out, $u \in \mathscr{L}(E)$ is one-dimensional if and only if to any $v \in \mathscr{F}(E)$ corresponds a number α such that $(uv)^2 = \alpha uv$. Now, taking an arbitrary $v \in \mathscr{F}(E)$ and, since $1+v \in \mathscr{A}$, taking $f \in \mathscr{A}$ such that $\phi(f) = 1+v$, we have

$$\begin{aligned}\psi(u)v &= \psi(u)\phi(f)-\psi(u) = \phi(1+u)\phi(f)-\phi(f)-\phi(1+u)+1\\ &= \phi(f+uf)-\phi(f)-\phi(1+u)+1\\ &= \psi(uf)-\psi(u) = \psi(u(f-1)).\end{aligned}$$

Since $u(f-1) \in \mathscr{F}(E)$, $(u(f-1))^2 = \alpha u(f-1)$ for some α. Hence,

$$\begin{aligned}(\psi(u)v)^2 &= \psi((u(f-1))^2) = \psi(\alpha u(f-1))\\ &= \alpha\psi(u(f-1)) = \alpha\psi(u)v.\end{aligned}$$

(v) *ψ is an automorphism of the ring $\mathscr{F}(E)$.*

By (iii) and (iv), we have that ψ maps $\mathscr{F}(E)$ into itself. Therefore, we have only to show that it is onto. For any $v \in \mathscr{F}(E)$ we take $f \in \mathscr{A}$ such that $\phi(f) = 1+v$. Then, for $u = f-1$,

$$\phi(1+u)-1 = v \in \mathscr{F}(E).$$

Therefore, by the same method as above, we can show that $u \in \mathscr{F}(E)$, and it is obvious that $v = \psi(u)$.

(vi) Thus, by the Lemma, there exists a topological linear isomorphism h such that, for every $u \in \mathscr{F}(E)$,

$$\psi(u) = huh^{-1}.$$

Hence, for any $f \in \mathscr{A}$ and for any $u \in \mathscr{F}(E)$,

$$\begin{aligned}\phi(f)\psi(u) &= \phi(f(1+u))-\phi(f) = \phi(f+fu)-\phi(f)\\ &= \psi(fu) = hfuh^{-1} = hfh^{-1}\psi(u),\end{aligned}$$

from which (4) follows.

In order to obtain a non-linear version of this theorem, starting with the semi-group defined above, *we consider the set $d^{-1}(\mathscr{A})$ of (non-linear) mappings $f: E \to E$ which satisfy the following two conditions* (see [5]):

(5) *f is Fréchet-differentiable at every point of E;*

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(6) *$f'(x) \in \mathscr{A}$ for every $x \in E$,*

where $f'(x)$ denotes the Fréchet-derivative of f at x.

Obviously, $d^{-1}(\mathscr{A})$ is a semigroup with respect to the composition and $\mathscr{A} \subset d^{-1}(\mathscr{A})$. It is also obvious that $d^{-1}(\mathscr{F}r(E))$ is the set of all non-linear Fredholm mappings in the sense of S. Smale [4].

THEOREM. *If ϕ is an additive automorphism of the semigroup $d^{-1}(\mathscr{A})$, there is a unique topological linear isomorphism h of E such that, for every $f \in \mathscr{A}$,*

$$\phi(f) = hfh^{-1}.$$

PROOF. (i) *$\phi(1) = 1$ and $\phi(2) = 2$.*

Since ϕ is onto and $1 \in d^{-1}(\mathscr{A})$, there exists $f \in d^{-1}(\mathscr{A})$ such that $\phi(f) = 1$. Then, for any $x \in E$ we have

$$\phi(1)(x) = \phi(1)(\phi(f)(x)) = (\phi(1)\phi(f))(x) = \phi(f)(x) = x.$$

The additivity of ϕ implies the second equality.

(ii) *Let c_x be a constant mapping such that*

$$c_x(y) = x \text{ for any } y \in E.$$

Then, $1+c_x \in d^{-1}(\mathscr{A})$ and $\phi(1+c_x)-1$ is also constant.

Since $c'_x(y) = 0$ for any $y \in E$, it is obvious that

$$1+c_x \in d^{-1}(\mathscr{A}),$$

and

$$\begin{aligned}(\phi(1+c_x)-1)(y) &= \phi(1+c_x)(1+c_y)(0)-y \\ &= \phi(1+c_x)\phi(f)(0)-y \text{ where } \phi(f) = 1+c_y \\ &= \phi(f+c_x)(0)-y \\ &= \phi(f)(0)+\phi(1+c_x)(0)-\phi(1)(0)-y \\ &= (\phi(1+c_x)-1)(0)\end{aligned}$$

for any $y \in E$. Hence, $\phi(1+c_x)-1$ is constant.

(iii) *There exists a bijection $h: E \to E$ such that for any $x \in E$,*

$$\phi(1+c_x) = 1+c_{h(x)}.$$

The following method of defining h is due to K. D. Magill, Jr. (see [2] or [6]). Since $\phi(1+c_x)-1$ is a constant mapping and ϕ is injective, there is a unique element y such that

$$\phi(1+c_x)-1 = c_y.$$

We define h by

$$y = h(x).$$

Then, h is obviously injective, and, for any $y \in E$, by the same method as in (ii)

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we see that $\phi^{-1}(1+c_y)-1$ is a constant mapping which we denote by c_x. Then,

$$c_y = \phi(1+c_x)-1, \text{ or } y = h(x).$$

Therefore, h is a bijection.

(iv) *$\phi(f)(0) = 0$ iff $f \in \mathscr{A}$.*
In fact,

$$\begin{aligned}\phi(f)(0) = \phi(f)(20) = \phi(f)\phi(2)(0) = \phi(f2)(0)\\ = \phi(2f)(0) = 2\phi(f)(0).\end{aligned}$$

(v) *$\phi(f) = hfh^{-1}$ for any $f \in \mathscr{A}$.*
From the additivity of ϕ it follows that, for any $x \in E$ and any $f \in \mathscr{A}$,

$$\phi(f+c_x) = \phi(f)+c_{h(x)}.$$

Therefore, replacing x by $f(x)$, we have

$$\phi(f+c_{f(x)}) = \phi(f)+c_{hf(x)}.$$

Hence, since $f \in \mathscr{A}$,

$$\begin{aligned}hf(x) &= \phi(f+c_{f(x)})(0)-\phi(f)(0)\\ &= \phi(f(1+c_x))(0) = \phi(f)\phi(1+c_x)(0)\\ &= \phi(f)(1+c_{h(x)})(0) = \phi(f)h(x)\end{aligned}$$

for every $x \in E$.

(vi) *ϕ is an additive automorphism of $\mathscr{A}$.*
At first, we shall show that, for $x, y \in E$,

$$h(x+y) = h(x)+h(y).$$

To do this, we use the fact that

$$c_{h(x)} = \phi(1+c_x)-1 = \phi(2+c_x)-2$$

for every $x \in E$, which follows from (i) and the additivity of ϕ. Then,

$$\begin{aligned}h(x+y) &= c_{h(x+y)}(0) = (\phi(2+c_{x+y})-2)(0)\\ &= \phi(1+c_x+1+c_y)(0)\\ &= \phi(1+c_x)(0)+\phi(1+c_y)(0)\\ &= h(x)+h(y).\end{aligned}$$

Thus, for $f \in \mathscr{A}$, $\phi(f) = hfh^{-1}$ is additive as well as differentiable, which implies that

$$\phi(f)(\alpha x) = \alpha\phi(f)(x)$$

for any $x \in E$ and any real number α. Therefore, by the condition (6),

$$\phi(f) = \phi(f)'(x) \in \mathscr{A}.$$

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In order to prove that $\phi : \mathscr{A} \to \mathscr{A}$ is onto, we have only to use ϕ^{-1} instead of ϕ. The rest of the proof is obvious.

(vii) *h is a topological linear isomorphism.*

By the previous theorem, there is a topological linear isomorphism h_1 such that (4) is satisfied. Combining this fact with v), we see that $h = \alpha h_1$ for some real number α. Therefore, h is bicontinuous.

We do not know whether or not every additive automorphism ϕ of the semigroup $d^{-1}(\mathscr{A})$ is inner. Here, we present some conditions which are equivalent to that ϕ is inner.

THEOREM. *Let ϕ be an additive automorphism of the semigroup $d^{-1}(\mathscr{A})$. The following conditions are equivalent:*

(7) *ϕ is inner.*

(8) *If $f_n \in d^{-1}(\mathscr{A})$ $(n = 0, 1, 2, \cdots)$ and $\lim_{n\to\infty} f_n(x) = f_0(x)$ at every $x \in E$, then $\lim_{n\to\infty} \phi(f_n)(0) = \phi(f_0)(0)$.*

(9) *If $f(0) = 0$ and $f \in d^{-1}(\mathscr{A})$, then $\phi(f)(0) = 0$.*

PROOF. It is evident that (7) implies (8). Let us assume that ϕ satisfies (8) and $f(0) = 0$ for $f \in d^{-1}(\mathscr{A})$. Then, the limit

$$\lim_{\varepsilon_n \to 0} \varepsilon_n^{-1}[f(\varepsilon_n x)] = f'(0)(x)$$

exists and

$$\varepsilon_n^{-1} f \varepsilon_n \in d^{-1}(\mathscr{A}) \text{ and } f'(0) \in \mathscr{A} \subset d^{-1}(\mathscr{A}).$$

Therefore, (8) implies that the limit

$$\phi(f'(0))(0) = \lim_{\varepsilon_n \to 0} \phi(\varepsilon_n^{-1} f \varepsilon_n)(0) = \lim_{\varepsilon_n \to 0} \varepsilon_n^{-1}\, \phi(f)(0)$$

exists, hence it follows that $\phi(f)(0) = 0$.

Finally, if ϕ satisfies (9), since

$$(f - c_{f(0)})(0) = 0 \text{ and } f - c_{f(0)} \in d^{-1}(\mathscr{A}),$$

we have

$$(\phi(f) - c_{hf(0)})(0) = \phi(f - c_{f(0)})(0) = 0,$$

or, $\phi(f)(0) = hf(0)$ for every $f \in d^{-1}(\mathscr{A})$. Therefore, for every $x \in E$,

$$\begin{aligned}\phi(f)(x) &= \phi(f(1 + c_{h^{-1}(x)}))(0) \\ &= hf(1 + c_{h^{-1}(x)})(0) = hfh^{-1}(x),\end{aligned}$$

which implies that ϕ is inner.

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