J. Inst. Math. Jussieu (2024), 1–81
doi:10.1017/S1474748024000021

WILES DEFECT OF HECKE ALGEBRAS VIA LOCAL-GLOBAL ARGUMENTS

WITH AN APPENDIX BY NAJMUDDIN FAKHRUDDIN AND CHANDRASHEKHAR B. KHARE

GEBHARD BÖCKLE[1], CHANDRASHEKHAR B. KHARE[2] AND JEFFREY MANNING[3]

[1] *Interdisciplinary Center for Scientific Computing, Universität Heidelberg, Heidelberg, Germany*
(gebhard.boeckle@iwr.uni-heidelberg.de)

[2] *Department of Mathematics, UCLA, Los Angeles, CA 90095-1555 USA*
(shekhar@math.ucla.edu)

[3] *Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2RH, UK*
(jeffrey.manning@imperial.ac.uk)

(Received 29 November 2021; revised 24 December 2023; accepted 8 January 2024)

Abstract In his work on modularity of elliptic curves and Fermat's last theorem, A. Wiles introduced two measures of congruences between Galois representations and between modular forms. One measure is related to the order of a Selmer group associated to a newform $f \in S_2(\Gamma_0(N))$ (and closely linked to deformations of the Galois representation ρ_f associated to f), whilst the other measure is related to the congruence module associated to f (and is closely linked to Hecke rings and congruences between f and other newforms in $S_2(\Gamma_0(N))$). The equality of these two measures led to isomorphisms $R = \mathbf{T}$ between deformation rings and Hecke rings (via a numerical criterion for isomorphisms that Wiles proved) and showed these rings to be complete intersections.

We continue our study begun in [BKM21] of the *Wiles defect* of deformation rings and Hecke rings (at a newform f) acting on the cohomology of Shimura curves over $\mathbf{Q}$: It is defined to be the difference between these two measures of congruences. The Wiles defect thus arises from the failure of the Wiles numerical criterion at an augmentation $\lambda_f : \mathbf{T} \to \mathcal{O}$. In situations we study here, the Taylor–Wiles–Kisin patching method gives an isomorphism $R = \mathbf{T}$ without the rings being complete intersections. Using novel arguments in commutative algebra and patching, we generalize significantly and give different proofs of the results in [BKM21] that compute the Wiles defect at $\lambda_f : R = \mathbf{T} \to \mathcal{O}$, and explain in an *a priori* manner why the answer in [BKM21] is a sum of *local defects*. As a curious application of our work we give a new and more robust approach to the result of Ribet–Takahashi that computes change of degrees of optimal parametrizations of elliptic curves over $\mathbf{Q}$ by Shimura curves as we vary the Shimura curve. The results we prove are not attainable using only the methods of Ribet–Takahashi.

2020 Mathematics subject classification: Primary 11F80
Secondary 11F33; 14G35

Contents

1. Introduction

In the work on modularity of elliptic curves, Wiles pioneered methods to prove $R = \mathbf{T}$ theorems where R is a deformation ring and $\mathbf{T}$ a Hecke algebra, thus proving an equality of moduli spaces of Galois representations to pro-p Artinian rings arising from modular forms with the *a priori* larger moduli space of corresponding abstract Galois representations, both with certain prescribed local (ramification) behavior.

The injectivity of the *a priori* surjective map $R \twoheadrightarrow \mathbf{T}$ was proven by using two different types of criteria/methods:

(i) the numerical criterion of [Wil95, Proposition 2 of Appendix];

(ii) the patching method of [TW95].

In [TW95], the local conditions imposed on the deformations were smooth. Kisin [Kis09] later generalized the patching method to allow local conditions on the deformations that were not necessarily smooth. The generic fiber of the local deformation rings in question was smooth and Kisin proved a $R[1/p] = \mathbf{T}[1/p]$ theorem, thus proving a coarser equality of moduli spaces of p-adic Galois representations arising from modular forms with the *a priori* larger moduli space of corresponding abstract Galois representations, both with certain prescribed local behavior. When the local conditions are Cohen–Macaulay, one sees *a posteriori* that R has no p-torsion (see [KW09, paragraph before Corollary 4.7], [Sno18, §5] or [BKM21, Theorem 6.3] for instance) and thus as $\mathbf{T}$ is also torsion-free one can promote an $R[1/p] = \mathbf{T}[1/p]$ theorem to an integral $R = \mathbf{T}$ theorem, without the rings in question turning out to be complete intersections.

Wiles used his numerical criterion for maps between rings to be isomorphisms of complete intersections to deduce $R = \mathbf{T}$ theorems in the *nonminimal case* from $R = \mathbf{T}$ theorems in the *minimal case* (see [Wil95, Theorem 2.17 of §2]). The minimal case was proved via the patching method of [TW95]. The numerical criterion has been used subsequently in [Kha03] to prove $R = \mathbf{T}$ theorems without any reliance on patching. The numerical criterion of Wiles has not as yet been generalized to give a criterion for maps between rings to be an isomorphism when the rings are known to not be complete intersections.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

The work of this paper, like that of the previous paper [BKM21] of this series, arises when considering situations when we have $R = \mathbf{T}$ theorems proved by patching, but R and $\mathbf{T}$ fail to be complete intersections. In [BKM21] and the present paper, we seek to study the failure (quantified in a numerical quantity called the *Wiles defect* introduced in [TU22], see also [BKM21, Definition 3.10]) of the numerical criterion for being a complete intersection locally at an augmentation $\lambda_f : \mathbf{T} \to \mathcal{O}$ induced by a newform f. The term defect is justified since, as we shall explain in Proposition 3.28, for a complete Noetherian $\mathcal{O}$-algebra R with an augmentation $\lambda : R \to \mathcal{O}$, the Wiles defect $\delta_\lambda(R)$ vanishes if and only if R is a complete intersection ring.

In [BKM21] we studied the Wiles defect (at λ_f of a certain Hecke ring $\mathbf{T}$ acting on the cohomology of a Shimura curve) using a combination of patching and level lowering results of Ribet–Takahashi [RT97]. In the present paper, we combine the new results in commutative algebra that we prove here with patching to determine the Wiles defect. The patching method allows one to show that the Wiles defect of a global deformation ring at an augmentation λ_f depends only on the induced augmentations of the corresponding local deformation rings. This gives yet another illustration of the versatility of the patching method and its ability to reduce proving properties of global deformation rings to proving properties of the corresponding local deformation rings.

As a curious consequence, we derive and strengthen the results of Ribet–Takahashi in [RT97] on degrees of optimal parametrizations of elliptic curves over $\mathbf{Q}$ by Shimura curves, via a new argument. The methods of Ribet–Takahashi use arithmetic geometry, while the method here uses patching. Our strengthening of their results is not accessible using only the methods of their paper as we explain below in the introduction.

1.1. A particular case of our main theorem

In [BKM21, Theorem 10.1], we determined the Wiles defect associated to a newform $f \in S_2(\Gamma_0(NQ))$ of squarefree level NQ that arises by the Jacquet–Langlands correspondence from a newform in $S_2(\Gamma_0^Q(N))$. Here, $\Gamma_0^Q(N)$ is the congruence subgroup of a quaternion algebra that is ramified at the set of primes dividing Q, of level $\Gamma_0(N)$ and the maximal compact subgroup at the primes in Q.

We state an improvement of [BKM21, Theorem 10.1] referring to it for any of the unexplained notation in the statement below (we do recall the definition of the Wiles defect below). The proof relies on the Taylor–Wiles–Kisin patching method, but not on [RT97], and also explains *en passant* why the Wiles defect computed below is a sum of *local defects* in a sense we make precise later in the introduction.

Theorem 1.1. *Let N and Q be relatively prime squarefree integers. Let $p > 2$ be a prime not dividing NQ, and let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k. Let $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a Galois representation arising from a newform $f \in S_2(\Gamma_0(NQ))$, and let $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ be the residual representation. Assume that $\overline{\rho}_f$ is irreducible and $N|N(\overline{\rho}_f)$.*

Let R^{st} be the Galois deformation ring of $\overline{\rho}_f$ parameterizing lifts of $\overline{\rho}_f$ of fixed determinant which are Steinberg at each prime dividing Q, finite flat at p and minimal at all other primes.

Let D be the quaternion algebra with discriminant Q, and let $\Gamma_0^Q(N)$ be the level N congruence subgroup for D. Let $\mathbf{T}^Q(N)$ and $S^Q(\Gamma_0^Q(N))$ be the Hecke algebra and cohomological Hecke module at level $\Gamma_0^Q(N)$, and let $\mathfrak{m} \subseteq \mathbf{T}^Q(N)$ be the maximal ideal corresponding to f. Let $\mathbf{T}^{\mathrm{st}} = \mathbf{T}^Q(N)_{\mathfrak{m}}$, and let $\lambda : \mathbf{T}^{\mathrm{st}} \to \mathcal{O}$ be the augmentation corresponding to f.

Then the Wiles defects of $\mathbf{T}^{\mathrm{st}}$ and $S^Q(\Gamma_0^Q(N))$ with respect to the map $R^{\mathrm{st}} \twoheadrightarrow \mathbf{T}^{\mathrm{st}}$ and the augmentation λ are

$$\delta_\lambda(R^{\mathrm{st}}) = \delta_\lambda(\mathbf{T}^{\mathrm{st}}) = \delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}}) = \sum_{q|Q} \frac{2n_q}{e},$$

where e is the ramification index of $\mathcal{O}$ and for each $q|Q$, n_q is the largest integer for which $\rho_f|_{G_{\mathbf{Q}_q}} \pmod{\varpi^{n_q}}$ is unramified and $\rho_f(\mathrm{Frob}_q) \equiv \pm\mathrm{Id} \pmod{\varpi^{n_q}}$.

The improvement as far as the statement of the theorem is concerned, if one compares to [BKM21, Theorem 10.1], is that the assumptions needed there on Q:

1. Q is a product of an even number of primes (i.e., D is indefinite), and $(N(\overline{\rho}), Q) > 1$;
2. Q is a product of an odd number of primes (i.e., D is definite), and $N > 1$;
3. $N(\overline{\rho})$ is divisible by at least two primes,

which arose from our relying on delicate results in [RT97], are no longer needed because of the innovations introduced in this paper. We prove a much more general theorem below; see Theorem 6.5, that works with more general local conditions than being Steinberg at *trivial primes* (see [BKM21, §2]) and with the field $\mathbf{Q}$ replaced by any totally real field F, but focus on this special case for the purposes of the introduction to more easily explain the novelty of our methods in comparison to [BKM21].

If we look at the shape of the formula

$$\delta_\lambda(\mathbf{T}^{\mathrm{st}}) = \delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}}) = \sum_{q|Q} \frac{2n_q}{e}$$

we see that the Wiles defect $\delta_\lambda(\mathbf{T}^{\mathrm{st}})$, that is defined as a *global* quantity arising from the augmentation $\lambda_f : \mathbf{T}^{\mathrm{st}} \to \mathcal{O}$ is expressed as a sum over the primes dividing Q of terms $2n_q/e$. Furthermore, each of the integers n_q depends only on $\rho_f|_{G_{\mathbf{Q}_q}}$. In [BKM21], it is only after having proved the theorem that one observes that the formula depends only on $(\rho_f|_{G_{\mathbf{Q}_q}})_{q \in Q}$. In this paper, we show that the Wiles defect $\delta_\lambda(R^{\mathrm{st}})$ is *a priori* local, and in fact is a sum of the defects of local deformation rings (equivalently, *local defects*) at primes in Q that we define below. The proof of [BKM21, Theorem 10.1] did not shed light on the local-global aspect of the statement of the theorem.

Further, the proof of [BKM21, Theorem 10.1] computed the Wiles defect using a combination of patching and arguments related to level lowering results of [RT97]. The latter was used to first show that

$$\delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}}) = \sum_{q|Q} \frac{2n_q}{e}.$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Then delicate results from [Man21] were used to prove [BKM21, Theorem 3.10, Theorem 8.1, Corollary 8.3] that

$$\delta_\lambda(\mathbf{T}^{\mathrm{st}}) = \delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}}).$$

(As the referee has remarked, the inequality $\delta_\lambda(\mathbf{T}^{\mathrm{st}}) \leq \delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}})$ is easier and follows from [BKM21, Theorem 3.12].) One deduces that

$$\delta_\lambda(\mathbf{T}^{\mathrm{st}}) = \sum_{q|Q} \frac{2n_q}{e}.$$

Here, we reverse the logic of the proof in [BKM21] and show using patching and the new commutative algebra results about the Wiles defect that are proven here (see Theorem 6.5) that

$$\delta_\lambda(R^{\mathrm{st}}) = \delta_\lambda(\mathbf{T}^{\mathrm{st}}) = \sum_{q|Q} \frac{2n_q}{e}$$

and deduce from this (see Theorem 7.5 (ii) and Proposition 7.7) that

$$\delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}}) = \sum_{q|Q} \frac{2n_q}{e}.$$

Thus, our determination of $\delta_\lambda(\mathbf{T}^{\mathrm{st}})$ no longer relies on [RT97]. Indeed, we show how to use defects of Hecke rings to compute the defects of their 'cohomological' modules (arising from the first cohomology of modular curves and Shimura curves that they act on). Besides the intrinsic interest in having methods that work for modules over rings rather than just for rings, the computations of defects $\delta_\lambda(S^Q(\Gamma_0^Q(N))_{\mathfrak{m}})$ of modules such as $S^Q(\Gamma_0^Q(N))$, turns out to be key to the next application that we outline below. It relies on exact computations of changes of lengths of congruence modules that arise from cohomology groups of modular curves and Shimura curves.

1.2. Application to change of degree formula of parametrizations of elliptic curves by Shimura curves

Our results and methods allow one to give a more robust approach (see Theorem 7.5(ii), Corollary 7.9 and Corollary 7.10 below and the remarks that follow) to the main result proved by Ribet and Takahashi [RT97, Theorem 1] that computes changes of p-parts of degrees of optimal parametrizations of semistable elliptic curves E over $\mathbf{Q}$ by Shimura curves as one varies the Shimura curve for a prime p such that $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module. The methods of [RT97, Theorem1] rely at a crucial point (see proof of second assertion of [RT97, Theorem 1]) on the following consequence of E being defined over $\mathbf{Q}$ and semistable:

- (*) There is a prime q dividing the conductor of E (of semistable bad reduction) at which the order of the group of components at q is not divisible by p. Equivalently, the mod p representation $\overline{\rho}$ arising from E is such that $\overline{\rho}(I_q)$ is either not finite flat (in the case $q = p$), and ramified (in the case $q \neq p$), with I_q an inertia group at q.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

This is used to show that certain maps on the p-primary parts of components groups are surjective (by a clever trick of permuting primes around, see [RT97, pg. 11113]) which is the key to computing change of degrees of parametrizations in [RT97, Theorem 1]. We generalize the results of [RT97] (Corollary 7.9 and Corollary 7.10 below) to elliptic curves over $\mathbf{Q}$ which need not be semistable and for which (*) may not necessarily hold. Our methods should also extend to situations where we replace $\mathbf{Q}$ by a totally real number field F, and E is an elliptic curve over F which need not be semistable (outside the set of primes at which the quaternion algebra giving rise to the Shimura X curve that parametrizes E is ramified and at which both X and E have multiplicative reduction at these primes), provided that the mod p representation $\overline{\rho}$ arising from E is irreducible when restricted to $G_{F(\zeta_p)}$.

Our very indirect method to compute change of degrees, that is arithmetic and global in nature, seems necessary to get results of [RT97] in general situations. We note that the surjectivity of maps on component groups arising from optimal quotients of abelian varieties with multiplicative reduction defined over a finite extension K of $\mathbf{Q}_q$ is not generally true. More precisely, there are A, A' be abelian varieties defined over a finite extension K of $\mathbf{Q}_q$ that have multiplicative reduction at q, and $f : A \to A'$ is an optimal quotient over K (i.e., $\ker(f)$ is connected) such that the induced map $\phi(A) \to \phi(A')$ on component groups is not surjective on the p-primary parts for a prime p. (K. Ribet showed us an example due to Raynaud.) It is easy to show that the map is surjective when the pth roots of unity are not in K. Our global methods show that the surjectivity holds even when K contains pth roots of unity in the situations we consider; namely, when A arises from Jacobian of Shimura curves over F and K is a completion of F, A' is an optimal GL_2-abelian variety quotient, and with p a prime so that that the residual characteristic p representations $\overline{\rho}_\lambda$ arising from A' satisfy the Taylor–Wiles hypothesis that $\overline{\rho}_\lambda|_{F(\zeta_p)}$ is irreducible.

1.3. Main ideas of proof of Theorem 1.1

We consider in this paper the category $C_{\mathcal{O}}$ of tuples (R,λ), with $R \in \mathrm{CNL}_{\mathcal{O}}$ (with $\mathrm{CNL}_{\mathcal{O}}$ the usual category; see §1.6) that is flat over $\mathcal{O}$ and Cohen–Macaulay, together with an augmentation $\lambda : R \to \mathcal{O}$ (that is by definition a continuous surjective $\mathcal{O}$-algebra homomorphism) that is formally smooth over the generic fiber.

We take a cue from a formula discovered by Venkatesh [Ven16, Ven20] (see Proposition A.6 of the appendix) and define in §2 the *Wiles defect* $\delta_\lambda(R)$ for $(R,\lambda) \in C_{\mathcal{O}}$. The defect $\delta_\lambda(R)$ is expressed in terms of two invariants first introduced by Venkatesh (for rings R finite over $\mathcal{O}$):

(i) the length of the $\mathcal{O}$-module $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ which can be directly defined using a continuous version of the André–Quillen cohomology of rings (cf. §3.3), (which will agree with the standard André–Quillen cohomology module $\mathrm{Der}^1_{\mathcal{O}}(R,E/\mathcal{O})$ in the case when R has dimension 1) and

(ii) the length of the $\mathcal{O}$-module $C_{1,\lambda}(R)$ (cf. §3.2, in particular Corollary 3.12).

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

The Wiles defect $\delta_\lambda(R)$ is then defined (cf. Definition 3.24) to be

$$\delta_\lambda(R) = \frac{\log|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})| - \log|C_{1,\lambda}(R)|}{\log|\mathcal{O}/p|} = \frac{\ell_{\mathcal{O}}(\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})) - \ell_{\mathcal{O}}(C_{1,\lambda}(R))}{\ell_{\mathcal{O}}(\mathcal{O}/p)}.$$

This definition of the defect we give for $R \in C_{\mathcal{O}}$ agrees, by Proposition A.6 and Proposition 3.27, in the case when $R \in C_{\mathcal{O}}$ is of dimension one with the definition of the Wiles defect given in [BKM21] as

$$\delta_\lambda(R) = \frac{\log|\Phi_\lambda(R)| - \log|\Psi_\lambda(R)|}{\log|\mathcal{O}/p|}.$$

We note that this latter definition makes sense only for rings $R \in C_{\mathcal{O}}$ of dimension one as only then are the modules $\Phi_\lambda(R), \Psi_\lambda(R)$ of finite cardinality (see Lemma 2.4).

Our main technique for the proof of Theorem 1.1 is the Taylor–Wiles–Kisin patching method. Specifically, under some mild global hypotheses, one can write R^{st} as a quotient $R^{\mathrm{st}}_{\mathrm{loc}}[[x_1,\dots,x_g]]/(y_1,\dots,y_d)$ (see Theorem 6.4 and Theorem 6.5), where $R^{\mathrm{st}}_{\mathrm{loc}}$ is a completed tensor product of local Galois deformation rings and is thus determined by local Galois theoretic information. In the case when $R^{\mathrm{st}}_{\mathrm{loc}}$ is Cohen–Macaulay[1] we prove general results (see Theorem 3.9 and Theorem 3.20) that imply that $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ and $C_{1,\lambda}(R)$ are independent of the choice of ideal $(y_1,\dots,y_d)$ and thus depend only on the ring $R^{\mathrm{st}}_{\mathrm{loc}}$ and the induced composite map $R^{\mathrm{st}}_{\mathrm{loc}} \to R^{\mathrm{st}} \xrightarrow{\lambda} \mathcal{O}$, which shows that

$$\delta_\lambda(R^{\mathrm{st}}) = \delta_\lambda(R^{\mathrm{st}}_{\mathrm{loc}}) = \sum_{q|Q} \delta_\lambda(R^{\mathrm{st}}_q),$$

where $\delta(R^{\mathrm{st}}_q)$ is the defect of the local deformation ring $R^{\mathrm{st}}_q \in C_{\mathcal{O}}$. Thus, to determine $\delta_\lambda(R^{\mathrm{st}})$, we have to compute the defects $\delta_\lambda(R^{\mathrm{st}}_q)$ of the local deformation rings R^{st}_q. These computations are quite elaborate and are done in Theorem 5.18 of §5 (Theorems 5.26 and 5.33 do analogous computations for local deformation rings defined by conditions of being unipotent and unipotent together with a choice of Frobenius eigenvalue). One of the contributions of this paper is to show that these subtle invariants $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ and $C_{1,\lambda}(R)$ are computable for fairly complicated rings: The local deformation rings $R \in C_{\mathcal{O}}$ that we consider below at trivial primes are not Gorenstein (for the Steinberg and unipotent local conditions) and Gorenstein but not complete intersections (for the unipotent condition with choice of Frobenius eigenvalue). The computations are delicate.

1.4. Broader context

We make some more informal remarks about the broader context of our work and further questions to pursue in this context.

Our work is in the general context of understanding deformation rings R when they are 'obstructed' and are thus not expected to be complete intersections. The Wiles defect

[1] Which happens in many cases in which the relevant deformation rings have been explicitly computed, including the case considered in [BKM21], and is conjectured to hold far more generally.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

is a measure of the obstructedness of R at a given augmentation $\lambda: R \to \mathcal{O}$. In the context of the present paper, the obstructions are local in nature. The Wiles defect is a global quantity which in our case turns out to be a sum of local defects. This is proved by patching and showing that that the invariants $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$ and $C_{1,\lambda}(R)$ remain invariant under going modulo regular sequences. In other situations (as in [TU22]), the obstructedness of deformation rings R is because of global reasons, in that one is in a situation of positive defect $\ell_0 > 0$, and the natural 'automorphic cohomology' to consider lives in more than one degree. The work in [GV18] gives a framework to understand this more deeply via considering derived deformation rings $\mathcal{R}$ such that $R = \pi_0(\mathcal{R})$, and $\pi_*(\mathcal{R})$ acts as a graded ring on the 'automorphic cohomology'. It seems interesting to explore these ideas in the context of the paper, and for instance 'derive' the local deformation rings at trivial primes. One of the points of our work both here and in [BKM21] is that in the cases which we study, the Wiles defect of Hecke algebras can be calculated unconditionally and one can determine it explicitly.

We have not dealt with cases when the local deformation ring at p is not a complete intersection in this paper, but our results will still be applicable provided that the local deformation rings are Cohen–Macaulay. For example, [Sno18] considers a fixed weight ordinary deformation ring when the residual representation is trivial at p and shows that this ring is Cohen–Macaulay but not a complete intersection (or even Gorenstein). Our methods show that the global Wiles defect is again a sum of local defects in this case. However, we have not been able to determine the local defect at p in this case (due largely to the fact that [Sno18] only computes the special fiber of the ring, while computing the local defect would require the integral version of the ring).

In the tame cases we have considered here and in [BKM21, §11], the local defect at q is related to tame regulators (in the sense of Mazur–Tate) of the q-adic Mumford–Raynaud–Tate periods of the corresponding abelian variety A_f which has multiplicative reduction at q. In the wild case, one imagines that the local defect will be related to p-adic regulators.

Our work should also help in formulating and proving Bloch–Kato conjectures for newforms $f \in S_2(\Gamma_0(N))$ (say N squarefree) and the p-part of special value of the $L(1, \mathrm{Ad})$ for the adjoint L-function of f for suitable primes p. The algebraic part of the L-value is traditionally related to congruence modules of f by the work of Hida [Hid81]. The Selmer group for the adjoint motive of f can be related to the cotangent space at the augmentation $\lambda_f : R \to \mathcal{O}$ where the local deformation problem at primes dividing N is the unipotent condition. The Wiles defect here by Theorem 6.5 is $\sum_{q|N} n_q$ and is the discrepancy between the length of the congruence module for f and the Selmer group for the adjoint motive of f. It will be interesting to see this defect emerge from automorphic considerations. We believe that the Selmer group we are alluding to here is the natural (primitive) Selmer group to consider for the adjoint motive of f, reflecting nature of π_f locally at primes dividing N. (See [TU22, Theorem 5.20] that relates the ratio of different integral normalizations of periods (cohomological and motivic) of the adjoint motive of a Bianchi form to the Wiles defect, and to Bloch–Kato conjectures.) Note that if we relax the Selmer conditions at primes dividing N to be unrestricted of fixed determinant and consider the corresponding imprimitive Selmer group, then the Wiles defect becomes 0 and one is in a setting where Wiles-type methods prove the Bloch–Kato conjecture for this imprimitive Selmer group.

We could also consider a Bloch–Kato conjectures in this context with the local condition at primes dividing N to be Steinberg. The Wiles defect in this case by Theorem 6.5 is $\sum_{q|N} 2n_q$, and the automorphic cohomology to consider here is $H^1(X^Q,\mathcal{O})$, where X^Q is a Shimura curve over $\mathbf{Q}$ arising from the quaternion algebra D_Q ramified at places Q dividing N (which we assume here is a set of even cardinality). If we consider the Jacquet–Langlands correspondent g of f on D_Q, normalized (as in [Pra06]) using the schematic structure over $\mathbf{Z}_p$ of the corresponding Shimura curve X^Q over $\mathbf{Q}$, with p a prime such that $(p,N)=1$, then one sees easily that the ratio of Petersson inner products

$$\frac{(f,f)}{(g,g)} = \frac{\deg(\phi)}{\deg(\phi')},$$

where ϕ,ϕ' are optimal parametrizations of abelian varieties in the isogeny class $\mathcal{A}_f$ over $\mathbf{Q}$ associated to A_f. We could ask for a different 'natural' normalization g' such that

$$\frac{(f,f)}{(g',g')} = \frac{\deg(\phi)}{\deg(\phi')}\Pi_{q\in Q}\omega^{-2n_q}$$

would be the change of the corresponding Selmer groups (when we change the local conditions at primes in Q from Steinberg to unrestricted with fixed determinant) and thus would incorporate the Wiles defect $\sum_{q\in Q}\frac{2n_q}{e}$.

Our method to compute p-parts of change of degrees of parametrizations of elliptic curves over $\mathbf{Q}$ by Shimura curves gives results that are stronger than the ones which can be obtained using the arithmetic-geometric methods of [RT97]. To have these results in the fullest possible generality should be important for applications (see [Pas24] for Diophantine applications of [RT97]).

1.5. Structure of this paper

We begin by developing the commutative algebra tools that are needed for our main theorem Theorem 6.5. In §2, we state a formula for Wiles defects of rings of dimension one that is proved in Appendix A. In the key §3, we define and prove properties of the invariants $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ and $C_{1,\lambda}(R)$ for rings $R\in C_{\mathcal{O}}$. In §4, we summarize information about local and global deformation rings. In §5, we compute the invariants defined in §3 for the local deformation rings we consider. This is a key input in computing the Wiles defect of global deformation rings in Theorem 6.5. In §6, we use patching and the work in §3 to show that the Wiles defect of global deformation rings and Hecke rings we consider is the sum of local defects. As the local defects have been computed in §5, this allows us to complete the proof of our main Theorem 6.5. In §7, we apply Theorem 6.5 to compute the Wiles defect for modules over Hecke algebras that arise from their action on the cohomology of modular and Shimura curves. This also leads to a new approach to, and strengthening of, the results in [RT97] about change of degrees of optimal parametrizations of elliptic curves by Shimura curves as one changes the Shimura curve.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

In Appendix A (written by Najmuddin Fakhruddin and CBK), a formula stated in a particular case by Venkatesh is proven; it was previously proved in a special case in [TU22, Proposition 4].

1.6. Notation

By F we denote a totally real number field, our base field, by F_v its completion at any place v of F, and we choose algebraic closures $\overline{F}$ of F and $\overline{F}_v$ if F_v for all places v. These choices define the absolute Galois groups $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$. We write $I_v \subset G_{F_v}$ for the inertia subgroup. We also fix embeddings $\overline{F} \to \overline{F}_v$, extending the canonical embeddings $F \to F_v$. This determines for each place v of F an embedding $G_{F_v} \to G_F$. By Frob_v we denote a Frobenius automorphism in G_{F_v}, that is unique up to I_v, and we also write Frob_v for its image in G_F. All representations of G_F or of G_{F_v} will be assumed to be continuous. If v is a finite place of F, then we write q_v for the cardinality of its residue field.

Throughout the paper, we fix a prime $p > 2$, and we denote by $\overline{\mathbf{Q}}_p$ an algebraic closure of $\mathbf{Q}_p$. We will call a finite extension E of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ a coefficient field. For a coefficient field E, we let $\mathcal{O}$ be its ring of integers, k its residue field and $\varpi \in \mathcal{O}$ a uniformizer. We write Σ_p for the set of places of F above p, and we assume throughout the paper that F over $\mathbf{Q}$ is unramified at all places above p. It is likely that this hypothesis could be weakened.

The category of complete Noetherian local $\mathcal{O}$-algebras with residue field k is denoted by $\mathrm{CNL}_{\mathcal{O}}$, and for any object R in $\mathrm{CNL}_{\mathcal{O}}$, we write $\mathfrak{m}_R \subset R$ for its maximal ideal. Each object $R \in \mathrm{CNL}_{\mathcal{O}}$ will be endowed with its profinite ($\mathfrak{m}_R$-adic) topology. By a complete Noetherian local $\mathcal{O}$-algebra, we implicitly mean that its residue field is equal to k; we feel justified because our rings typically have an augmentation to $\mathcal{O}$.

We denote by ε_p the p-adic cyclotomic character $\varepsilon_p : G_F \to \mathbf{Z}_p^\times$; if we compose ε_p on the right with any map $G_{F_v} \to G_F$ or on the left with $\mathbf{Z}_p^\times \to R^\times$, induced from any morphism $\mathbf{Z}_p \to R$ in $\mathrm{CNL}_{\mathbf{Z}_p}$, then we also write ε_p by slight abuse of notation.

For an $\mathcal{O}$-algebra R, an augmentation λ of R will always mean a surjective $\mathcal{O}$-algebra homomorphism $\lambda : R \to \mathcal{O}'$, where $\mathcal{O}'$ is the ring of integers in a finite extension of E (we will almost always take $\mathcal{O} = \mathcal{O}'$). For an $\mathcal{O}$-module M that is a finite abelian group, we denote by $\ell_{\mathcal{O}}(M)$ the length of M as an $\mathcal{O}$-module. For $\alpha \in \mathcal{O}$, we denote by $\mathrm{ord}_{\mathcal{O}}(\alpha) = \ell_{\mathcal{O}}(\mathcal{O}/(\alpha))$.

For a Galois representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which is finite flat at p, we will let $N(\overline{\rho})$ represent its Artin conductor.

2. Wiles defect for rings of dimension one

In this section, we state results from the Appendix A in the form in which they are used in the paper, and also with a view to generalizing these results to higher-dimensional rings in §3.

For any ring R, any ideal $I \subseteq R$ and any R-module M, we will always use $M[I] \subseteq M$ for the submodule of I-torsion elements of M. In particular, $R[I] = \mathrm{Ann}_R(I) \subseteq R$ is the annihilator of the ideal I.

If M is a finitely generated R-module, with generating set $m_1,\dots,m_n$ inducing a surjection $R^n \twoheadrightarrow M$, then we will let $\mathrm{Fitt}_R(M) \subseteq R$ (called the 0^{th} *fitting ideal*) denote the ideal generated by all elements of the form $\det(v_1,\dots,v_n) \in R$ for $v_1,\dots,v_n \in \ker(R^n \twoheadrightarrow M)$. It is well known that this is independent of the choice of generating set $m_1,\dots,m_n$ and moreover that $\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M)$. When the ring R is clear from context, we will sometimes write $\mathrm{Fitt}(M)$ in place of $\mathrm{Fitt}_R(M)$.

Let R be a complete, local Noetherian $\mathcal{O}$-algebra with $\dim(R) = 1$ and assume that R is finite over $\mathcal{O}$. Let $\lambda : R \twoheadrightarrow \mathcal{O}$ be any *augmentation* (i.e., surjective $\mathcal{O}$-algebra homomorphism). Let R^{tf} be the maximal ϖ-torsion free quotient of R, which is automatically finite free over $\mathcal{O}$.[2] Also, use λ to denote the augmentation $R^{\mathrm{tf}} \twoheadrightarrow \mathcal{O}$ induced by λ. Define

$$\Phi_\lambda(R) = (\ker\lambda)/(\ker\lambda)^2 = \Omega_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}$$

and

$$\Psi_\lambda(R) = \mathcal{O}/\eta_\lambda(R) = \mathcal{O}/(\lambda(R^{\mathrm{tf}}[\ker\lambda])),$$

which we will call the *cotangent space* and *congruence module* of R (with respect to λ). From now on, we will assume that $\Phi_\lambda(R)$ is finite, which geometrically means that λ is smooth on the generic fiber of R.

In [BKM21], we define the *Wiles defect* of R with respect to λ to be

$$\delta_\lambda(R) = \frac{\log|\Phi_\lambda(R)| - \log|\Psi_\lambda(R)|}{\log|\mathcal{O}/p|} = \frac{\ell_{\mathcal{O}}(\Phi_\lambda(R)) - \ell_{\mathcal{O}}(\Psi_\lambda(R))}{\ell_{\mathcal{O}}(\mathcal{O}/p)}, \tag{2.1}$$

which is known to be a nonnegative rational number. The reason for the normalization factor of $\log|\mathcal{O}/p|$ is to ensure that $\delta_\lambda(R)$ is invariant under expanding the coefficient ring $\mathcal{O}$. Moreover, we have the following standard result (cf. [Wil95, Len95]):

Lemma 2.1. *For R as above, we have $\delta_\lambda(R) = 0$ if and only if $R = R^{\mathrm{tf}}$ and R is a complete intersection*

Proof. From $\delta_\lambda(R) = 0$, we see by [FKR21, Proposition A.6] that the map $R \to R^{\mathrm{tf}}$ is an isomorphism of complete intersections. □

Venkatesh, in an unpublished note [Ven16], observed that $\delta_\lambda(R)$ can be expressed in terms of two other invariants of R (see Appendix A of this paper for a detailed proof of a more general version of Venkatesh's observation).

First, let R act on $E/\mathcal{O}$ through its quotient $R \xrightarrow{\lambda} \mathcal{O}$. Venkatesh's first invariant is simply the first André–Quillen cohomology group $\mathrm{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})$.

To define Venkatesh's second invariant, we will fix an $\mathcal{O}$-algebra $\widetilde{R}$ and a surjection $\varphi : \widetilde{R} \twoheadrightarrow R$ with the properties that

- $\widetilde{R}$ is a complete intersection of dimension 1, finite free over $\mathcal{O}$.
- $\Phi_{\lambda\circ\varphi}(\widetilde{R})$ is finite.

[2] In the rest of this paper, we will always work in the case where $R = R^{\mathrm{tf}}$, but we still state the general version in this section for the sake of completeness.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

(such a ring always exists, as explained in Appendix A). When there is no chance of confusion we will also use λ to denote the induced map $\lambda \circ \varphi : \widetilde{R} \twoheadrightarrow R \twoheadrightarrow \mathcal{O}$.

Now, write $I = \ker \varphi$ so that $\lambda(I) = 0$. As $\widetilde{R}$-modules, we have that $\mathrm{Fitt}(I) \subseteq \widetilde{R}[I]$, and hence $\lambda(\mathrm{Fitt}(I)) \subseteq \lambda(\widetilde{R}[I])$ as ideals of $\mathcal{O}$ (and in fact, both of these ideals are nonzero as explained in Appendix A). We then define Venkatesh's second invariant to be the cyclic $\mathcal{O}$-module

$$C_{1,\lambda}(R) = \lambda\left(\widetilde{R}[I]\right)/\lambda(\mathrm{Fitt}(I)).$$

A priori, this looks like it will depend on the choice of complete intersection $\widetilde{R}$, but the work of Appendix A shows that it in fact depends only on R and λ. The main result Proposition A.6 of Appendix A is the following formula for the Wiles defect $\delta_\lambda(R)$. We recall as noted earlier that [TU22, Proposition 4] proves a particular case (when $C_{1,\lambda}(R)$ is trivial) of this formula.

Theorem 2.2 (see A.6). *If R and $\lambda : R \twoheadrightarrow \mathcal{O}$ are as described above, and $\Phi_\lambda(R)$ is finite, then*

$$\frac{|\mathrm{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})|}{|C_{1,\lambda}(R)|} = \frac{|\Phi_\lambda(R)|}{|\Psi_\lambda(R)|}.$$

In particular, $\delta_\lambda(R) = \dfrac{\log|\mathrm{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})| - \log|C_{1,\lambda}(R)|}{\log|\mathcal{O}/p|}$.

Remark 2.3. In practice, one is often interested in the Wiles defect $\delta_\lambda(M)$ (as defined in [BKM21, Section 3]) of a particular module M over R, as well as, or instead of $\delta_\lambda(R)$. However, in many cases relevant to us, the results of [BKM21] imply that $\delta_\lambda(R) = \delta_\lambda(M)$, so we will focus mainly on $\delta_\lambda(R)$ in this paper, except in §7 in which we apply Theorem 6.5 which determines defects of Hecke rings to detect the defect of modules that they act on.

We do suspect that there may exist some generalization of Theorem 2.2 which would directly express $\delta_\lambda(M)$ in terms of similar invariants. Such a generalization would allow us to directly study $\delta_\lambda(M)$ in cases when we can not prove it is equal to $\delta_\lambda(R)$, and could possibly work in cases when the results of this paper do not apply. The results of [BIK23, Theorem 1.2] support such a suspicion.

We end this section by remarking that the definition of the Wiles defect $\delta_\lambda(R)$ in [BKM21], which depends on finiteness of $\Phi_\lambda(R)$, makes sense for a complete Noetherian, Cohen–Macaulay local $\mathcal{O}$-algebra R only when R is of dimension one.

Lemma 2.4. *Let R be a complete Noetherian local $\mathcal{O}$-algebra together with an augmentation $\lambda : R \to \mathcal{O}$ such that $\Phi_\lambda(R)$ is a finite abelian group then* $\ker(\lambda)$ *is a minimal prime ideal. If we further assume that R is Cohen–Macaulay then R is of dimension one.*

Proof. Let $\ker(\lambda) = \mathfrak{p}$, and we observe that the localization $R_\mathfrak{p}$ is a local ring with maximal ideal $m = \mathfrak{p}R_\mathfrak{p}$ and infinite residue field E, and by our assumption that $\ker(\lambda)/\ker(\lambda)^2$ is finite we deduce that $m = m^2$ and thus $m = 0$. This implies that $R_\mathfrak{p}$ is a field, and thus $\mathfrak{p}$ is a minimal prime ideal of R. As Cohen–Macaulay rings are equidimensional, we deduce the last statement of the lemma. □

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

3. Wiles defect for higher-dimensional Cohen–Macaulay rings

We define and prove properties of the Wiles defect for (higher-dimensional) rings in the category $C_{\mathcal{O}}$. The category $C_{\mathcal{O}}$ was alluded to in the introduction.

Definition 3.1. The category $C_{\mathcal{O}}$ consists of tuples (R,λ_R) such that:

- R a complete, Noetherian local $\mathcal{O}$-algebra, with maximal ideal $\mathfrak{m}$ and residue field $k = \mathcal{O}/\varpi$, which is flat over $\mathcal{O}$ and Cohen–Macaulay;
- $\lambda_R : R \to \mathcal{O}$ is an augmentation (that is, a continuous surjective $\mathcal{O}$-algebra homomorphism) such that $\operatorname{Spec} R[1/\varpi]$ is formally smooth at the point corresponding to λ.

The morphisms in the category $C_{\mathcal{O}}$ are local homomorphisms of $\mathcal{O}$-algebras compatible with the augmentation, namely local $\mathcal{O}$-algebra maps $f\colon R \to S$ such that $\lambda_S \circ f = \lambda_R$. (As the augmentation considered will be clear from the context, we will often denote λ_R by just λ and also given a pair $(R,\lambda) \in C_{\mathcal{O}}$ we will sometimes write $R \in C_{\mathcal{O}}$.)

In light of Lemma 2.4, the definition of the Wiles defect as given in [BKM21] can be applied to $R \in C_{\mathcal{O}}$ only when R is of dimension 1. Thus, we define the *Wiles defect* $\delta_\lambda(R)$ for $R \in C_{\mathcal{O}}$ (cf. Definition 3.24) motivated by the Venkatesh formula of the defect $\delta_\lambda(R)$ for $R \in C_{\mathcal{O}}$ when R is one-dimensional. This requires some preliminary work that we undertake first. To orient the reader, we indicate the main steps towards the definition.

The Wiles defect is expressed in terms of:

(i) the invariant $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ which can be directly defined using a continuous version of the André–Quillen cohomology of rings (cf. §3.3, in particular Theorem 3.20);

(ii) the invariant $C_{1,\lambda}(R)$ that is defined in terms of an auxiliary complete intersection $\widetilde{R}$ surjecting onto R (cf. §3.2, in particular Corollary 3.12, which shows that this does not depend on the choice of $\widetilde{R}$).

The Wiles defect $\delta_\lambda(R)$ is then defined (cf. Definition 3.24) via the formula

$$\delta_\lambda(R) = \frac{\log|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})| - \log|C_{1,\lambda}(R)|}{\log|\mathcal{O}/p|}.$$

We show below the key property of independence of the invariants we define under forming quotients by regular sequences (see §3.2 and 3.3). We also provide formulas for the invariants in terms of certain complete intersection rings that surject onto $R \in C_{\mathcal{O}}$, similar to the treatment in the appendix, but in higher dimensions.

In the case when R is of dimension 1, this definition of the defect for $R \in C_{\mathcal{O}}$ agrees, by Theorem 2.2 and Proposition 3.27, with the definition of the Wiles defect defined in [BKM21] as

$$\delta_\lambda(R) = \frac{\log|\Phi_\lambda(R)| - \log|\Psi_\lambda(R)|}{\log|\mathcal{O}/p|}.$$

(Note that when R is of dimension one, the finiteness of $|\Phi_\lambda(R)|$ is equivalent to saying that $\lambda : R \to \mathcal{O}$ has formally smooth generic fiber.)

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

For the remainder of this section, we will fix $(R,\lambda_R) \in C_{\mathcal{O}}$, and let $\lambda = \lambda_R : R \to \mathcal{O}$ denote the augmentation. Recall that by the definition of $C_{\mathcal{O}}$, R is Cohen–Macaulay. We will let $d = \dim_{\mathcal{O}} R$, and consider the power series ring $S = \mathcal{O}[[y_1,\ldots,y_d]]$.

We will introduce a number of other auxiliary rings and morphisms which will be used to define the invariants $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ and $C_{1,\lambda}(R)$ and to prove the key property of invariance under regular sequences. For ease of reading, we will summarize all of this in the following commutative diagram:

$$\begin{tikzcd}
& & S \arrow[dl, hook', "\widetilde{\theta}"'] \arrow[d, hook, "\theta"] & \\
I \arrow[r, hook] \arrow[d] & \widetilde{R} \arrow[r, "\varphi"] \arrow[d, "\pi_\theta"] \arrow[drr, dashed, "\widetilde{\lambda}"] & R \arrow[d] \arrow[dr, "\lambda"] & \\
I_\theta \arrow[r, hook] & \widetilde{R}_\theta \arrow[r, "\varphi_\theta"'] \arrow[rr, dashed, bend right, "\widetilde{\lambda}_\theta"'] & R_\theta \arrow[r, "\lambda_\theta"'] & \mathcal{O}
\end{tikzcd}$$

Here:

- θ is an injective map $\mathcal{O}$-algebra map, satisfying Property (P);
- $\widetilde{R}$ is a complete intersection with $\dim_{\mathcal{O}} \widetilde{R} = \dim_{\mathcal{O}} R = d$ and $\varphi : \widetilde{R} \to R$ is a continuous surjective map with kernel $I \subseteq \widetilde{R}$ (the precise properties satisfied by $(\widetilde{R}, I, \varphi)$ are outlined in Property (CI));
- $\widetilde{\theta} : S \hookrightarrow \widetilde{R}$ is a lift of θ along φ, satisfying certain properties, which is proven to exist in Lemma 3.7;
- We will usually identify S with its images in R and $\widetilde{R}$ so that in particular $y_1,\ldots,y_d \in R$ and $y_1,\ldots,y_d \in \widetilde{R}$;
- Treating R and $\widetilde{R}$ as S-modules via θ and $\widetilde{\theta}$, we have $R_\theta = R \otimes_S \mathcal{O}$ and $\widetilde{R}_\theta = \widetilde{R} \otimes_S \mathcal{O}$. Equivalently, $R_\theta = R/(\theta(y_1),\ldots,\theta(y_d))$ and $\widetilde{R}_\theta = \widetilde{R}/(\widetilde{\theta}(y_1),\ldots,\widetilde{\theta}(y_d))$;
- $\varphi_\theta : \widetilde{R}_\theta \to R_\theta$ is the map induced by φ;
- $\lambda_\theta : R_\theta \to \mathcal{O}$ is the augmentation induced by λ;
- $\pi_\theta : \widetilde{R} \twoheadrightarrow \widetilde{R}_\theta$ is the quotient map;
- $I_\theta = \ker \varphi_\theta$. From the surjectivity of π_θ, it also follows that $I_\theta = \pi_\theta(I)$;
- $\widetilde{\lambda} : \widetilde{R} \twoheadrightarrow \mathcal{O}$ and $\widetilde{\lambda}_\theta : \widetilde{R}_\theta \twoheadrightarrow \mathcal{O}$ are simply the induced augmentations $\widetilde{\lambda} = \lambda \circ \varphi$ and $\widetilde{\lambda}_\theta = \lambda_\theta \circ \varphi_\theta$.

We say that the inclusion $\theta : S \hookrightarrow R$ satisfies (P) if the following conditions hold:

Property (P).

- *$\theta : S \hookrightarrow R$ is a continuous $\mathcal{O}$-algebra homomorphism.*
- *θ makes R into a finite free S-module (so that $(\theta(y_1),\ldots,\theta(y_d),\varpi)$ is a regular sequence for R).*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

- $(\theta(y_1),\dots,\theta(y_d)) \subseteq \ker\lambda$.
- *If* $R_\theta = R/(\theta(y_1),\dots,\theta(y_d)) = R\otimes_S \mathcal{O}$ *and* $\lambda_\theta : R_\theta \twoheadrightarrow \mathcal{O}$ *is the map induced by* λ, *then* $\Phi_{\lambda_\theta}(R_\theta)$ *is finite.*

We will say that the triple $(\widetilde{R},I,\varphi)$ satisfies (CI) if:

Property (CI).

- $\widetilde{R}$ *is a complete, Noetherian local* $\mathcal{O}$*-algebra, flat and equidimensional over* $\mathcal{O}$ *of relative dimension* d.
- $\widetilde{R}$ *is a complete intersection.*
- $\varphi : \widetilde{R} \to R$ *is a continuous surjection of* $\mathcal{O}$*-algebras with* $I = \ker\varphi$.
- *The point corresponding to* $\lambda\circ\varphi$ *in* $\operatorname{Spec}\widetilde{R}[1/\varpi]$ *is a formally smooth point.*

We note the following two results, which will be proved in Section 3.1:

Proposition 3.2. *For any* $(R,\lambda)\in C_{\mathcal{O}}$ *with* $\dim_{\mathcal{O}} R = d$, *a map* θ *satisfying property (P) exists.*

Proposition 3.3. *For any* $(R,\lambda)\in C_{\mathcal{O}}$ *with* $\dim_{\mathcal{O}} R = d$, *there exists a triple* $(\widetilde{R},I,\varphi)$ *satisfying Property (CI).*

We will give the proof of Proposition 3.2 in Section 3.1, after the proof of Lemma 3.6. Proposition 3.3 will be a direct consequence of Lemma 3.6.

Note that Property (P) implies that R_θ is finite free over $\mathcal{O}$ and that $\Phi_{\lambda_\theta}(R_\theta)$ is finite. Thus, it satisfies the conditions of Section 2, and so we may consider the Wiles defect $\delta_{\lambda_\theta}(R_\theta)$ and the Venkatesh invariants $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$ and $C_{1,\lambda_\theta}(R_\theta)$. The main result of this section is Theorem 3.25, which shows that all three of these quantities depend only on R and λ and not on the choice of θ.

This section is structured as follows: In Section 3.1, we prove Propositions 3.2 and 3.3 and establish the basic properties of all of the auxiliary rings we are considering; Section 3.2 proves the invariance of $C_{1,\lambda_\theta}(R_\theta)$; Section 3.3 proves the invariance of $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$; Section 3.4 uses the prior work to give a definition the invariants $D_{1,\lambda}(R)$ and $c_{1,\lambda}(R)$ and of the Wiles defect $\delta_\lambda(R)$, for any $(R,\lambda)\in C_{\mathcal{O}}$; lastly, Section 3.5 proves a key property of these invariants – that they are compatible with completed tensor products (see Proposition 3.32).

Remark 3.4. In our main number theoretic applications in Section 6, the rings R and S will typically be the rings R_∞ (or R^τ_∞ in our notation) and S_∞ appearing the classical in the Taylor–Wiles–Kisin patching method – see Theorem 6.4 for specifics. The ring R_θ will be a global Galois deformation ring, denoted R^τ_0, and the augmentation $\lambda_\theta : R_\theta \to \mathcal{O}$ will be induced by a Galois representation $\rho_\lambda : G_F \to \mathrm{GL}_2(\mathcal{O})$, where F is a (totally real) number field. The augmentation $\lambda : R \to \mathcal{O}$ will simply be the pullback of λ_θ.

The map $\theta : S \to R$, and hence the ring R_θ and the augmentation $\lambda_\theta : R_\theta \to \mathcal{O}$, will depend on subtle global Galois theoretic information involving the representation ρ_λ. However, the ring R and the augmentation $\lambda : R \to \mathcal{O}$ will depend only on the restrictions $\rho_\lambda|_{G_{F_v}}$ for a finite collection of places v of F – that is, only on local information. Thus,

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Theorem 3.25 will imply the the Wiles defect $\delta_{\lambda_\theta}(R_\theta)$, a priori a global invariant, will depend only on local information. See Theorem 6.5 for a precise result.

The ring $\widetilde{R}$ will have no particular number theoretic significance. It will be chosen in Section 5 in order to facilitate computations of the Wiles defects of various local deformation rings.

3.1. Complete intersection (CI) covers

We begin with the following lemma:

Lemma 3.5. *Let S be a complete, Noetherian local $\mathcal{O}$-algebra with an augmentation $\lambda\colon S \to \mathcal{O}$, and let $d > 0$. Suppose that $S[1/\varpi]$ is formally smooth at λ of dimension $n \geq d$ and that there are elements $f_1,\dots,f_d \in \ker\lambda$ such that $f_1,\dots,f_d,\varpi$ is a regular sequence in S. Then there exist $h_1,\dots,h_d \in (\ker\lambda \cap (f_1,\dots,f_d,\varpi))$ such that $h_1,\dots,h_d,\varpi$ is a regular sequence in S and such that for $A = S/(h_1,\dots,h_d)$ and the induced augmentation $\lambda_A\colon A \to \mathcal{O}$, the ring $A[1/\varpi]$ is formally smooth at λ_A of dimension $n-d$.*

Proof. By replacing $(f_1,\dots,f_d)$ by $(f_1^2,\dots,f_d^2)$, we may assume that $(f_1,\dots,f_d) \subset \ker\lambda^2$; see [Mat80, 15.A, Theorem 26]. Write $S[1/\varpi]$ for the localization of S at ϖ and $\widehat{S[1/\varpi]}$ for the completion of the latter at the point corresponding to λ. By our hypothesis, the ring $\widehat{S[1/\varpi]}$ is a power series ring over E in $n \geq d$ indeterminates. Let $\widehat{I}$ denote its maximal ideal. Choose $g_1,\dots,g_d$ in $\ker\lambda$ whose images in $\widehat{I}/\widehat{I}^2$ are linearly independent over E. Then $(h_1,\dots,h_d)$ with $h_i = f_i + \varpi g_i$ has all properties required. □

Lemma 3.6. *Suppose B is a complete, Noetherian local $\mathcal{O}$-algebra with $\dim B = d+1$ and $\dim B/\varpi = d$ and $\lambda : B \to \mathcal{O}$ is an augmentation such that $\operatorname{Spec} B[1/\varpi]$ is formally smooth at λ of dimension d. Then there exists a Noetherian $\mathcal{O}$-algebra A and a surjective homomorphism $\pi\colon A \to B$ such that the following holds:*

1. *The ring A is local and complete, a complete intersection, flat over $\mathcal{O}$ and of relative dimension d.*
2. *The map $\pi[1/\varpi]\colon A[1/\varpi] \to B[1/\varpi]$, obtained from π by inverting ϖ, induces an isomorphism after completion at the points corresponding to the augmentations λ and $\mu = \lambda \circ \pi\colon A \to \mathcal{O}$, respectively. In particular, $\operatorname{Spec} A[1/\varpi]$ is formally smooth at μ of dimension d.*

Proof. Let $\Pi\colon S = \mathcal{O}[[z_1,\dots,z_n]] \to B$ be a surjective ring homomorphism. Let $\mathfrak{p}_\lambda \subset \mathfrak{m}_B$ be the prime ideal $\ker\lambda$, and denote by $\mathfrak{q}_\lambda \subset \mathfrak{m}_S$ its inverse image under Π, that is, $\mathfrak{q}_\lambda = \ker\lambda \circ \Pi$. Let $m = n - d \geq 0$.

By hypothesis B/ϖ has dimension d. Because S is ϖ-torsion free and S/ϖ is regular, we can find a regular sequence $(f_1,\dots,f_m)$ in $\ker\Pi \subset S$ such that $(f_1,\dots,f_m,\varpi)$ is a regular sequence. Because $S[1/\varpi]$ is regular of dimension n and $\mathfrak{q}_\lambda[1/\varpi]$ is a maximal ideal of that ring, the ring $S[1/\varpi]$ is formally smooth at $\mathfrak{q}_\lambda[1/\varpi]$ of dimension n.

It follows from Lemma 3.5 that there exist $h_1,\dots,h_m \in \ker\Pi + \varpi S$ such that $h_1,\dots,h_m,\varpi$ is a regular sequence in S and such that for $A = S/(h_1,\dots,h_m)$ and induced augmentation $\lambda_A\colon A \to \mathcal{O}$ the ring $A[1/\varpi]$ is formally smooth at λ_A of dimension

$n-m=d$. It follows that one has an induced surjection $A \to B$, where A is a local complete, complete intersection $\mathcal{O}$-algebra, flat over $\mathcal{O}$ of relative dimension d and that the induced surjection $A[1/\varpi] \to B[1/\varpi]$ becomes an isomorphism after completion at $\mathfrak{q}_\lambda[1/\varpi]$. □

Proof of Proposition 3.2. Because R is Cohen–Macaulay and flat over $\mathcal{O}$ of relative dimension d, we can find a regular sequence $\varpi, f_1, \ldots, f_d$ in R. If we replace each f_i by an element in $f_i + \varpi R$ the resulting sequence is again regular. Now, using that $\ker\lambda$ together with ϖ generate the maximal ideal of R, we may assume that $f_1, \ldots, f_d$ lie in $\ker\lambda$. Again by hypothesis $R[1/\varpi]$ is Cohen–Macaulay of dimension d and formally smooth at λ, and hence it is formally smooth at λ of dimension d.

Then by Lemma 3.5, there exist $h_1, \ldots, h_d \in \ker\lambda$ such that $h_1, \ldots, h_d, \varpi$ is a regular sequence in R and such that for $B = R/(h_1, \ldots, h_d)$ and the induced augmentation $\lambda_B \colon B \to \mathcal{O}$ the ring $B[1/\varpi]$ is formally smooth at λ_B of dimension 0. It follows that the continuous $\mathcal{O}$-algebra map $\theta \colon S = \mathcal{O}[[y_1, \ldots, y_d]] \to R$ with $y_i \mapsto f_i$ makes R into a finite free S-module such that in the notation of (P), we have $B = R_\theta$ and $\lambda_B = \lambda_\theta$, and moreover R_θ is finite free over $\mathcal{O}$. Hence, $R_\theta[1/\varpi]$ is a product of Artin E-algebras, and the smoothness at λ_θ shows that the component corresponding to λ_θ is equal to E. From this, it follows that $\Phi_{\lambda_\theta}(R_\theta) = \ker\lambda_\theta/(\ker\lambda_\theta)^2$ is of finite $\mathcal{O}$-length, as it is finitely generated over $\mathcal{O}$ and $\mathcal{O}$-torsion. □

Next, we observe that we can lift regular sequences of R along $\widetilde{R} \to R$.

Lemma 3.7. *Assume that $\theta : S \hookrightarrow R$ satisfies (P) and $(\widetilde{R}, I, \varphi)$ satisfies (CI). Then θ lifts to a morphism $\widetilde{\theta} : S \to \widetilde{R}$ (making φ into a S-algebra homomorphism) which makes $\widetilde{R}$ into a finite free S-module. That is, identifying S with its image in $\widetilde{R}$, that $(y_1, \ldots, y_d, \varpi)$ is a regular sequence for both $\widetilde{R}$ and R.*

Moreover, if $\widetilde{R}_\theta = \widetilde{R}/(y_1, \ldots, y_d)$ and $\widetilde{\lambda}_\theta : \widetilde{R}_\theta \twoheadrightarrow \mathcal{O}$ is the map induced by $\widetilde{\lambda}$, then $\widetilde{R}_\theta$ is a complete intersection of dimension 1, finite free over $\mathcal{O}$ and $\Phi_{\widetilde{\lambda}_\theta}(\widetilde{R}_\theta)$ is finite.

This will follow from the following lemma:

Lemma 3.8. *Let A be a Noetherian local ring, and let $B = A/I$ for some ideal I of A. Let $x \in \mathfrak{m}_B$ be an element not contained in any minimal prime of B. Then x lifts to an element $\widetilde{x} \in \mathfrak{m}_A$ which is not contained in any minimal primes of A.*

Proof. Pick any lift $\widetilde{x}_0 \in \mathfrak{m}_A$ of x. Let the set of minimal primes of A be $\{P_1, \ldots, P_n\}$, labeled so that there is some $0 \le a \le n$ for which $\widetilde{x}_0 \notin P_1, P_2, \ldots, P_a$, and $\widetilde{x}_0 \in P_{a+1}, \ldots, P_n$.

Now, fix any $i > a$, so that $\widetilde{x}_0 \in P_i$. Note that if $I \subseteq P_i$ then P_i/I would be a minimal prime of B containing x, contradicting our assumption. Hence, $I \not\subseteq P_i$, and so there is some $r_i \in I \setminus P_i$.

Also, for any $j \ne i$, $P_j \not\subseteq P_i$, and so there is some $s_{ij} \in P_j \setminus P_i$. Now, define

$$y_i := r_i \prod_{j \ne i} s_{ij}$$

so that $y_i \in I$, $y_i \in P_j$ for $j \neq i$ and $y_i \notin P_i$. Finally, let

$$\widetilde{x} = \widetilde{x}_0 + y_{a+1} + y_{b+2} + \cdots + y_n.$$

Then we have $\widetilde{x} \equiv \widetilde{x}_0 \equiv x \bmod I$, $\widetilde{x} \equiv \widetilde{x}_0 \not\equiv 0 \bmod P_i$ for $i \leq a$ and $\widetilde{x} \equiv y_i \not\equiv 0 \bmod P_i$ for $i > a$. So $\widetilde{x}$ is our desired lift. □

Proof of Lemma 3.7. Identifying S with its image in R, we get that $(y_1,\dots,y_d,\varpi)$, and thus $(\varpi,y_1,\dots,y_d)$, is a regular sequence for R. We claim that we can inductively construct a sequence $\widetilde{y}_1,\widetilde{y}_2,\dots,\widetilde{y}_d \in \widetilde{R}$ such that $\varphi_\infty(\widetilde{y}_i) = y_i$ for all i and $\dim \widetilde{R}/(\varpi,\widetilde{y}_1,\dots,\widetilde{y}_j) = d - j = \dim R/(\varpi,y_1,\dots,y_j)$ for all $0 \leq j \leq d$.

As $\widetilde{R}$ and R are both flat over $\mathcal{O}$ of relative dimension d, we have $\dim \widetilde{R}/(\varpi) = d = \dim R/(\varpi)$. Now, assume that $\widetilde{y}_1,\dots,\widetilde{y}_j$ have been constructed for some $j < d$. Let $A_j = \widetilde{R}/(\varpi,\widetilde{y}_1,\dots,\widetilde{y}_j)$ and $B_j = R/(\varpi,y_1,\dots,y_j)$ so that $\varphi_\infty : \widetilde{R} \to R$ induces a map $\varphi_j : A_j \to B_j$. As $(\varpi,y_1,\dots,y_d)$ is a regular sequence for R, y_{j+1} is by definition not a zero divisor in B_j, and so in particular cannot be contained in any minimal primes of B_j. By Lemma 3.8 it follows there is some $y'_{j+1} \in A_j$ with $\varphi_j(y'_{j+1}) = y_{j+1}$ which is not contained in any minimal prime of A_j. Let $\widetilde{y}_{j+1} \in \widetilde{R}$ be any lift of y'_{j+1}. But now

$$\widetilde{R}/(\varpi,\widetilde{y}_1,\dots,\widetilde{y}_j,\widetilde{y}_{j+1}) \cong A_j/(y'_{j+1})$$

which has dimension $\dim A_j - 1 = d - (j+1)$, by the assumption that y'_{j+1} is not contained in any minimal prime of A_j. This completes the induction.

Now, $(\varpi,\widetilde{y}_1,\dots,\widetilde{y}_d)$ is a system of parameters for $\widetilde{R}$. As $\widetilde{R}$ is a complete intersection and thus Cohen–Macaulay, it follows that $(\varpi,\widetilde{y}_1,\dots,\widetilde{y}_d)$, and thus $(y_1,\dots,y_d,\varpi)$, is a regular sequence for $\widetilde{R}$.

So now defining $\widetilde{\theta} : S \to \widetilde{R}$ by $\widetilde{\theta}(y_i) = \widetilde{y}_i$ makes $\widetilde{R}$ into a finite free S module, as desired.

The fact that $\widetilde{R}_\theta$ is a complete intersection of dimension 1, and finite free over $\mathcal{O}$, now follows immediately from the fact that $\widetilde{R}$ is a complete intersection. For the last assertion, the proof of [BKM21, Theorem 7.16] gives rise to a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathcal{O}^d \arrow[r, "\widetilde{\Theta}"] \arrow[d, "="] & \Phi_{\widetilde{\lambda}}(\widetilde{R}) \arrow[r] \arrow[d] & \Phi_{\widetilde{\lambda}_\theta}(\widetilde{R}_\theta) \arrow[r] \arrow[d] & 0 \\
\mathcal{O}^d \arrow[r, "\Theta"] & \Phi_\lambda(R) \arrow[r] & \Phi_{\lambda_\theta}(R_\theta) \arrow[r] & 0
\end{tikzcd}$$

where $\Phi_\lambda(R) = (\ker\lambda)/(\ker\lambda)^2 = \widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}$ and $\Phi_{\widetilde{\lambda}}(\widetilde{R}) = (\ker\widetilde{\lambda})/(\ker\widetilde{\lambda})^2 = \widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes_{\widetilde{\lambda}} \mathcal{O}$, and the maps Θ and $\widetilde{\Theta}$ are given in terms of differentials by $e_i \mapsto dy_i$.

Now, as in [BKM21, Theorem 7.16], the fact that $\operatorname{Spec} R[1/\varpi]$ and $\operatorname{Spec} \widetilde{R}[1/\varpi]$ are both equidimensional of dimension d and λ and $\widetilde{\lambda}$, respectively, correspond to formally smooth points on these schemes, implies that $\Phi_\lambda(R)$ and $\Phi_{\widetilde{\lambda}}(\widetilde{R})$ both have rank d as $\mathcal{O}$-modules.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

But now the fact that $\Phi_{\lambda_\theta}(R_\theta)$ is finite implies that Θ must be injective. By commutativity, this implies that $\widetilde{\Theta}$ is also injective, which in turn implies that $\Phi_{\widetilde{\lambda}_\theta}(\widetilde{R}_\theta)$ is also finite. $\square$

3.2. Invariance of $C_{1,\lambda_\theta}(R_\theta)$ of θ

For this section, we will fix θ satisfying (P) and $(\widetilde{R},I,\varphi)$ satisfying (CI). We will let $\widetilde{\theta}: S \hookrightarrow \widetilde{R}$ be a lift of θ satisfying the conclusion of Lemma 3.7, and we will identify S with its images in R and $\widetilde{R}$.

Let $\widetilde{R}_\theta$ and $\widetilde{\lambda}_\theta$ be as in Lemma 3.7, and let $\varphi_\theta = \varphi \otimes_S \mathcal{O} : \widetilde{R}_\theta \twoheadrightarrow R_\theta$ (so that $\widetilde{\lambda} = \lambda \circ \varphi$), and let $I_\theta = \ker \varphi_\theta \subseteq \widetilde{R}_\theta$. Also, let $\pi_\theta : \widetilde{R} \to \widetilde{R}_\theta$ be the quotient map so that $\widetilde{\lambda} = \widetilde{\lambda}_\theta \circ \pi_\theta$ and $I_\theta = \pi_\theta(I)$.

The ring $\widetilde{R}_\theta$ now satisfies the conditions from Section 2, so we have

$$C_{1,\lambda_\theta}(R_\theta) = \widetilde{\lambda}_\theta\left(\widetilde{R}_\theta[I_\theta]\right)/\widetilde{\lambda}_\theta(\mathrm{Fitt}(I_\theta)).$$

The main result of this subsection is the following:

Theorem 3.9. *We have the following:*

1. $\widetilde{R}_\theta[I_\theta] = \pi_\theta(\widetilde{R}[I])$
2. $\mathrm{Fitt}(I_\theta) = \pi_\theta(\mathrm{Fitt}(I))$

So in particular,

$$C_{1,\lambda_\theta}(R_\theta) = \widetilde{\lambda}_\theta\left(\widetilde{R}_\theta[I_\theta]\right)/\widetilde{\lambda}_\theta(\mathrm{Fitt}(I_\theta)) = \widetilde{\lambda}_\theta\left(\pi_\theta(\widetilde{R}[I])\right)/\widetilde{\lambda}_\theta(\pi_\theta(\mathrm{Fitt}(I))) = \widetilde{\lambda}\left(\widetilde{R}[I]\right)/\widetilde{\lambda}(\mathrm{Fitt}(I)),$$

which depends only on $\widetilde{R}$, R and $\widetilde{\lambda}: R \to \mathcal{O}$, all of which are independent of θ.

Thus, if we define $C_{1,\widetilde{\lambda}}(\widetilde{R}) = \widetilde{\lambda}\left(\widetilde{R}[I]\right)/\widetilde{\lambda}(\mathrm{Fitt}(I))$, then we have

$$C_{1,\widetilde{\lambda}}(\widetilde{R}) = C_{1,\lambda_\theta}(R_\theta).$$

Proof of Theorem 3.9(1). Clearly, we have $\pi_\theta(\widetilde{R}[I]) \subseteq \widetilde{R}_\theta[I_\theta]$ (since $I_\theta = \pi_\theta(I)$ and so $\widetilde{R}_\theta[I_\theta] = \widetilde{R}[I]$), so it suffices to prove that $\pi_\theta|_{\widetilde{R}[I]} : \widetilde{R}[I] \to \widetilde{R}_\theta[I_\theta]$ is surjective.

We first note that as $\widetilde{R}$ and $\widetilde{R}_\theta$ are complete intersections, and thus are Gorenstein, we get the following:

Lemma 3.10. *There are isomorphisms $\Psi : \widetilde{R} \xrightarrow{\sim} \mathrm{Hom}_S(\widetilde{R},S)$ and $\Psi_\theta : \widetilde{R}_\theta \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta,\mathcal{O})$ of $\widetilde{R}$-modules, fitting into a commutative diagram:*

$$\begin{tikzcd}
\widetilde{R} \arrow[r, "\Psi"] \arrow[d, "\pi_\theta"] & \mathrm{Hom}_S(\widetilde{R},S) \arrow[d, "\sigma"] \\
\widetilde{R}_\theta \arrow[r, "\Psi_\theta"] & \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta,\mathcal{O})
\end{tikzcd}$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

where the vertical map $\sigma : \mathrm{Hom}_S(\widetilde{R}, S) \to \mathrm{Hom}_S(\widetilde{R}, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta, \mathcal{O})$ *is just composition with the map* $S \to S/(y_1, \ldots, y_d) = \mathcal{O}$.

Proof. As $\widetilde{R}$ is Cohen–Macaulay and free of finite rank over S, we have $\omega_{\widetilde{R}} \cong \mathrm{Hom}_S(\widetilde{R}, S)$. But as $\widetilde{R}$ is a complete intersection, it is Gorenstein, and so $\omega_{\widetilde{R}} \cong \widetilde{R}$. Composing these isomorphisms gives the desired isomorphism $\Psi : \widetilde{R} \xrightarrow{\sim} \mathrm{Hom}_S(\widetilde{R}, S)$.

Now, note that (as $\widetilde{R}$ is a free S-module):

$$\begin{aligned}\Psi(\ker \pi_\theta) &= \Psi(y_1 \widetilde{R} + \cdots + y_d \widetilde{R}) = y_1 \Psi(\widetilde{R}) + \cdots + y_d \Psi(\widetilde{R}) \\ &= y_1 \mathrm{Hom}_S(\widetilde{R}, S) + \cdots + y_d \mathrm{Hom}_S(\widetilde{R}, S) = \mathrm{Hom}_S(\widetilde{R}, y_1 S + \cdots + y_d S) \\ &= \ker \sigma,\end{aligned}$$

which implies that there is an injection $\Psi_\theta : \widetilde{R} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}, \mathcal{O})$ making the above diagram commute. As σ is clearly surjective (since $\widetilde{R}$ is a projective S-module), it follows that Ψ_θ is also surjective. □

Lemma 3.11. *We have*

$$\Psi(\widetilde{R}[I]) = \{f : \widetilde{R} \to S | f(I) = 0\} = \mathrm{Hom}_S(\widetilde{R}/I, S)$$

and

$$\Psi_\theta(\widetilde{R}_\theta[I_\theta]) = \{f : \widetilde{R}_\theta \to \mathcal{O} | f(I_\theta) = 0\} = \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O}) = \mathrm{Hom}_S(\widetilde{R}/I, \mathcal{O}).$$

Proof. As Ψ is an isomorphism of $\widetilde{R}$-modules, we have $\Psi(\widetilde{R}[I]) = \mathrm{Hom}_S(\widetilde{R}, S)[I]$ and thus

$$\begin{aligned}\Psi(\widetilde{R}[I]) &= \{f : \widetilde{R} \to S | rf = 0 \text{ for all } r \in I\} \\ &= \{f : \widetilde{R} \to S | (rf)(x) = 0 \text{ for all } r \in I \text{ and } x \in \widetilde{R}\} \\ &= \{f : \widetilde{R} \to S | f(rx) = 0 \text{ for all } r \in I \text{ and } x \in \widetilde{R}\} \\ &= \{f : \widetilde{R} \to S | f(I) = 0\} \\ &= \mathrm{Hom}_S(\widetilde{R}/I, S).\end{aligned}$$

The proof for $\Psi_\theta(\widetilde{R}_\theta[I_\theta])$ is identical. □

Now, since $\widetilde{R}/I \cong R$ is a projective S-module, σ induces a surjective map $\mathrm{Hom}_S(\widetilde{R}/I, S) \to \mathrm{Hom}_S(\widetilde{R}/I, \mathcal{O})$. By Lemma 3.11, this is a surjective map $\sigma|_{\Psi(\widetilde{R}[I])} : \Psi(\widetilde{R}[I]) \to \Psi_\theta(\widetilde{R}_\theta[I_\theta])$, so the commutative diagram from Lemma 3.10 gives that $\pi_\theta|_{\widetilde{R}[I]} : \widetilde{R}[I] \to \widetilde{R}_\theta[I_\theta]$ is surjective. This completes the proof of (1). □

Proof of Theorem 3.9(2). By the definition of I, we have a short exact sequence of S-modules

$$0 \to I \to \widetilde{R} \xrightarrow{\varphi} R \to 0.$$

Applying $- \otimes_S \mathcal{O}$ to this gives an exact sequence

$$\mathrm{Tor}_1^S(R, \mathcal{O}) \to I \otimes_S \mathcal{O} \to \widetilde{R}_\theta \xrightarrow{\varphi} R_\theta \to 0.$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

and so as $I_\theta = \ker\varphi_\theta$, this gives as exact sequence

$$\mathrm{Tor}_1^S(R,\mathcal{O}) \to I \otimes_S \mathcal{O} \to I_\theta \to 0.$$

But now as R is a finite free S-module, $\mathrm{Tor}_1^S(R,\mathcal{O}) = 0$ and so we have an isomorphism $I \otimes_S \mathcal{O} \cong I_\theta$ of $\widetilde{R}_\theta$-modules.

Now, by [Sta19, Lemma 07ZA] we indeed have:

$$\pi_\theta(\mathrm{Fitt}(I)) = \mathrm{Fitt}(I \otimes_S \mathcal{O}) = \mathrm{Fitt}(I_\theta),$$

as desired. This completes the proof of (2) and hence of Theorem 3.9. □

We note the following corollary.

Corollary 3.12. *With notation as above*

$$C_{1,\widetilde{\lambda}}(\widetilde{R}) = \widetilde{\lambda}(\widetilde{R}[I])/\widetilde{\lambda}(\mathrm{Fitt}(I))$$

depends only on its quotient $\widetilde{R}/I \simeq R$ *and we define*

$$C_{1,\lambda}(R) \stackrel{\text{def}}{=} C_{1,\widetilde{\lambda}}(\widetilde{R}).$$

Proof. This follows from Theorem 3.9 which shows that

$$C_{1,\lambda_\theta}(R_\theta) = C_{1,\widetilde{\lambda}}(\widetilde{R}),$$

and the results of Appendix A which show that $C_{1,\lambda_\theta}(R_\theta)$ is well defined and independent of $\widetilde{R}_\theta$. □

Remark 3.13. The above Corollary 3.12 can also be proved directly by using the proof of Lemma A.5 instead of reducing to the statement of Lemma A.5.

For later use, we also state the following result.

Lemma 3.14. *As* R*-modules one has* $\widetilde{R}[I] \cong \omega_R$.

Proof. As R is Cohen–Macaulay and $\widetilde{R}$ is Gorenstein, we have that $\omega_R \cong \mathrm{Hom}_S(R,S)$ and $\widetilde{R} \cong \mathrm{Hom}_S(\widetilde{R},S)$ as $\widetilde{R}$-modules. Now, by [Sta19, Lemma 08YP]:

$$\widetilde{R}[I] \cong \mathrm{Hom}_{\widetilde{R}}(R,\widetilde{R}) \cong \mathrm{Hom}_{\widetilde{R}}(R,\mathrm{Hom}_S(\widetilde{R},S)) \cong \mathrm{Hom}_S(R,S) \cong \omega_R$$

as R-modules. □

3.3. Invariance of $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$

In this section, we will let $R \in C_{\mathcal{O}}$ and $S = \mathcal{O}[[y_1,\ldots,y_d]]$ be as above. We shall show that for any inclusion $\theta : S \hookrightarrow R$ satisfying (P), the André–Quillen cohomology group $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$ does not depend on the choice of θ.

In order to do this, we will need to make use of a continuous version of André–Quillen cohomology, as the classical version does not behave well for rings that are not of finite type but only topologically of finite type over the base. We will define this in terms of the analytic cotangent complex defined in [GR03, Chapter 7].

For any ring A, we will let Mod_A denote the category of A-modules, $D(\mathrm{Mod}_A)$ its derived category, and $D^-(\mathrm{Mod}_A) \subseteq D(\mathrm{Mod}_A)$ the subcategory of bounded above complexes.

For any map of rings $A \to B$, let $L_{B/A} \in D^-(\mathrm{Mod}_B)$ denote the relative cotangent complex.

Now, consider any $A \in \mathrm{CNL}_{\mathcal{O}}$ and let $\wedge : \mathrm{Mod}_A \to \mathrm{Mod}_A$ denote the $\mathfrak{m}_A$-adic completion functor. As in [GR03, Chapter 7.1], let $\wedge : D^-(\mathrm{Mod}_A) \to D^-(\mathrm{Mod}_A)$ denote its left-derived functor.

If $A \to B$ is a *continuous* map of rings in $\mathrm{CNL}_{\mathcal{O}}$, then define the analytic relative cotangent complex to be $L^{\mathrm{an}}_{B/A} = (L_{B/A})^\wedge$. For any B-module M and any $i \geq 0$ we may then define the i^{th} continuous André–Quillen cohomology group to be

$$\widehat{\mathrm{Der}}^i_A(B,M) = H^i(\mathrm{RHom}_B(L^{\mathrm{an}}_{B/A}, M)).$$

Similarly, if $A \to B$ is any ring map and M is any B-module, the i^{th} André–Quillen cohomology group is just

$$\mathrm{Der}^i_A(B,M) = H^i(\mathrm{RHom}_B(L_{B/A}, M)).$$

We will begin by recording the basic properties of continuous André–Quillen cohomology we will need in our arguments.

Proposition 3.15. *Given any $A, B, C \in \mathrm{CNL}_{\mathcal{O}}$, and continuous ring homomorphisms $A \to B \to C$ and any C-module M, there is a long exact sequence:*

$$0 \to \widehat{\mathrm{Der}}^0_B(C,M) \to \widehat{\mathrm{Der}}^0_A(C,M) \to \widehat{\mathrm{Der}}^0_A(B,M) \to \widehat{\mathrm{Der}}^1_B(C,M) \to \widehat{\mathrm{Der}}^1_A(C,M) \to \widehat{\mathrm{Der}}^1_A(B,M) \to \cdots$$

Proof. This follows from the distinguished triangle

$$C \otimes^{\mathbf{L}}_B L^{\mathrm{an}}_{B/A} \to L^{\mathrm{an}}_{C/A} \to L^{\mathrm{an}}_{C/B} \to C \otimes^{\mathbf{L}}_B L^{\mathrm{an}}_{B/A}[1]$$

from [GR03, Theorem 7.1.33]. □

Proposition 3.16. *If $A \to B$ is a continuous map of rings in* $\mathrm{CNL}_{\mathcal{O}}$ *which makes B into a finite A-module, then $L^{\mathrm{an}}_{B/A} \cong L_{B/A}$, and so $\widehat{\mathrm{Der}}^i_A(B,M) \cong \mathrm{Der}^i_A(B,M)$ for all $i \geq 0$ and all $M \in \mathrm{Mod}_B$.*

Proof. As the map $A \to B$ is finite, it is finite type (and not merely *topologically* finite type). By [Iye07, 6.11], $L_{B/A}$ is quasi-isomorphic to a bounded above complex of finite free B-modules $\mathcal{L}^\bullet$. Using $\mathcal{L}^\bullet$ to compute $(L_{B/A})^\wedge$, we get

$$L^{\mathrm{an}}_{B/A} = (L_{B/A})^\wedge \cong (\mathcal{L}^\bullet)^\wedge = \mathcal{L}^\bullet \cong L_{B/A},$$

as finitely generated B-modules are already $\mathfrak{m}_B$-adically complete. The last claim now follows from the definition of $\widehat{\mathrm{Der}}^i_A(B,M)$ and $\mathrm{Der}^i_A(B,M)$. □

Proposition 3.17. *If $A \to B$ is a continuous map of rings in* $\mathrm{CNL}_{\mathcal{O}}$*, then the module $\widehat{\Omega}_{B/A} = \varprojlim \Omega_{(B/\mathfrak{m}^n_B)/A}$ of continuous Kähler differentials defined in [BKM21, Section 7.1]*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

is the $\mathfrak{m}_B$*-adic completion of* $\Omega_{B/A}$ *and we have* $\widehat{\mathrm{Der}}^0_A(B,M) \cong \mathrm{Hom}_A(\widehat{\Omega}_{B/A},M)$ *for any* B*-module* M*.*

Proof. For the first claim, we argue as in [BKM21, Lemma 7.1] (and note that the assumption that $\mathcal{R}$ is finitely generated over A in that lemma was used only in the last step, to conclude that $\Omega_{\mathcal{R}/A}$ was finitely generated over A). Specifically, for any $n > k$ we have $\Omega_{B/A}/\mathfrak{m}_B^k\Omega_{B/A} = \Omega_{B/A} \otimes_B B/\mathfrak{m}_B^k \cong \Omega_{(B/\mathfrak{m}_B^n)/A} \otimes B/\mathfrak{m}_B^k$ and so taking inverse limits gives

$$\Omega_{B/A}/\mathfrak{m}_B^k\Omega_{B/A} \cong \varprojlim_n \left(\Omega_{(B/\mathfrak{m}_B^n)/A} \otimes_B B/\mathfrak{m}_B^k\right) \cong \varprojlim_n \left(\Omega_{(B/\mathfrak{m}_B^n)/A}\right) \otimes_B B/\mathfrak{m}_B^k = \widehat{\Omega}_{B/A} \otimes_B B/\mathfrak{m}_B^k.$$

Taking inverse limits again and using the fact that $\widehat{\Omega}_{B/A}$ is finite over B, and hence $\mathfrak{m}_B$-adically complete gives

$$\widehat{\Omega}_{B/A} \cong \varprojlim \widehat{\Omega}_{B/A} \otimes_B B/\mathfrak{m}_B^k \cong \varprojlim_k \Omega_{B/A}/\mathfrak{m}_B^k\Omega_{B/A}$$

as desired.

In particular, this shows that the module $\widehat{\Omega}_{B/A}$ is simply the module $\Omega^{\mathrm{an}}_{B/A} = (\Omega_{B/A})^\wedge$ from [GR03], and so the second claim follows from [GR03, Lemma 7.1.27(iii)] and the definition of $\widehat{\mathrm{Der}}^i_A(B,M)$. □

We will also need the following specific computations of continuous André–Quillen cohomology:

Lemma 3.18. *For any* $n \geq 0$ *and any* $\mathcal{O}[[x_1,\ldots,x_n]]$*-module* M*, we have*

$$\widehat{\mathrm{Der}}^i_{\mathcal{O}}(\mathcal{O}[[x_1,\ldots,x_n]],M) = \begin{cases} M^n & i = 0 \\ 0 & i \geq 1. \end{cases}$$

Proof. By [GR03, Proposition 7.1.29], we have $L^{\mathrm{an}}_{\mathcal{O}[[x_1,\ldots,x_n]]/\mathcal{O}} = \widehat{\Omega}_{\mathcal{O}[[x_1,\ldots,x_n]]/\mathcal{O}}[0] = \mathcal{O}[[x_1,\ldots,x_n]]^n[0]$ and so

$$\mathrm{RHom}_{\mathcal{O}[[x_1,\ldots,x_n]]}(L^{\mathrm{an}}_{\mathcal{O}[[x_1,\ldots,x_n]]/\mathcal{O}},M) = \mathrm{RHom}_{\mathcal{O}[[x_1,\ldots,x_n]]}(\mathcal{O}[[x_1,\ldots,x_n]]^n[0],M) = M^n[0]$$

so the claim follows. □

Lemma 3.19. *If* A *is a ring and* $B = A/I$ *for some ideal* $I \subseteq A$*, then for any* B*-module* M*,* $\widehat{\mathrm{Der}}^0_A(B,M) = 0$ *and* $\widehat{\mathrm{Der}}^1_A(B,M) = \mathrm{Hom}_B(I/I^2,M)$*.*

Proof. As $B = A/I$ is clearly finite over A, Proposition 3.16 gives $\widehat{\mathrm{Der}}^i_A(B,M) = \mathrm{Der}^i(B,M)$ for all $i \geq 0$ and all M. The claim now follows from [Iye07, 6.12]. □

For the remainder of this section, we always treat $E/\mathcal{O}$ as an R_θ-module (and hence as an R-module) via $\lambda_\theta : R_\theta \to \mathcal{O}$. Our main result is the following:

Theorem 3.20. *We have* $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$*.*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

This implies that $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$ depends only on R and the R-module structure on $E/\mathcal{O}$, which is induced by $\lambda : R \to \mathcal{O}$, and not on θ. This will thus complete the proof of Theorem 3.25.

We first observe the following:

Lemma 3.21. *For any $i \geq 0$ and any R_θ-module M, we have*

$$\widehat{\mathrm{Der}}^i_S(R, M) \cong \mathrm{Der}^i_S(R, M) \cong \mathrm{Der}^i_{\mathcal{O}}(R_\theta, M) \cong \widehat{\mathrm{Der}}^i_{\mathcal{O}}(R_\theta, M).$$

Proof. The first and last isomorphisms follow from Proposition 3.16, as R is finite over S and R_θ is finite over $\mathcal{O}$.

For the second isomorphism, first note that as R is a finite free S-module, it is a projective resolution for itself in $D(S)$, and so we have $R \otimes^{\mathbf{L}}_S \mathcal{O} = R \otimes_S \mathcal{O} \cong R_\theta$. By [Sta19, Lemma 08QQ], this implies that $L_{R/S} \otimes^{\mathbf{L}}_R R_\theta \cong L_{R_\theta/\mathcal{O}}$. But now [Sta19, Lemma 0E1W] gives that

$$\mathrm{RHom}_R(L_{R/S}, M) = \mathrm{RHom}_{R_\theta}(L_{R/S} \otimes^{\mathbf{L}}_R R_\theta, M) \cong \mathrm{RHom}_{R_\theta}(L_{R_\theta/\mathcal{O}}, M)$$

so the claim follows by definition. □

So to prove Theorem 3.20, it will suffice to prove the following:

Proposition 3.22. $\widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$.

Proof. Applying Proposition 3.15 to the ring maps $\mathcal{O} \to S \to R$ gives an exact sequence:

$$\begin{aligned} 0 &\to \widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(S, E/\mathcal{O}) \\ &\to \widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(S, E/\mathcal{O}). \end{aligned}$$

By Lemma 3.18, $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(S, E/\mathcal{O}) = (E/\mathcal{O})^d$ and $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(S, E/\mathcal{O}) = 0$.

But now by the assumption that $\lambda : R \to \mathcal{O}$ represents a smooth point of $\operatorname{Spec} R[1/\varpi]$ we get that $\widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}$ has rank d as an $\mathcal{O}$-module (as in [BKM21, Theorem 7.16]), and so

$$\widehat{\mathrm{Der}}^0_{\mathcal{O}}(R, E/\mathcal{O}) = \mathrm{Hom}_R(\widehat{\Omega}_{R/\mathcal{O}}, E/\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}, E/\mathcal{O}) = (E/\mathcal{O})^d \oplus G,$$

for some finite group G. Also as $\Phi_{\lambda_\theta}(R_\theta) = \widehat{\Omega}_{R^\theta/\mathcal{O}} \otimes_{\lambda_\theta} \mathcal{O}$ is finite (as θ satisfies (P)),

$$\widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R_\theta, E/\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R_\theta/\mathcal{O}}, E/\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\Phi_{\lambda_\theta}(R_\theta), E/\mathcal{O})$$

is finite as well. Now, the exact sequence simplifies to

$$0 \to \widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \to (E/\mathcal{O})^d \oplus G \to (E/\mathcal{O})^d \to \widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \to 0.$$

But comparing coranks in the sequence $0 \to \widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \to (E/\mathcal{O})^d \oplus G \to (E/\mathcal{O})^d$ implies that $(E/\mathcal{O})^d \oplus G \to (E/\mathcal{O})^d$ has finite cokernel and hence must be surjective, as $E/\mathcal{O}$ does not have any nontrivial finite quotients. This implies that the map $\widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$ is indeed an isomorphism. This completes the proof of Theorem 3.20. □

We note that in Theorem 3.20 and Corollary 3.12, we have proved that

$$\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O}). \tag{3.1}$$

$$C_{1,\lambda}(R) = C_{1,\lambda_\theta}(R_\theta). \tag{3.2}$$

In order to actually compute $\delta_{\lambda_\theta}(R_\theta)$, we will need a method for computing $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$. For this, take any triple $(\widetilde{R},I,\varphi)$ satisfying (CI). Then we now have the following generalization of Equation (A.3):

Theorem 3.23. *There is a four-term exact sequence:*

$$0 \to \mathrm{Hom}_R(\widehat{\Omega}_{R/\mathcal{O}},E/\mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}},E/\mathcal{O}) \to \mathrm{Hom}_R(I/I^2,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O}) \to 0.$$

Proof. Applying Proposition 3.15 to the ring maps $\mathcal{O} \to \widetilde{R} \to R$ gives an exact sequence:

$$\begin{aligned} 0 &\to \widehat{\mathrm{Der}}^0_{\widetilde{R}}(R,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \\ &\to \widehat{\mathrm{Der}}^1_{\widetilde{R}}(R,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}), \end{aligned}$$

and Lemma 3.19 implies that $\widehat{\mathrm{Der}}^0_{\widetilde{R}}(R,E/\mathcal{O}) = 0$ and $\widehat{\mathrm{Der}}^1_{\widetilde{R}}(R,E/\mathcal{O}) = \mathrm{Hom}_R(I/I^2,E/\mathcal{O})$, so it's enough to prove that $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) = 0$ (since by Proposition 3.17, $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(R,E/\mathcal{O}) = \mathrm{Hom}_R(\widehat{\Omega}_R,E/\mathcal{O})$ and $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) = \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}},E/\mathcal{O})$).

Since $\widetilde{R}$ is a complete intersection, we can write $\widetilde{R} = P/J$, where $P = \mathcal{O}[[x_1,\dots,x_{d+n}]]$ and $J = (f_1,\dots,f_n)$ is generated by a regular sequence. Applying Proposition 3.15 to the ring maps $\mathcal{O} \to P \to \widetilde{R}$ gives an exact sequence:

$$\begin{aligned} 0 &\to \widehat{\mathrm{Der}}^0_P(\widetilde{R},E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(P,E/\mathcal{O}) \\ &\to \widehat{\mathrm{Der}}^1_P(\widetilde{R},E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(P,E/\mathcal{O}). \end{aligned}$$

Now, Lemma 3.18 gives the identification $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(P,E/\mathcal{O}) = (E/\mathcal{O})^{d+n}$ and $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(P,E/\mathcal{O}) = 0$ and Lemma 3.19 gives $\widehat{\mathrm{Der}}^0_P(\widetilde{R},E/\mathcal{O}) = 0$ and $\widehat{\mathrm{Der}}^1_P(\widetilde{R},E/\mathcal{O}) = \mathrm{Hom}_{\widetilde{R}}(J/J^2,E/\mathcal{O})$. Moreover, as J is generated by a regular sequence of length n, it follows that $J/J^2 \cong (\widetilde{R})^n$ as $\widetilde{R}$-modules, and so $\widehat{\mathrm{Der}}^1_P(\widetilde{R},E/\mathcal{O}) = \mathrm{Hom}_{\widetilde{R}}(J/J^2,E/\mathcal{O}) \cong (E/\mathcal{O})^n$. Thus, the above exact sequence simplifies to

$$0 \to \widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \to (E/\mathcal{O})^{n+d} \to (E/\mathcal{O})^n \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \to 0.$$

But now, just as in the proof of Proposition 3.22 above, the fact that $\mathrm{Spec}\,\widetilde{R}[1/\varpi]$ is smooth of dimension d at $\widetilde{\lambda}$ implies that $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) \cong (E/\mathcal{O})^d \oplus H$ for some finite group H, and so comparing ranks gives that $(E/\mathcal{O})^{n+d} \to (E/\mathcal{O})^n$ has finite cokernel, and hence is surjective. Thus, $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R},E/\mathcal{O}) = 0$, and so the claim follows. □

3.4. Wiles defect for augmented rings $(R,\lambda) \in C_{\mathcal{O}}$

We make the following definitions and in particular define the *Wiles defect* for tuples $(R,\lambda) \in C_{\mathcal{O}}$. Recall that by definition all such R are Cohen–Macaulay and flat over $\mathcal{O}$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Definition 3.24. Let R be a complete, Noetherian local $\mathcal{O}$-algebra which is Cohen–Macaulay and flat over $\mathcal{O}$ of relative dimension d and with an augmentation $\lambda : R \to \mathcal{O}$ such that $\operatorname{Spec} R[1/\varpi]$ is formally smooth at the point corresponding to λ.

- Define
$$D_{1,\lambda}(R) = \frac{\log|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})|}{\log|\mathcal{O}/p|}$$
(see Theorem 3.20).
- Define
$$c_{1,\lambda}(R) = \frac{\log|C_{1,\lambda}(R)|}{\log|\mathcal{O}/p|} = \frac{\log\left|\widetilde{\lambda}(\widetilde{R}[I])/\widetilde{\lambda}(\mathrm{Fitt}(I))\right|}{\log|\mathcal{O}/p|},$$
for any triple $(\widetilde{R},I,\varphi)$ satisfying (CI).
- The *Wiles defect* $\delta_\lambda(R)$ of R at λ is defined to be
$$\delta_\lambda(R) = D_{1,\lambda}(R) - c_{1,\lambda}(R).$$

Here is the main theorem of this section which uses all the work we have done here.

Theorem 3.25. *Let R and $\lambda : R \twoheadrightarrow \mathcal{O}$ be as above, and let $\theta : S \hookrightarrow R$ be a map satisfying (P). Then the invariants $C_{1,\lambda_\theta}(R_\theta)$, $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$ and $\delta_{\lambda_\theta}(R_\theta)$ are independent of the choice of θ.*

Proof. The proofs of the independence statements for $C_{1,\lambda_\theta}(R_\theta)$ and $\mathrm{Der}^1_{\mathcal{O}}(R_\theta,E/\mathcal{O})$ follow from Theorems 3.9 and 3.20, respectively. The assertion for the Wiles defect $\delta_{\lambda_\theta}(R_\theta)$ is then immediate from Theorem 2.2. □

Corollary 3.26. *The numbers*
$$D_{1,\lambda}(R), c_{1,\lambda}(R), \delta_\lambda(R)$$
are all nonnegative rational numbers depending only on R and λ. Moreover, if E'/E is any finite extension and $\mathcal{O}'$ is the ring of integers of E', then we have
$$D_{1,\lambda\otimes\mathcal{O}'}(R\otimes_{\mathcal{O}}\mathcal{O}') = D_{1,\lambda}(R), \quad c_{1,\lambda\otimes\mathcal{O}'}(R\otimes_{\mathcal{O}}\mathcal{O}') = c_{1,\lambda}(R), \quad \text{and} \quad \delta_{\lambda\otimes\mathcal{O}'}(R\otimes_{\mathcal{O}}\mathcal{O}') = \delta_\lambda(R).$$
That is, $D_{1,\lambda}(R), c_{1,\lambda}(R), \delta_\lambda(R)$ are all unaffected by changing the coefficient ring.

Proof. This is a consequence of Theorem 3.25, Theorem 3.9, Corollary 3.12 and Theorem 3.20, combined with Remark A.7 which confirms the finiteness of length of the terms involved in the one-dimensional case.

The final claim about changing the coefficient ring is easy to verify in the case when R is finite free over $\mathcal{O}$ (this fact was already noted in [BKM21, Section 3] for $\delta_\lambda(R)$), and the general claim follows from this. □

We note the consistency of this definition with the definition of Wiles defect for tuples $(R,\lambda) \in C_{\mathcal{O}}$ when R is of dimension one.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Proposition 3.27. *In the case when $(R,\lambda) \in C_{\mathcal{O}}$ and R is of dimension one, then*

$$\delta_\lambda(R) = D_{1,\lambda}(R) - c_{1,\lambda}(R) = \frac{\log|\Phi_\lambda(R)| - \log|\Psi_\lambda(R)|}{\log|\mathcal{O}/p|}.$$

Proof. This follows from Proposition 3.16 and Proposition A.6 of Appendix A (cf. Theorem 2.2). $\square$

Proposition 3.28. *For $(R,\lambda) \in C_{\mathcal{O}}$, $\delta_\lambda(R) = 0$ if and only if R is a complete intersection. In particular, $\delta_\lambda(\mathcal{O}[[x_1,\dots,x_n]]) = 0$ for any $n \geq 1$ and any $\lambda : \mathcal{O}[[x_1,\dots,x_n]] \twoheadrightarrow \mathcal{O}$.*

Proof. If R is a complete intersection, then $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O}) = 0$ by the argument given in the proof of Theorem 3.23. Further $C_{1,\lambda}(R) = 0$ (as we can take the CI cover $\widetilde{R} = R$). This gives that $\delta_\lambda(R) = 0$.

Conversely, assume $\delta_\lambda(R) = 0$. Then by our results we have a quotient $(R_\theta,\lambda_\theta) \in C_{\mathcal{O}}$ of $(R,\lambda) \in C_{\mathcal{O}}$ by a regular sequence $(y_1,\dots,y_d)$, namely $R_\theta = R/(y_1,\dots,y_d)$ and $\lambda_\theta : R \to R_\theta \to \mathcal{O}$ (the last map being λ) with R_θ of dimension one. Further, $\delta_{\lambda_\theta}(R_\theta) = \delta_\lambda(R) = 0$. Thus, by Lemma 2.1, R_θ is a complete intersection, which implies that R is a complete intersection. $\square$

Remark 3.29. For $(R,\lambda) \in C_{\mathcal{O}}$ and R of dimension 1, by Lemma 2.1 note that the vanishing of $\mathrm{Der}^1_{\mathcal{O}}(R,E/\mathcal{O})$ implies that R is a complete intersection because of the inequality $|\Phi_\lambda(R)| \geq |\Psi_\lambda(R)|$ which follows from the usual Fitting ideals argument (cf. [Len95]). From this, we again deduce, by invariance of $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ quotienting by regular sequences, that in general for $(R,\lambda) \in C_{\mathcal{O}}$, the vanishing of $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R,E/\mathcal{O})$ implies that R is a complete intersection.

Example 3.30. At the suggestion of the referee, we now compute the Wiles defect for a simple example of a pair $(R,\lambda) \in \mathcal{C}_{\mathbf{Z}_p}$ with R a non-CI ring. Much more elaborate computations will be done in Section 5. We consider

$$\lambda : R = \mathbf{Z}_p[[X,Y]]/(X(X-p),Y(Y-p),XY) \longrightarrow \mathbf{Z}_p, f(X,Y) \longmapsto f(0,0).$$

As a $\mathbf{Z}_p$-module R is free of rank 3, and possible bases are $\{1,X,Y\}$ and $\{1,X-p,Y-p\}$. In particular, R is of Krull dimension 1 and we have $R = R^{\mathrm{tf}}$. The most direct way to compute the Wiles defect $\delta_\lambda(R)$ is via formula (2.1): We have $\ker(\lambda) = (X,Y) = \mathbf{Z}_pX \oplus \mathbf{Z}_pY$, $R[\ker(\lambda)] = \mathbf{Z}_p(X+Y-p)$, and hence

$$\Psi_\lambda(R) = \mathbf{Z}_p/(\lambda(R^{\mathrm{tf}}[\ker\lambda])) = \mathbf{Z}_p/p\mathbf{Z}_p.$$

Moreover $\Omega_{R/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p = (\mathbf{Z}_p\mathrm{d}X \oplus \mathbf{Z}_p\mathrm{d}Y)/\langle \frac{\partial f_i}{\partial X}(0,0)\mathrm{d}X + \frac{\partial f_i}{\partial Y}(0,0)\mathrm{d}Y : i = 1,2,3\rangle_{\mathbf{Z}_p}$ for $f_1 = X(X-p)$, $f_2 = Y(Y-p)$, $f_3 = XY$, and computing the Jacobian of the f_i relative to X and Y at $(0,0)$ shows that

$$\Phi_\lambda(R) = \Omega_{R/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p = \mathbf{Z}_p/p\mathbf{Z}_p\mathrm{d}X \oplus \mathbf{Z}_p/p\mathbf{Z}_p\mathrm{d}Y.$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

This gives

$$\delta_\lambda(R) = \frac{\log|\Phi_\lambda(R)| - \log|\Psi_\lambda(R)|}{\log|\mathcal{O}/p|} = \frac{2-1}{1} = 1.$$

Alternatively, one may compute $\delta_\lambda(R)$ via Definition 3.24: A possible choice of CI-cover is the quotient map

$$\varphi : \widetilde{R} = \mathbf{Z}_p[[X,Y]]/(X(X-p),Y(Y-p)) \to R = \mathbf{Z}_p[[X,Y]]/(X(X-p),Y(Y-p),XY)$$

with kernel $I = \mathbf{Z}_pXY$. Then $\widetilde{R}[I] = \mathbf{Z}_p(X-p) \oplus \mathbf{Z}_p(Y-p) \oplus \mathbf{Z}_p(X-p)(Y-p)$. To obtain the fitting ideal of I, we consider the right exact sequence

$$\widetilde{R}^2 \xrightarrow{(f,g)\mapsto f(X-p)+g(Y-p)} \widetilde{R} \xrightarrow{h\mapsto hXY} I \longrightarrow 0.$$

The fitting ideal is the ideal generated by the 1×1-minors of the matrix describing the map on the left, that is, $\mathrm{Fitt}(I) = (X-p,Y-p)$. One deduces that

$$c_{1,\lambda}(R) = \frac{\log\left|\widetilde{\lambda}(\widetilde{R}[I])/\widetilde{\lambda}(\mathrm{Fitt}(I))\right|}{\log|\mathbf{Z}_p/p|} = \frac{\log(1)}{\log(p)} = 0.$$

To compute $D_{1,\lambda}(R)$, we rely on the exact sequence from Theorem 3.23. An expression for $\Omega_{\widetilde{R}/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p$ is obtained in the same way as above for $\Omega_{R/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p$, and in fact one finds an isomorphism $\Omega_{\widetilde{R}/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p \to \Omega_{R/\mathbf{Z}_p} \otimes_\lambda \mathbf{Z}_p$. Theorem 3.23 now gives the isomorphism $\mathrm{Hom}_R(I/I^2, \mathbf{Q}_p/\mathbf{Z}_p) \xrightarrow{\cong} \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$. We compute

$$I/I^2 \otimes^\lambda_{\widetilde{R}} \mathbf{Z}_p = (XY)/(XY)\cdot(X,Y) = (XY)/(X^2Y, XY^2) = (XY)/(pXY) \cong \mathbf{Z}_p/p$$

so that $\mathrm{Hom}_R(I/I^2, E/\mathcal{O}) \cong \mathbf{Z}_p/p$. In turn this gives

$$D_{1,\lambda}(R) = \frac{\log|\widehat{\mathrm{Der}}^1_{\mathbf{Z}_p}(R, \mathbf{Q}_p/\mathbf{Z}_p)|}{\log|\mathbf{Z}_p/p|} = \frac{\log|\mathrm{Hom}_R(I/I^2, E/\mathcal{O})|}{1} = \frac{1}{1} = 1,$$

and we conclude (again)

$$\delta_\lambda(R) = D_{1,\lambda}(R) - c_{1,\lambda}(R) = 1 - 0 = 1.$$

3.5. Properties of the Wiles defect

Theorem 3.25 can be restated as:

Theorem 3.31. *If $(y_1,\ldots,y_d,\varpi)$ is a regular sequence for R with $y_1,\ldots,y_d \in \ker\lambda$, where we will also use λ to denote the induced map $R/(y_1,\ldots,y_d) \twoheadrightarrow \mathcal{O}$, then $\delta_\lambda(R) = \delta_\lambda(R/(y_1,\ldots,y_d))$. In particular, $\delta_\lambda(R/(y_1,\ldots,y_d))$ is independent of the choice of regular sequence.*

We now deduce some additivity properties of $\delta_\lambda(R)$ that we use later.

Proposition 3.32. *Let R_1 and R_2 be complete, Noetherian, Cohen–Macaulay, reduced $\mathcal{O}$-algebras, which are flat over $\mathcal{O}$ of relative dimensions d_1 and d_2. Pick augmentations*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$\lambda_i : R_i \to \mathcal{O}$ such that $\operatorname{Spec} R_i[1/\varpi]$ *is formally smooth at the point corresponding to λ_i. Let $R = R_1 \widehat{\otimes}_{\mathcal{O}} R_2$ and $\lambda = \lambda_1 \widehat{\otimes} \lambda_2 : R \to \mathcal{O}$.*

Then

1. $D_{1,\lambda}(R) = D_{1,\lambda_1}(R_1) + D_{1,\lambda_2}(R_2)$
2. $c_{1,\lambda}(R) = c_{1,\lambda_1}(R_1) + c_{1,\lambda_2}(R_2)$
3. $\delta_\lambda(R) = \delta_{\lambda_1}(R_1) + \delta_{\lambda_2}(R_2)$.

Proof. By definition, (3) will follow from (1) and (2).

For (1), we will first reduce to dimension 1. Let $S_1 = \mathcal{O}[[x_1,\ldots,x_{d_1}]]$ and $S_2 = \mathcal{O}[[y_1,\ldots,y_{d_2}]]$. By Proposition 3.2, we may find maps $\theta_1 : S_1 \hookrightarrow R_1$ and $\theta_2 : S_2 \hookrightarrow R_2$ satisfying (P). Then the map $\theta = \theta_1 \widehat{\otimes}_{\mathcal{O}} \theta_2 : S_1 \widehat{\otimes}_{\mathcal{O}} S_2 \hookrightarrow R$ satisfies (P) as well. So consider the rings

$$R_{1,\theta_1} = R_1 \otimes_{S_1} \mathcal{O}, \quad R_{2,\theta_1} = R_1 \otimes_{S_1} \mathcal{O}, \quad \text{and} \quad R_\theta = R \otimes_{S_1 \widehat{\otimes}_{\mathcal{O}} S_2} \mathcal{O} = R_{1,\theta_1} \otimes_{\mathcal{O}} R_{2,\theta_2}$$

and note that these are all finite free over $\mathcal{O}$.

By Theorem 3.20, we now have that

$$\begin{aligned} \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_1, E/\mathcal{O}) &= \mathrm{Der}^1_{\mathcal{O}}(R_{1,\theta_1}, E/\mathcal{O}), \\ \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_2, E/\mathcal{O}) &= \mathrm{Der}^1_{\mathcal{O}}(R_{2,\theta_2}, E/\mathcal{O}), \\ \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) &= \mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}). \end{aligned}$$

But now by [Sta19, Lemma 09DA], as R_1 and R_2 are both free over $\mathcal{O}$, and hence Tor-independent, we have

$$L_{R_\theta/\mathcal{O}} \cong L_{R_{1,\theta_1} \otimes_{\mathcal{O}} R_{2,\theta_2}/\mathcal{O}} \cong L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{1,\theta_1}} R_\theta \oplus L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{2,\theta_1}} R_\theta.$$

Thus,

$$\begin{aligned} &\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \\ &\quad \cong \mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) = H^1(\mathrm{RHom}_{R_\theta}(L_{R/\mathcal{O}}, E/\mathcal{O})) \\ &\quad \cong H^1(\mathrm{RHom}_{R_\theta}(L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{1,\theta_1}} R_\theta \oplus L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{2,\theta_1}} R_\theta, E/\mathcal{O})) \\ &\quad \cong H^1(\mathrm{RHom}_{R_\theta}(L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{1,\theta_1}} R_\theta, E/\mathcal{O})) \oplus H^1(\mathrm{RHom}_{R_\theta}(L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathbf{L}}_{R_{2,\theta_1}} R_\theta, E/\mathcal{O})) \\ &\quad \cong H^1(\mathrm{RHom}_{R_{1,\theta_1}}(L_{R_{1,\theta_1}/\mathcal{O}}, E/\mathcal{O})) \oplus H^1(\mathrm{RHom}_{R_{2,\theta_2}}(L_{R_{2,\theta_2}/\mathcal{O}}, E/\mathcal{O})) \\ &\quad = \mathrm{Der}^1_{\mathcal{O}}(R_{1,\theta_1}, E/\mathcal{O}) \oplus \mathrm{Der}^1_{\mathcal{O}}(R_{2,\theta_2}, E/\mathcal{O}) = \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_1, E/\mathcal{O}) \oplus \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_2, E/\mathcal{O}) \end{aligned}$$

and so (1) follows.

It remains to prove (2). Consider triples $(\widetilde{R}_1, I_1, \varphi_1)$ and $(\widetilde{R}_2, I_2, \varphi_2)$ satisfying (CI) (with (R_1, λ_1) and (R_2, λ_2), respectively, in place of (R, λ)).

Define $\widetilde{R} = \widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2$, and note that $I_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2$ and $\widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} I_2$ are both ideals of $\widetilde{R}$. Let $\varphi = \varphi_1 \otimes \varphi_2 : \widetilde{R} = \widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2 \twoheadrightarrow R_1 \widehat{\otimes}_{\mathcal{O}} R_2 = R$, and note that $\ker \varphi = \left(I_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2\right) + \left(\widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} I_2\right)$. Denoting this ideal $I \subseteq \widetilde{R}$, the triple $(\widetilde{R}, I, \varphi)$ satisfies (CI). So by the definition of $c_{1,\lambda}$,

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$$c_{1,\lambda_1}(R_1)\log|\mathcal{O}/p| = \log\left|\lambda_1(\widetilde{R}_1[I_1])/\lambda_1(\mathrm{Fitt}(I_1))\right|$$
$$c_{1,\lambda_2}(R_1)\log|\mathcal{O}/p| = \log\left|\lambda_2(\widetilde{R}_2[I_2])/\lambda_2(\mathrm{Fitt}(I_2))\right|$$
$$c_{1,\lambda}(R)\log|\mathcal{O}/p| = \log\left|\lambda(\widetilde{R}[I])/\lambda(\mathrm{Fitt}(I))\right|.$$

The desired equality will now follow from Lemma 3.33. □

Lemma 3.33. *If $\widetilde{R}_1,\widetilde{R}_2,\widetilde{R}$, I_1,I_2,I, λ_1,λ_2 and λ are as in the proof of Proposition 3.32, then we have*

$$\lambda\left(\widetilde{R}[I]\right) = \lambda_1\left(\widetilde{R}_1[I_1]\right)\lambda_2\left(\widetilde{R}_2[I_2]\right), \qquad \textit{and} \qquad \lambda(\mathrm{Fitt}(I)) = \lambda_1(\mathrm{Fitt}(I_1))\,\lambda_2(\mathrm{Fitt}(I_2))$$

as ideals of $\mathcal{O}$.

Proof. For the first claim, standard properties of annihilators imply that

$$\begin{aligned}\widetilde{R}[I] &= \widetilde{R}\left[\left(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)+\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2\right)\right] = \widetilde{R}\left[\left(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)\right]\cap\widetilde{R}\left[\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2\right)\right]\\ &= \left(\widetilde{R}_1[I_1]\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)\cap\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2[I_2]\right) = \widetilde{R}_1[I_1]\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2[I_2]\end{aligned}$$

(where we've used that fact that $\left(A\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)\cap\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}B\right) = \left(A\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}B\right) = A\widehat{\otimes}_{\mathcal{O}}B$ for any ideals $A\subseteq\widetilde{R}_1$ and $B\subseteq\widetilde{R}_2$). Thus,

$$\lambda\left(\widetilde{R}[I]\right) = (\lambda_1\otimes\lambda_2)\left(\widetilde{R}_1[I_1]\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2[I_2]\right) = \lambda_1\left(\widetilde{R}_1[I_1]\right)\lambda_2\left(\widetilde{R}_2[I_2]\right).$$

For the statement about fitting ideals, fix presentations

$$0\to K_1\to\widetilde{R}_1^m\xrightarrow{A}I_1\to 0$$
$$0\to K_2\to\widetilde{R}_2^n\xrightarrow{B}I_2\to 0,$$

where K_i is a finitely generated $\widetilde{R}_i$-module. Then A and B induce surjective maps $A\otimes\mathrm{Id}:\widetilde{R}^m=\widetilde{R}_1^m\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\to I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2$ and $\mathrm{Id}\otimes B:\widetilde{R}^n=\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2^n\to\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2$, and so we may combine them to produce a surjective map

$$C=(A\otimes\mathrm{Id})-(\mathrm{Id}\otimes B):\widetilde{R}^{m+n}=\widetilde{R}^m\oplus\widetilde{R}^n\to\left(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)+\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2\right)=I.$$

Write $K\subseteq\widetilde{R}^{m+n}$ for the kernel of C.

By definition: $\mathrm{Fitt}(I_1)$ is the ideal of $\widetilde{R}_1$ generated by all elements of the form $\det(u_1,\ldots,u_m)\in\widetilde{R}_1$ for $u_1,\ldots,u_m\in K_1\subseteq\widetilde{R}_1^m$; $\mathrm{Fitt}(I_2)$ is the ideal of $\widetilde{R}_2$ generated by all elements of the form $\det(v_1,\ldots,v_n)\in\widetilde{R}_1$ for $v_1,\ldots,v_n\in K_2\subseteq\widetilde{R}_2^n$; and lastly $\mathrm{Fitt}(I)$ is the ideal of $\widetilde{R}$ generated by all elements of the form $\det(w_1,\ldots,w_{m+n})\in\widetilde{R}$ for $w_1,\ldots,w_{m+n}\in K\subseteq\widetilde{R}^{m+n}$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Now, given any $u_1,\dots,u_m \in K_1$ and $v_1,\dots,v_n \in K_2$ it's easy to see that $\begin{pmatrix} u_i\otimes 1 \\ 0\end{pmatrix}$, $\begin{pmatrix} 0 \\ 1\otimes v_j\end{pmatrix} \in K$ for all i and j, and so $\mathrm{Fitt}(I)$ contains the element

$$\det\begin{pmatrix} u_1\otimes 1 & \cdots & u_m\otimes 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1\otimes v_1 & \otimes & 1\otimes v_n\end{pmatrix} = \det(u_1,\dots,u_m)\otimes\det(v_1,\dots,v_m).$$

It follows that $\mathrm{Fitt}(I_1)\widehat{\otimes}_{\mathcal{O}}\mathrm{Fitt}(I_2)\subseteq \mathrm{Fitt}(I)$ and so $\lambda_1(\mathrm{Fitt}(I_1))\,\lambda_2(\mathrm{Fitt}(I_2))\subseteq\lambda(\mathrm{Fitt}(I))$.

For the reverse inclusion, we will use the following simple lemma:

Lemma 3.34. *For any $w=\begin{pmatrix} w_1 \\ w_2\end{pmatrix}\in K$, for $w_1\in\widetilde{R}^m$ and $w_2\in\widetilde{R}^n$, there exist $u\in K_1$ and $v\in K_2$ for which $\lambda(w_1)=\lambda_1(u)$ and $\lambda(w_2)=\lambda_2(v)$.*

Proof. As $w\in K$, we have $(A\otimes\mathrm{Id})(w_1)-(\mathrm{Id}\otimes B)(w_2)=C(w)=0$ so let $r=(A\otimes\mathrm{Id})(w_1)=(\mathrm{Id}\otimes B)(w_2)\in\widetilde{R}$. By the definitions of A and B, we have $r=(A\otimes\mathrm{Id})(w_1)\in I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2$ and $r=(\mathrm{Id}\otimes B)(w_2)\in\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2$ and so

$$r\in\left(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2\right)\cap\left(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2\right)=I_1\widehat{\otimes}_{\mathcal{O}}I_2$$

Now, as $\lambda_1(I_1)=\lambda_2(I_2)=0$ by assumption, we get that $(\lambda_1\otimes\mathrm{Id})(r)=(\mathrm{Id}\otimes\lambda_2)(r)=0$. Now, let $u=(\mathrm{Id}\otimes\lambda_2)(w_1)\in\widetilde{R}_1^m$ and $v=(\lambda_1\otimes\mathrm{Id})(w_2)\in\widetilde{R}_2^n$ so that

$$\begin{aligned}\lambda_1(u)&=\lambda_1((\mathrm{Id}\otimes\lambda_2)(w_1))=(\lambda_1\otimes\lambda_2)(w_1)=\lambda(w_1)\\ \lambda_2(v)&=\lambda_2((\lambda_1\otimes\mathrm{Id})(w_2))=(\lambda_1\otimes\lambda_2)(w_2)=\lambda(w_2)\end{aligned}$$

and

$$\begin{aligned}A(u)&=(A\otimes\mathrm{Id})(\mathrm{Id}\otimes\lambda_2)(w_1)=(\mathrm{Id}\otimes\lambda_2)(A\otimes\mathrm{Id})(w_1)=(\mathrm{Id}\otimes\lambda_2)(r)=0\\ B(v)&=(\mathrm{Id}\otimes B)(\lambda_1\otimes\mathrm{Id})(w_2)=(\lambda_1\otimes\mathrm{Id})(\mathrm{Id}\otimes B)(w_2)=(\lambda_1\otimes\mathrm{Id})(r)=0.\end{aligned}$$

So now $w_1\in\ker A=K_1$ and $w_2\in\ker B=K_2$, as desired. □

So now take any $w_1,\dots,w_{m+n}\in K$. The lemma allows us to write $\lambda(w_i)=\begin{pmatrix}\lambda_1(u_i)\\ \lambda_2(v_1)\end{pmatrix}$ for $u_i\in K_1$ and $v_i\in K_2$, which gives

$$\lambda(\det(w_1,\dots,w_{m+n}))=\det\begin{pmatrix}\lambda_1(u_1) & \cdots & \lambda_1(u_{m+n})\\ \lambda_2(v_1) & \cdots & \lambda_2(v_{m+n})\end{pmatrix}.$$

But now by standard properties of determinants, the determinant of this $(m+n)\times(m+n)$ matrix may be written as an alternating sum in the form

$$\sum_{X,Y}(\pm1)\det\left((\lambda_1(u_i))_{i\in X}\right)\det\left((\lambda_2(v_j))_{j\in Y}\right)=\sum_{X,Y}(\pm1)\lambda_1\left(\det\left((u_i)_{i\in X}\right)\right)\lambda_2\left(\det\left((v_j)_{j\in Y}\right)\right)$$

(where the sum is taken over partitions $X\sqcup Y=\{1,\dots,m+n\}$ with $|X|=m$ and $|Y|=n$). As this sum is in $\lambda_1(\mathrm{Fitt}(I_1))\,\lambda_2(\mathrm{Fitt}(I_2))$, it follows that $\lambda(\mathrm{Fitt}(I))\subseteq\lambda_1(\mathrm{Fitt}(I_1))\,\lambda_2(\mathrm{Fitt}(I_2))$, giving the desired equality $\lambda(\mathrm{Fitt}(I))=\lambda_1(\mathrm{Fitt}(I_1))\,\lambda_2(\mathrm{Fitt}(I_2))$, and completing the proof. □

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

4. Galois deformation theory

This section recalls basic results on Galois deformation theory and fixes some notation for the remainder of this work. Our main references are [Tho16, §5] and [BKM21, §4].

Recall the notation from the end of Section 1. We fix a continuous, absolutely irreducible residual representation

$$\overline{\rho}: G_F \to \mathrm{GL}_2(k)$$

with $\det \overline{\rho} = \varepsilon_p$, for simplicity. We will assume that k contains the eigenvalues of all elements in the image of $\overline{\rho}$. We also fix a finite set Σ of finite places v of F disjoint from Σ_p that contains all places $v \notin \Sigma_p$ at which $\overline{\rho}$ is ramified and possibly further places of F.

Local deformation rings

Let $v \in \Sigma$. We write $\mathcal{D}_v^{\square} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Sets}$ for the functor that associates to $R \in \mathrm{CNL}_{\mathcal{O}}$ the set of all continuous homomorphisms $r : G_{F_v} \to \mathrm{GL}_2(R)$ such that $r \pmod{\mathfrak{m}_R} = \overline{\rho}|_{G_{F_v}}$ and $\det r = \varepsilon_p$. The functor $\mathcal{D}_v^{\square}$ is representable by an object $R_v^{\square} \in \mathrm{CNL}_{\mathcal{O}}$, a *framed deformation ring*. We will write $\rho_v^{\square} : G_{F_v} \to \mathrm{GL}_2(R_v^{\square})$ for the *universal framed deformation*.

A *local deformation problem* for $\overline{\rho}|_{G_{F_v}}$ is a subfunctor $\mathcal{D}_v \subset \mathcal{D}_v^{\square}$ satisfying the following conditions:

1. The functor $\mathcal{D}_v$ is represented by a quotient R_v of $R_v^{\square}$.
2. For all $R \in \mathrm{CNL}_{\mathcal{O}}$, $g \in \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$ and $r \in \mathcal{D}_v(R)$, we have $grg^{-1} \in \mathcal{D}_v(R)$.

The ring R_v will be called the *local deformation ring* representing $\mathcal{D}_v$.

If a quotient R_v of $R_v^{\square}$ corresponding to a local deformation problem $\mathcal{D}_v$ has been fixed, we will write $\rho_v : G_{F_v} \to \mathrm{GL}_2(R_v)$ for *the universal framed deformation of type* $\mathcal{D}_v$. A sufficient condition for a quotient R_v of $R_v^{\square}$ to be a local deformation ring is the following; see [Tho16, Lemma 5.12].

Lemma 4.1. *Let $\pi: R_v^{\square} \to R_v$ be a surjective morphism in $\mathrm{CNL}_{\mathcal{O}}$ with specialization $r : G_{F_v} \to \mathrm{GL}_2(R_v)$ induced from the universal framed deformation, and assume the following conditions:*

1. *The ring R_v is reduced, and not isomorphic to k.*
2. *For all $g \in \ker(\mathrm{GL}_2(R_v) \to \mathrm{GL}_2(k))$, the homomorphism $R_v^{\square} \to R_v$ associated to the representation grg^{-1} by universality factors through π.*

Then the subfunctor of $\mathcal{D}_v^{\square}$ defined by R_v is a local deformation problem.

Below, we consider quotients of $R_v^{\square}$ which are defined as in [Kis09] as reduced, $\mathcal{O}$-flat quotients R_v of $R_v^{\square}$, that are the Zariski closure of a set of $\overline{\mathbf{Q}}_p$-valued points of $R_v^{\square}$; in each case the set forms the closed points of a Zariski closed subset of the generic fiber Jacobson ring $R_v^{\square}[1/p]$ and thus the generic fiber $R_v[1/p]$ has this set as its $\overline{\mathbf{Q}}_p$-points; in particular, these R_v satisfy Lemma 4.1 and thus give rise to a local deformation problem.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

[Kis09] computes the dimension of generic fibers of the quotients we consider and proves that they are regular.

Modified local deformation rings

We shall also consider modified deformation problems as introduced in [Cal18]. For this, one fixes an eigenvalue α_v of $\overline{\rho}(\mathrm{Frob}_v)$. Note that $\alpha_v \in k$ by our hypothesis that the eigenvalues of all matrices in the image $\overline{\rho}(G_F) \subset \mathrm{GL}_2(k)$ lie in k.

Definition 4.2. The functor $\widetilde{\mathcal{D}}_v^{\square} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Sets}$ of *modified framed deformations* associates to $R \in \mathrm{CNL}_{\mathcal{O}}$ a pair (r,a) with $r \in \mathcal{D}_v^{\square}(R)$ and $a \in R$ a root of the characteristic polynomial of $r(\mathrm{Frob}_v)$ such that $a \equiv \alpha_v \mathrm{mod} \mathfrak{m}_R$.

There is an obvious natural transformation $u_v \colon \widetilde{\mathcal{D}}_v^{\square} \Rightarrow \mathcal{D}_v^{\square}$, and $\widetilde{\mathcal{D}}_v^{\square}$ is representable by the localization $\widetilde{R}_v^{\square}$ of the ring $R_v^{\square}[x]/(x^2 - x\,\mathrm{tr}\,\rho_v^{\square}(\mathrm{Frob}_v) + \det\rho_v^{\square}(\mathrm{Frob}_v))$ at the maximal ideal generated by $\mathfrak{m}_{R_v^{\square}}$ and $(x-\alpha_v)$. If $\overline{\rho}(\mathrm{Frob}_v)$ has a multiple eigenvalue, the ring $R_v^{\square}[x]/(x^2 - x\,\mathrm{tr}\,\rho_v^{\square}(\mathrm{Frob}_v) + \det\rho_v^{\square}(\mathrm{Frob}_v))$ is local and hence isomorphic to $\widetilde{R}_v^{\square}$. This proves the following result; see [Cal18, Lemma 2.1].

Lemma 4.3. *If $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, the canonical map $R_v^{\square} \to \widetilde{R}_v^{\square}$ is an isomorphism. Otherwise, the extension $R_v^{\square} \to \widetilde{R}_v^{\square}$ is a finite flat extension of degree two.*

The following definition is extracted from [Cal18, §2].

Definition 4.4. A *modified local deformation problem* for $\overline{\rho}|_{G_{F_v}}$ is a subfunctor $\widetilde{\mathcal{D}}_v \subset \widetilde{\mathcal{D}}_v^{\square}$ satisfying the following conditions:

1. The functor $\widetilde{\mathcal{D}}_v$ is represented by a quotient $\widetilde{R}_v$ of $\widetilde{R}_v^{\square}$.
2. For all $R \in \mathrm{CNL}_{\mathcal{O}}$, $g \in \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$ and $(r,a) \in \widetilde{\mathcal{D}}_v(R)$, we have $(grg^{-1},a) \in \widetilde{\mathcal{D}}_v(R)$.

One has the following analog of Lemma 4.1.

Lemma 4.5. *Let $\widetilde{\pi} \colon \widetilde{R}_v^{\square} \to \widetilde{R}_v$ be a surjective morphism in $\mathrm{CNL}_{\mathcal{O}}$, with specialization $(r,a) \in \widetilde{\mathcal{D}}(\widetilde{\mathcal{R}}_v)$ induced from the universal pair via $\widetilde{\pi}$. Suppose that*

1. *The ring $\widetilde{R}_v$ is reduced, and not isomorphic to k.*
2. *The surjection $\widetilde{\pi}$ satisfies condition 2 of Lemma 4.1 with $\widetilde{R}_v$ and $\widetilde{R}_v^{\square}$ replacing R_v and $R_v^{\square}$, respectively.*

Then the subfunctor $\widetilde{\mathcal{D}}_v$ of $\widetilde{\mathcal{D}}_v^{\square}$ defined by $\widetilde{R}_v$ is a modified local deformation problem.

Proof. The proof follows from the aruguments given in the proof of [Lemma 3.2, BLGHT11] which contains a proof of Lemma 4.1. □

Local deformation conditions

We now define the local deformation conditions relevant to this work; the resulting framed deformation rings will be denoted by $R_v^{\tau_v}$, where the superscripts

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$\tau_v \in \{\mathrm{fl}, \min, \mathrm{st}, \mathrm{un}, \varphi\text{-uni}, \square\}$ indicate the type of condition used to define R_v, and the corresponding universal framed deformation by $\rho_v^{\tau_v}$. Our conditions for framed deformations r of $\overline{\rho}|_{G_v}$ will *always* include the condition $\det r = \varepsilon_p$; we shall not repeat this below. We shall be brief, as we closely follow [BKM21, §4].

For all $v \in \Sigma_p$ the extension $F_v/\mathbf{Q}_p$ is unramified by the hypotheses from Subsection 1.6 and we moreover assume that that $\overline{\rho}|_{G_v}$ is finite flat, so that Fontaine–Laffaille theory applies, and we let

- R_v^{fl} be the quotient of $R_v^\square$ parameterizing *flat framed deformations* of $\overline{\rho}|_{G_v}$.

For $v \in \Sigma$, we let

- $R_v^{\min}$ be the quotient of $R_v^\square$ parametrizing *minimally ramified framed deformations* of $\overline{\rho}|_{G_v}$. Concretely, a framed deformation ρ_v of $\overline{\rho}|_{G_v}$ parameterized by $R_v^{\min}$ is required to satisfy one of the following two conditions depending on $\overline{\rho}|_{I_v}$:

  (i) if $\overline{\rho}|_{I_v}$ is semisimple, then the restriction $\rho_v(I_v) \to \overline{\rho}(I_v)$ is an isomorphism,

  (ii) if $\overline{\rho}|_{I_v}$ is a nontrivial extension of a character ξ of I_v by itself, then $\rho_v|_{I_v}$ is an extension of $\widetilde{\xi}$ by itself, for $\widetilde{\xi}$ the Teichmüller lift of ξ.

  In particular, if $\overline{\rho}$ is unramified at v, then $R_v^{\min}$ parameterizes *unramified framed deformations*, and then, occasionally we write R_v^{unr} for $R_v^{\min}$.

Let $Q \subset \Sigma$ be the subset of those v such that the representation $\overline{\rho}|_{G_{F_v}}$ is of the form

$$\begin{pmatrix} \varepsilon_p\overline{\chi} & * \\ 0 & \overline{\chi} \end{pmatrix} \tag{4.1}$$

with respect to some basis e_1, e_2 of k^2 and where the character $\overline{\chi}$ is unramified;[3] we further assume that the basis is chosen so that $*$ is trivial whenever $\overline{\rho}|_{G_{F_v}}$ is split, which holds if $\overline{\rho}$ is unramified and ε_p is nontrivial. Also, $\overline{\chi}$ has to be quadratic and we let χ be its unique (quadratic) lift to $\mathcal{O}$. Let $\beta_v = \chi(\mathrm{Frob}_v) \in \{\pm 1\}$.

For $v \in Q$, we define the *Steinberg quotient* R_v^{st} of $R_v^\square$ as follows:

- If $\overline{\rho}$ is ramified at v, then R_v^{st} is defined to be $R_v^{\min}$.
- If $\overline{\rho}$ is unramified at v, we define R_v^{st} as the unique reduced quotient of $R_v^\square$ characterized by the fact that the L-valued points of its generic fiber, for any finite extension L/E, correspond to representations of the form

$$\begin{pmatrix} \varepsilon_p\chi & * \\ 0 & \chi \end{pmatrix},$$

  and with the additional condition $\chi(\mathrm{Frob}_v) = \beta_v$ in the case $q_v \equiv -1 \mod p$. In the case $q_v \equiv -1 \mod p$, without fixing β_v, $\operatorname{Spec} R_v^{\mathrm{st}}$ would have two irreducible components because here ε_p is quadratic and unramified; see also [BKM21, §4].

[3] Let us note that the set Q here and the sets Q in Sections 6 and 7 are (related but) in general not the same.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

For $v \in Q$ such that $\overline{\rho}|_{G_{F_v}}$ is unramified, we also define:

- The *unipotent quotient* R_v^{uni} of $R_v^{\square}$ is the unique reduced quotient such that $\operatorname{Spec} R_v^{\mathrm{uni}} = \operatorname{Spec} R_v^{\mathrm{st}} \cup \operatorname{Spec} R_v^{\mathrm{unr}}$ inside $\operatorname{Spec} R_v^{\square}$. If $q_v \equiv -1 \bmod p$, then note that R_v^{st} depends on β_v.
- The *modified unipotent quotient* $\widetilde{R}_v^{\mathrm{uni}}$ of $\widetilde{R}_v^{\square}$ is the unique reduced quotient of $\widetilde{R}_v^{\square}$ characterized by the fact that the L-valued points of its generic fiber, for any finite extension L/E, correspond to pairs (r,a), where r is a representation of the form
$$\begin{pmatrix} \varepsilon_p \chi & * \\ 0 & \chi \end{pmatrix}$$
with χ unramified, and such that $\chi(\mathrm{Frob}_v) = a$, and such that $\alpha_v = \beta_v \bmod p$ in the case $q_v \equiv -1 \bmod p$.

It is clear from the definitions that the natural map $R_v^{\square} \to \widetilde{R}_v^{\mathrm{uni}}$ factors via $R_v^{\square} \to R_v^{\mathrm{uni}} \to \widetilde{R}_v^{\mathrm{uni}}$, and by Lemma 4.3, the map $R_v^{\mathrm{uni}} \to \widetilde{R}_v^{\mathrm{uni}}$ is an isomorphism, unless $q_v \equiv 1 \bmod p$.

For a more uniform notation, from now on we write $R_v^{\varphi\text{-uni}}$ instead of $\widetilde{R}_v^{\mathrm{uni}}$.

The following result summarizes basic ring theoretic properties of the $R_v^{\tau_v}$.

Proposition 4.6. *The following hold:*

1. *We have $R_v^{\mathrm{fl}} \cong \mathcal{O}[[x_1,\dots,x_{3+[F_v:\mathbf{Q}_p]}]]$ for $v \in \Sigma_p$ and $R_v^{\min} \cong \mathcal{O}[[x_1,x_2,x_3]]$ for $v \in \Sigma$.*
2. *For $v \in \Sigma$, the ring $R_v^{\square}$ is a complete intersection, reduced and flat over $\mathcal{O}$ of relative dimension 3.*
3. *For $v \in Q$, the ring R_v^{st} is Cohen–Macaulay, flat of relative dimension 3 over $\mathcal{O}$ and geometrically integral and if v is not a trivial prime for $\overline{\rho}$, we in fact have $R_v^{\mathrm{st}} \cong \mathcal{O}[[x_1,x_2,x_3]]$.*
4. *For each $v \in Q$ and each minimal prime $\mathfrak{p}$ of $R_v^{\square}$, $R_v^{\square}/\mathfrak{p}$ is flat over $\mathcal{O}$ and geometrically integral.*
5. *For $v \in Q$ such that in addition $\overline{\rho}$ is unramified at v, the rings R_v^{uni} and $R_v^{\varphi\text{-uni}}$ are Gorenstein, reduced and flat over $\mathcal{O}$ of relative dimension 3.*

Moreover, the rings $R_v^{\tau_v}$ in 1.–5. are the completion of a finite type $\mathcal{O}$-algebra at a maximal ideal.

Proof. For all but 5, we refer to [BKM21, Prop. 4.3] and the references given in its proof. The proof of 5 is given in Lemmas 5.4 and 5.3 below. $\square$

For each $v \in \Sigma$, fix a $\tau_v \in \{\min, \mathrm{st}, \mathrm{un}, \varphi\text{-uni}, \square\}$, and let $\tau = (\tau_v)_{v \in \Sigma}$, and define[4]

$$R_{\mathrm{loc}}^{\tau} = \left(\widehat{\bigotimes_{v \in \Sigma}} R_v^{\tau_v} \right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes_{v|p}} R_v^{\mathrm{fl}} \right).$$

[4]We note that $\widehat{\bigotimes}_{v \in \Sigma}$ and $\widehat{\bigotimes}_{v|p}$ are formed over $\mathcal{O}$, but we do not add $\mathcal{O}$ into the notation.

We simply write R_{loc} for R^{τ}_{loc}, if $\tau_v = \square$ for all v. Note in particular, that for any τ there is a natural morphism $R_{\mathrm{loc}} \to R^{\tau}_{\mathrm{loc}}$ and that it factors via $R^{\tau'}_{\mathrm{loc}}$, where τ' is obtained from τ be replacing all φ-uni by uni.

Proposition 4.6 and [BKM21, Lemma 4.4] yield:

Proposition 4.7. *The ring R_{loc} is a complete intersection, the ring R^{τ}_{loc} is Cohen–Macaulay, and both are reduced and flat over $\mathcal{O}$. If $R^{\tau_v}_v$ is Gorenstein for all $v \in \Sigma$, then so is R^{τ}_{loc}.*

Moreover, each irreducible component of $\operatorname{Spec} R_{\mathrm{loc}}$ is of the form

$$\operatorname{Spec}\left[\widehat{\bigotimes_{v\in\Sigma}} R^{\square}_v/\mathfrak{p}^{(v)}\right]\widehat{\otimes}\, R^{\mathrm{fl}}_p,$$

where each $\operatorname{Spec} R^{\square}_v/\mathfrak{p}^{(v)}$ is an irreducible component of $\operatorname{Spec} R^{\square}_v$, that is, each $\mathfrak{p}^{(v)}$ is a minimal prime of $R^{\square}_v$.

Global deformation rings

Now, we set up the notation for the corresponding global deformation rings, following [BKM21, Section 4.3], where further details can be found.

Let R (resp. $R^{\square}$) denote the global unframed (resp. framed) deformation ring parameterizing lifts of $\overline{\rho}$ with determinant ε_p which are unramified outside $\Sigma \cup \Sigma_p$ (together with a choice of basis at every $v \in \Sigma \cup \Sigma_p$), One may noncanonically fix an isomorphism $R^{\square} = R[[X_1,\dots,X_{4\#(\Sigma\cup\Sigma_p)-1}]]$ so that we may treat the subring R of $R^{\square}$ also as a quotient of $R^{\square}$. One also has a natural map $R_{\mathrm{loc}} \to R^{\square}$ (and thus a map $R_{\mathrm{loc}} \to R$), by restricting the global framed deformation and performing locally a base change.

Let $\tau = (\tau_v)_{v\in\Sigma}$ be as in the previous subsection. Then we define

$$R^{\square,\tau} = R^{\tau}_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R^{\square} \text{ and } R^{\tau} = R^{\tau}_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R. \tag{4.2}$$

5. The Wiles defect for some local framed deformation rings

In this section, R will denote a ring $R^{\tau_v}_v$ as defined in Section 4 for a residual representation $\overline{\rho}_v = \overline{\rho}|_{G_{F_v}} : G_{F_v} \to \mathrm{GL}_2(k)$ as described in the displayed matrix (4.1) at a place $v \in Q$ of F, and a deformation condition $\tau_v \in \{\mathrm{st}, \varphi\text{-uni}, \mathrm{uni}\}$. We let $q = q_v$ be the cardinality of the residue field of F_v and e the ramification index of $\mathcal{O}$ over $W(k)$. We also fix an augmentation $\lambda : R \twoheadrightarrow \mathcal{O}$.

Throughout this section, we assume, in fact, that $q \equiv 1 \bmod p$ and that $\overline{\rho}_v$ is trivial.

Definition 5.1. Let $\rho_\lambda : G_{F_v} \to \mathrm{GL}_2(\mathcal{O})$ be the representation at v induced from the augmentation λ. We define the local monodromy invariant n_v of λ to be the largest integers n such that $\rho_\lambda(G_{F_v}) \bmod \varpi^n$ has trivial projective image.[5]

[5] This definition also applies in the case where our current setup is twisted by a character that is quadratic and unramified at v. The results of this section also apply to this twisted setup.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

The aim of this section is to compute the invariants $D_{1,\lambda}(R)$ and $c_{1,\lambda}(R)$ of Venkatesh and the Wiles defect $\delta_\lambda(R)$ as attached in Definition 3.24 to the pair (R,λ) for certain types of $\overline{\rho}_v$ and τ_v. The three types of deformation conditions that we shall investigate are weight 2 Steinberg representations, weight 2 unipotent representations and weight 2 unipotent representations with an additional choice of Frobenius eigenvalue; we call the corresponding cases (st), (un) and (φ-uni), respectively. We shall see that the invariants will only depend on the monodromy invariant n_v and on the type of deformation condition.

The overall strategy in each case is the same. The actual computations between case (st) and cases (un) and (φ-uni) differ greatly. In each case, we first give (or recall) an explicit description of R, as a quotient of a power series ring over $\mathcal{O}$ modulo some ideal given by explicit relations. Then we need to find a ring $\widetilde{R}$ and a morphism $\varphi : \widetilde{R} \twoheadrightarrow R$ that satisfy property (CI). In the unipotent cases, we also need a morphism $\widetilde{\theta} : S \to \widetilde{R}$ as in Lemma 3.7. We greatly benefit from the freedom in choosing $\widetilde{R}$ and φ. Venkatesh's invariants do not depend on this choice. So we do this in a way amenable to computation. Our choices are not 'natural', but they 'work'.[6] They allow us to explicitly compute at least the following objects that by Theorem 3.9 and Theorem 3.23 give Venkatesh's invariants:

(a) the first two steps in a finite free $\widetilde{R}$-resolution of $I = \ker\varphi$,

(b) the $\widetilde{R}$-annihilator $\widetilde{R}[I]$ of I and

(c) the modules of formal differentials $\widehat{\Omega}_R$ and $\widehat{\Omega}_{\widetilde{R}}$.

The computation of the quantities in (c) is done as in [BKM21]. They can be related to $\mathcal{O}$-linear subspaces of $\widehat{\Omega}_{\mathcal{O}[[x_1,\ldots,x_n]]/\mathcal{O}}$ formed by differentials in the kernel ideal of a surjective presentation $\mathcal{O}[[x_1,\ldots,x_n]] \to \widetilde{R}$ and are not difficult to compute. The resolutions needed for (a) turned out to be manageable, even by hand calculation. The most difficult quantity to compute was (b). In case (st), we can rely on the rich theory of determinantal rings. In the other cases, we needed explicit bases of $\widetilde{R}$ and R as free modules over S, and we need to understand the socle of the mod p fiber of the latter rings modulo the standard regular sequence of S and the chain of isomorphisms in the proof of Lemma 3.14.

In the Steinberg case, we were able to perform all computations by hand. For (a) we made use of a standard resolution from commutative algebra, the Eagon–Northcott complex; see the proof of Lemma 5.8. Also, (b) and (c) turned out to be directly computable. The reason is that the ring we consider is the completion of a certain determinantal variety of 2×2-minors of a 4×2-matrix. The equations defining such varieties possess many symmetries and have been much studied in commutative algebra.

[6]It is shown in [Sho18] that the unrestricted framed deformation ring $R_v^\square$ of any trivial $\overline{\rho}_v : G_{F_v} \to \mathrm{GL}_n(k)$ is a local complete intersection ring and so the induced surjection $\widetilde{R} = R_v^\square \to R$ might appear as a natural candidate for $\widetilde{\theta}$. However, for the purpose of computations, this seems not useful. The ring $R_v^\square$ can be significantly more complicated than R. For instance in case (st), the ring R can be defined entirely by quadratic polynomials, whereas the equations defining $R_v^\square$ involve expressions of degree q. The latter seem to make the sort of computations we need to preform impossible when using $\widetilde{R}$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

In the unipotent cases, the defining equations had no structure that we could link to well-studied classical varieties. In these two cases, we employed for nearly all computations the computer algebra system Macaulay2.[7] To do so, we modeled the sequence of maps $S \to \widetilde{R} \to R$ by a sequence of rings $S_{\mathbf{Z}} \to \widetilde{R}_{\mathbf{Z}} \to R_{\mathbf{Z}}$ of finite type over $\mathbf{Z}$, that obviously depend on the case (un) or (φ-uni). With the help of Macaulay2 and suitable choices of integral models, that we found by experiment, we were able to work out (a)–(c) in fact over $\mathbf{Z}$ (or over $\mathbf{Q}$ when this was sufficient). Using base change and completion, we convert these computations to answers to (a)–(c) for $S \to \widetilde{R} \to R$. Our models in fact work for all primes p simultaneously. The models we find satisfy in particular, that $\widetilde{R}_{\mathbf{Z}}$ and $R_{\mathbf{Z}}$ are finite free over $S_{\mathbf{Z}}$ and that certain related models for the mod p fibers of $S \to \widetilde{R} \to R$ have the analogous property with the same rank. Finding models that are in addition smooth at the augmentation point in the generic fiber of $\widetilde{R}_{\mathbf{Z}}$ posed an additional challenge. The code that performs our calculation can be found the the GitHub repository of the first author; see [Böc23].

Let us also mention here that in Subsection 5.5, at the end of this section, we gather some results on Cohen–Macaulay and Gorenstein rings that we use repeatedly. It also contains some elementary results on generating sets on dual modules that were useful in explicit computations in Subsections 5.3 and 5.4.

5.1. Presentations of and basic results on the rings R

Case (st)**.** In case (st), the ring R is the Steinberg quotient R_v^{st} defined in Section 4. The setup is as in [BKM21, §7.2] except for two minor differences: In [BKM21] the coordinates used for R_v^{st} were adapted to the augmentation, while here we chose the coordinates to better fit standard results on the Eagon–Northcott complex. Moreover, here F_v is an arbitrary l-adic field, there it was $\mathbf{Q}_l$, where l the prime divisor of q. As recalled in Proposition 4.6, the ring R_v^{st} is a reduced Cohen–Macaulay domain (but non-Gorenstein), and it is flat over $\mathcal{O}$ of relative dimension 3. From [BKM21, §7.3], we have the explicit presentation $R_v^{\mathrm{st}} = \mathcal{R}/J_{\mathrm{st}}$ where $\mathcal{R} = \mathcal{O}[[a,b,c,\alpha,\beta,\gamma]]$ and J_{st} is the ideal of $\mathcal{R}$ generated by the 2×2-minors of the matrix

$$\begin{pmatrix} \alpha & \beta & (q-1+a) & b \\ \gamma & -\alpha & c & -a \end{pmatrix}. \tag{5.1}$$

To describe various explicit calculations to be given below, we denote by $t_{i,j}$ the 2×2-minor for columns $i<j$, and we set

$$r_1^{\mathrm{st}} = -t_{1,2} = \alpha^2 + \beta\gamma, \quad r_2^{\mathrm{st}} = t_{2,3} = (q-1+a)\alpha + c\beta, \quad r_3^{\mathrm{st}} = -t_{3,4} = (q-1+a)a + bc,$$
$$r_4^{\mathrm{st}} = -t_{1,3} = (q-1+a)\gamma - c\alpha, \quad r_5^{\mathrm{st}} = -t_{1,4} = a\alpha + b\gamma, \quad r_6^{\mathrm{st}} = -t_{2,4} = a\beta - b\alpha,$$

so that $J_{\mathrm{st}} = (r_1^{\mathrm{st}},\dots,r_6^{\mathrm{st}})$.

As in [BKM21, §7.2], we consider the augmentation $\lambda\colon R_v^{\mathrm{st}} \to \mathcal{O}$ given by $\lambda(a) = \lambda(\alpha) = \lambda(c) = \lambda(\gamma) = 0$ and $\lambda(b) = s$, $\lambda(\beta) = t$, with $t \in \mathfrak{m}_{\mathcal{O}}$ nonzero.

[7] We thank Dan Grayson for answering some questions and the Macaulay developers for this useful software.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Case (φ-uni)**.** Fix a lift $\sigma \in G_{F_v}$ of Frobenius. In case (φ-uni), the ring R is the universal framed deformation ring $R_v^{\varphi\text{-uni}}$ defined in [Cal18, §2.1; called there R_ℓ^{mod}] for framed deformations ρ of $\overline{\rho}_v$ of trivial inertia type together with a choice of eigenvalue $(1+X)$ of $\rho(\sigma)$, and with $\det\rho(\sigma) = q$. In other words, the p-adic framed deformations parameterized by $R_v^{\varphi\text{-uni}}$ are those that can be made upper-triangular with unipotent inertia and with $q(1+X)^{-1}$ and $(1+X)$ as diagonal entries of $\rho(\sigma)$ for some X. It is shown in [Cal18, Lem. 2.4 and its proof] that we have

$$R_v^{\varphi\text{-uni}} = \mathcal{R}/\mathcal{I},$$

where $\mathcal{R} = \mathcal{O}[[\alpha,\beta,\gamma,X,a,b,c]]$ and $\mathcal{I} \subset \mathcal{R}$ is the ideal generated by the entries of the matrices

$$N^2, N(A-(1+X)I), (A-q(1+X)^{-1})N, AN-qNA, \det A - q$$

$$\text{with} \quad A := \begin{pmatrix} q(1+X)^{-1}+a & b \\ c & 1+X-a \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

The corresponding universal framed deformation factors through the tame quotient G_q^t of G_{F_v}, and if τ is a topological generator of the inertia subgroup of G_q^t such that $\sigma\tau\sigma^{-1} = \tau^q$, then this framed deformation is given by $\sigma \mapsto A$ and $\tau \mapsto I+N$.

Lemma 5.2. *We have* $\mathcal{I} = (r_1^{\varphi\text{-uni}} \ldots, r_9^{\varphi\text{-uni}})$ *for*

$$r_1^{\varphi\text{-uni}} = \alpha X,\ r_2^{\varphi\text{-uni}} = \beta X,\ r_3^{\varphi\text{-uni}} = \gamma X,\ r_4^{\varphi\text{-uni}} = aq + (a^2+bc)(1+X) - a(1+X)^2,\ r_5^{\varphi\text{-uni}} = \alpha^2+\beta\gamma,$$

$$r_6^{\varphi\text{-uni}} = \alpha c - \gamma(q-1+a),\ r_7^{\varphi\text{-uni}} = \alpha a + \gamma b,\ r_8^{\varphi\text{-uni}} = \beta c + \alpha(q-1+a),\ r_9^{\varphi\text{-uni}} = \beta a - \alpha b.$$

Proof. We claim that $\mathcal{I}$ is generated by the elements $\alpha X, \beta X, \gamma X$, $\det A - q$, $\det N$ together with the entries of the 2×2-matrix $N(A-(1+X)I)$ with X specialized to zero. From the claim and in particular $\alpha X, \beta X, \gamma X \in \mathcal{I}$, it is straightforward to see that the $r_i^{\varphi\text{-uni}}$, $i = 1,\ldots,9$ generate $\mathcal{I}$.

To show the claim, denote for a 2×2-matrix D over a ring R by D^ι the main involution applied to D as in the proof of [BKM21, Lem. 7.2]; recall that the map $D \mapsto D^\iota$ is R-linear and satisfies $D + D^\iota = \operatorname{tr} D \cdot I$, and that, up to sign, the set of entries of D and D^ι are the same.

It follows that $N^\iota = -N$ and $A^\iota = -A + (q(1+X)^{-1} + (1+X))I$, and from this one deduces that

$$(A - q(1+X)^{-1})N = \big(N(A-(1+X)I)\big)^\iota.$$

Hence, either the entries of $N(A-(1+X)I)$ or those of $(A-q(1+X)^{-1})N$ can be omitted when generating $\mathcal{I}$.

The vanishing of N^2 is easily be seen equivalent to that of $\det N$. It remains to show that assuming $N(A-(1+X)I) = 0$, we have $AN = qNA \iff \alpha X = \beta X = \gamma X = 0$: To see '$\Rightarrow$', we compute

$$0 = qN\cdot(A-(1+X)I) = qNA - q(1+X)N = AN - q(1+X)N = (A-q(1+X)I)N.$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Subtracting the latter from $(A-q(1+X)^{-1})N=0$ yields $q(1+X-(1+X)^{-1})N=0$, and from this and our hypothesis $p>2$ it is straightforward to see that $XN=0$, i.a., that $\alpha X=\beta X=\gamma X=0$. For '$\Leftarrow$', observe that the steps can be reverted. $\square$

Lemma 5.3. *The ring $R_v^{\varphi\text{-uni}}$ has the following properties:*

1. *It is reduced, flat over $\mathcal{O}$ and of relative dimension* 3*.*
2. *Its two minimal primes I_1 and I_2 can be labeled so that $R_v^{\varphi\text{-uni}}/I_1$ parameterizes unramified framed deformations of $\overline{\rho}$ with a choice of Frobenius eigenvalue, and $R_v^{\varphi\text{-uni}}/I_2$ is the Steinberg framed deformation ring R_v^{st} from case* (st)*.*
3. *The elements $\varpi, b-c, b-\beta, X-\gamma$ form a regular system of parameters and $R^{\varphi\text{-uni}}$ is Gorenstein.*

Proof. Part 1 is [Cal18, Lem. 2.2]. To see 2, set $\mathcal{I}_1=\mathcal{I}+(\alpha,\beta,\gamma)$ and $\mathcal{I}_2=\mathcal{I}+(X)$. From the description of $R_v^{\varphi\text{-uni}}$ and its universal framed deformation, it follows that the rings $R_1^{\varphi\text{-uni}}/\mathcal{I}_j$ have the moduli interpretation we claim in 2. It remains to show $\mathcal{I}\supseteq\mathcal{I}_1\cap\mathcal{I}_2$. Observe first that

$$\mathcal{R}/\mathcal{I}_1\cong\mathcal{O}[[a,b,c,X]]/(aq+(a^2+bc)(1+X)-a(1+X)^2)$$

is a domain because $aq+(a^2+bc)(1+X)-a(1+X)^2$ cannot be factored in the regular ring $\mathcal{O}[[a,b,c,X]]$. Hence, X is a nonzero divisor in the quotient $\mathcal{R}/\mathcal{I}_1$. Suppose now that we are given $r+r'X\in\mathcal{I}_1\cap\mathcal{I}_2$ with $r\in\mathcal{I}$ and $r'\in\mathcal{R}$. Reducing modulo $\mathcal{I}_1$ yields $r'\in\mathcal{I}_1$ and hence $r'X\in\mathcal{I}_1\cdot\mathcal{I}_2\subset\mathcal{I}$. This concludes 2.

We prove 3. The ring $\mathcal{R}/\mathcal{I}_2$ is isomorphic to R_v^{st} and hence Cohen–Macaulay of dimension 4. The ring $\mathcal{R}/\mathcal{I}_1$, given explicitly above, and its quotient by X, that is, the ring $\mathcal{R}/(\mathcal{I}_1+\mathcal{I}_2)$, are Cohen–Macaulay of dimension 4 and 3, respectively, by Proposition 5.35.3. Hence, $R_v^{\varphi\text{-uni}}=\mathcal{R}/(\mathcal{I}_1\cap\mathcal{I}_2)$ is Cohen–Macaulay of dimension 4 by [Eis95, Exerc. 18.13]. In particular systems of parameters of $R_v^{\varphi\text{-uni}}$ are regular sequences by Proposition 5.35.

Let now A be the quotient of $R_v^{\varphi\text{-uni}}$ modulo the sequence $\varpi,b-c,b-\beta,X-\gamma$. The relations allow one to eliminate the variables c,β,γ, and after some simple manipulations one finds

$$A\cong k[[a,b,X,\alpha]]/(\alpha X,bX,X^2,a^2-2aX,\alpha^2,\alpha b-aX,\alpha a,b^2,ab-aX).$$

It is a k-vector space of dimension 6 with basis $1,a,b,X,\alpha,a^2$ and one computes $\mathrm{socle}(A)=ka^2$. Hence, the sequence $\varpi,b-c,b-\beta,X-\gamma$ is regular and it follows from Proposition 5.35 that $R_v^{\varphi\text{-uni}}$ is Gorenstein. $\square$

We consider the 'same' augmentation as in case (st), namely the $\mathcal{O}$-algebra map $R_v^{\varphi\text{-uni}}\to\mathcal{O}$ that is the projection $R_v^{\varphi\text{-uni}}\to R_v^{\varphi\text{-uni}}/I_2=R_v^{\mathrm{st}}$ from Lemma 5.3.2. composed with the augmentation $R_v^{\mathrm{st}}\to\mathcal{O}$ from case (st). Concretely, λ is given by

$$a\mapsto 0, X\mapsto 0, c\mapsto 0, \alpha\mapsto 0, \gamma\mapsto 0, b\mapsto s, \beta\mapsto t$$

for some $s,t\in\mathfrak{m}_{\mathcal{O}}$ with t nonzero.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Case (un)**.** One has natural surjections $R_v^\square \to R_v^{\mathrm{st}}$ and $R_v^\square \to R_v^{\mathrm{unr}}$. Denote by I^{st} and I^{unr} the corresponding ideals of $R_v^\square$. Then in the case (un), we define R as the quotient

$$R_v^{\mathrm{uni}} = R_v^\square/(I^{\mathrm{st}} \cap I^{\mathrm{unr}}),$$

cf. [Sho16, Rem. 5.7] for a comparable definition. In other words, R_v^{uni} is the reduced quotient of $R_v^\square$ such that $\operatorname{Spec} R^{\mathrm{uni}} = \operatorname{Spec} R_v^{\mathrm{st}} \cup \operatorname{Spec} R_v^{\mathrm{unr}} \subset \operatorname{Spec} R_v^\square$; see Lemma 5.4.

The ring $R_v^\square$ is can be realized as the quotient $\mathcal{R}'/\mathcal{I}'$ for $\mathcal{R}' = \mathcal{O}[[\alpha,\beta,\gamma,\delta,a,b,c,X]]$ and $\mathcal{I}' \subset \mathcal{R}'$ as the ideal generated by the entries of the (2×2- and 1×1-) matrices

$$AB - B^q A, \det A - q, \det B - 1$$
$$\text{with } A := \begin{pmatrix} q+a & b \\ c & 1-a-X \end{pmatrix} \text{ and } B := \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix}.$$

The ideals I^{unr} and I^{st} both contain $\alpha+\delta$ since these quotient describe situations where either $N = B - I$ is zero, or N is of trace and determinant zero. Therefore, R_v^{uni} can be written as a quotient of $\mathcal{R} = \mathcal{O}[[\alpha,\beta,\gamma,a,b,c,X]]$ by an ideal $\mathcal{I}^{\mathrm{uni}} \subset \mathcal{R}$; with $\delta = -\alpha$.

We computed in Macaulay2 generators of I^{unr} and I^{st} by working inside the polynomial ring $\mathcal{R}_{\mathbf{Z}} = \mathbf{Z}[\underline{q},a,b,c,X,\alpha,\beta,\gamma]$, where we represent the prime power q in $\mathbf{Z}$ by the indeterminate $\underline{q}+1$ in the polynomial ring.[8] Let $I_{\mathbf{Z}}^{\mathrm{unr}}$ and $I_{\mathbf{Z}}^{\mathrm{st}}$ denote the corresponding ideals of $\mathcal{R}_{\mathbf{Z}}$. Then we let Macaulay2 also compute the intersection $I_{\mathbf{Z}}^{\mathrm{uni}} = I_{\mathbf{Z}}^{\mathrm{unr}} \cap I_{\mathbf{Z}}^{\mathrm{st}}$. The ideal $I_{\mathbf{Z}}^{\mathrm{uni}}$ is generated by the elements

$$r_1^{\mathrm{uni}} = X\gamma,\ r_2^{\mathrm{uni}} = X\beta,\ r_3^{\mathrm{uni}} = X\alpha,\ r_4^{\mathrm{uni}} = \alpha^2 + \beta\gamma,\ r_5^{\mathrm{uni}} = b\alpha - a\beta,\ r_6^{\mathrm{uni}} = a\alpha + b\gamma,$$
$$r_7^{\mathrm{uni}} = c\beta - b\gamma + \underline{q}\alpha,\ r_8^{\mathrm{uni}} = c\alpha - a\gamma - \underline{q}\gamma,\ r_9^{\mathrm{uni}} = a^2 + bc + aX + \underline{q}a + (\underline{q}+1)X.$$

We also have $I_{\mathbf{Z}}^{\mathrm{unr}} = (\alpha,\beta,\gamma)$ and $I_{\mathbf{Z}}^{\mathrm{st}} = (X, r_4^{\mathrm{uni}}, \ldots, r_9^{\mathrm{uni}})$. We shall use the same names r_i^{uni} for the corresponding elements in $\mathcal{R}$, with the silent assumption that in $\mathcal{R}$ we replace $\underline{q}$ by q.

Lemma 5.4. *The ring $R_v^{\mathrm{uni}} = \mathcal{R}/\mathcal{I}^{\mathrm{uni}}$ with $\mathcal{R} = \mathcal{O}[[\alpha,\beta,\gamma,X,a,b,c]]$ and $\mathcal{I}^{\mathrm{uni}} = (r_1^{\mathrm{uni}}, \ldots, r_9^{\mathrm{uni}})$ has the following properties:*

1. *We have $\mathcal{I}^{\mathrm{uni}} = \mathcal{I}^{\mathrm{unr}} \cap \mathcal{I}^{\mathrm{st}}$ for $\mathcal{I}^{\mathrm{unr}} = \mathcal{I} + (\alpha,\beta,\gamma)$ and $\mathcal{I}^{\mathrm{st}} = \mathcal{I} + (X)$ so that $\mathcal{R}/\mathcal{I}^{\mathrm{unr}}$ and $\mathcal{R}/\mathcal{I}^{\mathrm{st}}$ are identified with the unramified and the Steinberg quotient of R_v^{uni}, respectively.*
2. *The ring R_v^{uni} is Cohen–Macaulay, flat over $\mathcal{O}$ and of relative dimension* 3 *and reduced.*
3. *The elements $\varpi, b-c, b-\beta, X-\gamma$ form a regular system of parameters and R_v^{uni} is Gorenstein.*

Proof. Part 1 is clear, except for the containment $\mathcal{I}^{\mathrm{uni}} \supset \mathcal{I}^{\mathrm{unr}} \cap \mathcal{I}^{\mathrm{st}}$. Similar to Lemma 5.3, the quotient $\mathcal{R}/\mathcal{I}^{\mathrm{unr}} \cong \mathcal{O}[[X,a,b,c]]/(r_9^{\mathrm{uni}})$ is a Cohen–Macaulay domain of dimension 4.

[8] It might be worthwhile to remark that a computer algebra package cannot directly evaluate B^q, and so we cannot give explicit equations for $R_v^\square$. However, in the quotients R_v^{unr} and R_v^{st} the matrix B has characteristic polynomial $(T-1)^2$ so that $(B-I)^2 = 0$ by the Cayley–Hamilton theorem. This allows us to use $B^q = B + (q+1)I$ in when computing $\mathcal{I}^{\mathrm{unr}}$ and $\mathcal{I}^{\mathrm{st}}$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

The inclusion $\mathcal{I}^{\mathrm{uni}} \supset \mathcal{I}^{\mathrm{unr}} \cap \mathcal{I}^{\mathrm{st}}$ now follows as in the proof of Lemma 5.3, and this completes part 1. Because of part 1, the central factors in the short exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{R}/\mathcal{I}^{\mathrm{uni}} \to \mathcal{R}/\mathcal{I}^{\mathrm{unr}} \times \mathcal{R}/\mathcal{I}^{\mathrm{st}} \to \mathcal{R}/(\mathcal{I}^{\mathrm{unr}} + \mathcal{I}^{\mathrm{st}}) \to 0$$

are domains, and so R_v^{uni} is reduced. The two central factors and also $\mathcal{R}/(\mathcal{I}^{\mathrm{unr}} + \mathcal{I}^{\mathrm{st}}) \cong \mathcal{O}[[a,b,c]]/(r_3^{\mathrm{st}})$ are Cohen–Macaulay of dimensions 4, 4 and 3, respectively. As before we find that R_v^{uni} is Cohen–Macaulay of dimension 4 by [Eis95, Exerc. 18.13].

Finally, one verifies, by hand or via Macaulay2, that $\mathcal{R}_{\mathbf{Z}}/(\mathcal{I}_{\mathbf{Z}}^{\mathrm{uni}} + (\underline{q}, b-c, b-\beta, X-\gamma))$ is a free $\mathbf{Z}$-module of rank 6 with basis $1, a, b, b\alpha, X, \alpha$ and socle $b\alpha$. By reduction modulo any prime number p, one deduces that $R_v^{\mathrm{uni}}/(\varpi, b-c, b-\beta, X-\gamma)$ is a zero-dimensional ring. It follows that $\varpi, b-c, b-\beta, X-\gamma$ form a system of parameters, and hence a regular system of parameters by Proposition 5.35.3. In particular, R_v^{uni} is $\mathcal{O}$-flat. By computing the socle of $R_v^{\mathrm{uni}}/(\varpi, b-c, b-\beta, X-\gamma)$, which turns out to be of length 1, one deduces from Proposition 5.35.1 and Proposition 5.35.2 that R_v^{uni} is Gorenstein.

Let us indicate the relevant computation for $A := R_v^{\mathrm{uni}}/(\varpi, b-c, b-\beta, X-\gamma)$. Using the relations given by $\varpi, b-c, b-\beta, X-\gamma$ to eliminate c, β, γ, one finds

$$A \cong k[a,b,X,\alpha]/(X^2, bX, \alpha X, \alpha^2, b(\alpha-a), a\alpha, b^2, b\alpha - aX, a^2 + aX + X).$$

The last relation gives $X(1+a) = -a^2$, from the first it follows that $X^2 = 0$ and hence $a^4 = 0$ so that $1+a$ has inverse $1-a-a^2-a^3$, and with it $X = -a^2 + a^3$. This allows one to eliminate X and after elementary simplifications one finds: $A = k[a,b,\alpha]/(a^2, \alpha^2, b(a-\alpha), a\alpha, b^2, ba + a^3)$. Then a simple computation shows that a k-basis is given by $1, a, b, \alpha, a^2, ab$. To find the socle one computes the solution set of the equations $ax = bx = \alpha x = 0$ for x a general k-linear combination of the above k-basis of A. This gives $x \in kab$, and since $ab = b\alpha$ and $X = -a^2 - ab$ in A, this is a direct proof of the first sentence of the previous paragraph. □

Remark 5.5. One can also work out the above argument by first working out properties for $\mathcal{R}_{\mathbf{Z}}$, $\mathcal{I}^{\mathrm{st}}$, $\mathcal{I}^{\mathrm{unr}}$ and $\mathcal{I}^{\mathrm{uni}}$ and then completing at $\mathfrak{m}_Z = (p, \underline{q}, \alpha, \beta, \gamma, X, a, b, c)$ and then passing to the quotient modulo $\underline{q} - (q-1)$. The above direct argument is shorter.

5.2. Steinberg deformations at trivial primes

Lemma 5.6.

1. *The elements* $r_1^{\mathrm{st}}, r_2^{\mathrm{st}}, r_3^{\mathrm{st}}, \gamma-\beta, c+b, \beta+b, \varpi$ *of* $\mathcal{R} = \mathcal{O}[[a,b,c,\alpha,\beta,\gamma]]$ *form a regular sequence.*
2. *The complete intersection* $\widetilde{R} := \mathcal{R}/(r_1^{\mathrm{st}}, r_2^{\mathrm{st}}, r_3^{\mathrm{st}})$ *is flat over* $\mathcal{O}$ *and of relative dimension* 3.
3. *The point in* $\operatorname{Spec} \widetilde{R}[\frac{1}{\varpi}]$ *corresponding to the augmentation* $\widetilde{\lambda}\colon \widetilde{R} \to \mathcal{O}$ *given by the same prescription as* λ *is formally smooth.*

Proof. 1. It suffices to show that $\mathcal{R}$ modulo the ideal generated by the given sequence is finite. Modding out $\gamma-\beta, c+b, \beta+b, \varpi$ from $\mathcal{R}$, we need to show that $k[[a,b,\alpha]]$ modulo the 2×2-minors $t_{1,2}, t_{2,3}, t_{3,4}$ of the matrix

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$$\begin{pmatrix} \alpha & -b & a & b \\ -b & -\alpha & -b & -a \end{pmatrix}$$

is finite. Using the relation $\alpha a + b^2$ as a relation for b, it follows that the quotient ring is a degree 2 extension of $k[[a,\alpha]]/(\alpha^2 - a\alpha, a^2 + a\alpha)$, and the latter ring is finite, as $p > 2$; a k-basis is $1, a, \alpha, a\alpha$.

2. The regular sequence in 1. remains a regular sequence under any reordering and after any truncation. This shows that $\widetilde{R}$ is flat over $\mathcal{O}$ and of relative dimension 3 over $\mathcal{O}$.

3. To see the formal smoothness, we form the Jacobian matrix of $r_1^{\mathrm{st}}, r_2^{\mathrm{st}}, r_3^{\mathrm{st}}$ relative to the variables of $\mathcal{R}[\frac{1}{\varpi}]$ and evaluate at the augmentation. This gives

$$\begin{pmatrix} 0 & 0 & 0 & 2\alpha & \gamma & \beta \\ \alpha & \gamma & 0 & q-1+a & 0 & b \\ 2a+q-1 & c & b & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{eval. at } \widetilde{\lambda}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & q-1 & 0 & s \\ q-1 & 0 & s & 0 & 0 & 0 \end{pmatrix}.$$

Columns 1,4,6 witness the formal smoothness asserted for $\widetilde{\lambda}$. □

Remark 5.7. From the proof of Lemma 5.6.1, one deduces that as an $\mathcal{O}$-algebra map $S = \mathcal{O}[[y_1, y_1, y_3]] \to \widetilde{R}$ one can take

$$y \mapsto \gamma - \beta, \quad y_2 \mapsto c + b, \quad y_3 \mapsto \beta + b.$$

Similar to the proof of Lemma 5.6.1, one can show that $R_v^{\mathrm{st}}/(\varpi, y_1, y_2, y_3) \cong k[a,\alpha,\gamma]/(a,\alpha,\gamma)^2$. Its socle is obviously spanned by $\{a,\alpha,\gamma\}$ and has thus k-dimension 3. Using that R_v^{st} is local Cohen–Macaulay of dimension 4, by combining parts 3, 2 and 1 of Proposition 5.35 one deduces that R_v^{st} is not Gorenstein.

In the following, let $\widetilde{R} = \mathcal{R}/(r_1^{\mathrm{st}}, r_2^{\mathrm{st}}, r_3^{\mathrm{st}})$ and $I = \ker(\widetilde{R} \to R_v^{\mathrm{st}})$. We need some preparations to give a presentation of I as an $\widetilde{R}$-module. Recall that J_{st} was defined before formula (5.1).

Lemma 5.8. *The sequence of $\mathcal{R}$-modules $\mathcal{R}^8 \xrightarrow{A} \mathcal{R}^6 \xrightarrow{B} J_{\mathrm{st}} \to 0$ is exact, where B is the 1×6-matrix $(r_1^{\mathrm{st}}, r_2^{\mathrm{st}}, \dots, r_6^{\mathrm{st}})$ and A is the 8×6-matrix*

$$\begin{pmatrix} b & q-1+a & 0 & 0 & -a & c & 0 & 0 \\ 0 & -\alpha & -b & 0 & 0 & -\gamma & a & 0 \\ 0 & 0 & \beta & \alpha & 0 & 0 & -\alpha & \gamma \\ 0 & \beta & 0 & -b & 0 & -\alpha & 0 & a \\ \beta & 0 & 0 & q-1+a & -\alpha & 0 & 0 & c \\ -\alpha & 0 & q-1+a & 0 & -\gamma & 0 & c & 0 \end{pmatrix}.$$

Proof. The displayed presentation is part of the Eagon–Northcott complex attached to the 4×2-matrix from Equation (5.1), considered as an $\mathcal{R}$-linear map $\nu\colon \mathcal{R}^4 \to \mathcal{R}^2$, and in the present case, this complex is exact: The Eagon–Northcott complex is described in detail in [Eis05, §11H], which we now recall in parts. We follow the notation of [Eis05] and set $G = \mathcal{R}^2$ and $F = \mathcal{R}^4$, so that $\nu^*\colon F^* \to G^*$. Then in the case at hand, the Eagon–Northcott complex is the complex

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$$0 \longrightarrow (\mathrm{Sym}^2 G)^* \otimes \bigwedge^4 F \xrightarrow{d_2} (\mathrm{Sym}^1 G)^* \otimes \bigwedge^3 F \xrightarrow{d_1} (\mathrm{Sym}^0 G)^* \otimes \bigwedge^2 F \longrightarrow \bigwedge^2 G;$$

choosing bases $f_1,\ldots,f_4$ of F and g_1,g_2 of G, the complex is seen to be of the form $0 \to \mathcal{R}^3 \to \mathcal{R}^8 \to \mathcal{R}^6 \to \mathcal{R}$; the right most map of the complex sends the basis element $f_i \wedge f_j$, $i<j$, to the minor $t_{i,j}$ of Equation (5.1) formed by column i and column j, and thus its image is the ideal J_{st}.

To describe the maps d_i, let

$$\Gamma_j\colon (\mathrm{Sym}^j G)^* \to G^* \otimes (\mathrm{Sym}^{j-1} G)^*$$

be the map dual to the multiplication map $G \otimes \mathrm{Sym}^{j-1} G \to \mathrm{Sym}^j G$, and write $\Gamma_j(u) = \sum_l u'_l \otimes u''_l$ for $u \in (\mathrm{Sym}^j G)^*$. Let furthermore

$$\Phi_k\colon \bigwedge^k F \to F \otimes \bigwedge^{k-1} F$$

be the $\mathcal{R}$-linear map given on basis elements by

$$f_{i_1} \wedge \ldots \wedge f_{i_k} \mapsto \sum_{j=1}^{k} (-1)^{j-1} f_{i_j} \otimes f_{i_1} \wedge \ldots \wedge \widehat{f_{i_j}} \wedge \ldots \wedge f_{i_k},$$

and write $\Phi_k(v) = \sum_m v'_m \otimes v''_m$ for $v \in \bigwedge^k F$. Then for a pure tensor $u \otimes v$ in $(\mathrm{Sym}^j G)^* \otimes \bigwedge F^{j+2}$, and $k=j+2$, one has

$$d_j(u \otimes v) = \sum_{l,m} \big(\nu^*(u'_l)(v'_m)\big) u''_l \otimes v''_l \in (\mathrm{Sym}^{j-1} G)^* \otimes \bigwedge F^{j+1}.$$

This procedure can be applied to the basis $g_l \otimes f_{i_1} \wedge f_{i_2} \wedge f_{i_3}$, $1 \le i_1 < i_2 < i_3 \le 4$, of $(\mathrm{Sym}^1 G)^* \otimes \bigwedge^3 F$ to obtain the matrix A.

To complete the proof, it remains to show exactness of the Eagon–Northcott complex in the case at hand. By [Eis05, Thm. 11.35], this holds if and only if the grade of the ideal J_{st} attains the maximal value possible, namely the height of J_{st}; see [Mat80, p. 103]. Because $\mathcal{R}/J_{\mathrm{st}} = R_v^{\mathrm{st}}$ has Krull dimension 4, the height of J_{st} is 3. The grade of J_{st} is the maximal length of a regular sequence of $\mathcal{R}$ contained in J_{st}; see [Mat80, p. 103], and because of Lemma 5.6 this number is at least 3. □

Lemma 5.9. *Let $\mathcal{R}^m \xrightarrow{A} \mathcal{R}^n \xrightarrow{B} J \to 0$ be a right exact sequence of $\mathcal{R}$-modules for J an ideal of $\mathcal{R}$. We consider A as an $n\times m$-matrix and B as a $1\times n$-matrix over $\mathcal{R}$. Decompose $n = n'+n''$ with $n',n''>0$, and decompose correspondingly the matrix A into A' and A'' of size $n'\times m$ and $n''\times m$, and the matrix B into matrices B' of size $1\times n'$ and B'' of size $1\times n''$, respectively. Let $J' \subset J$ be the image of $\mathcal{R}^{n'}$ under B'. Then the induced sequence of $\mathcal{R}/J'$-modules*

$$(\mathcal{R}/J')^m \xrightarrow{A'' \pmod{J'}} (\mathcal{R}/J')^{n''} \xrightarrow{B'' \pmod{J'}} J/J' \to 0$$

is right exact.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Proof. By the definition of J', the map defined by B'' (mod J') is clearly surjective. Also, $BA=0$ implies $B'A'=-B''A''$ as maps on $\mathcal{R}^m$. But B' (mod J') is the zero map, and hence

$$(B'' \pmod{J'})(A'' \pmod{J'})=0.$$

It remains to show that $\ker(B'' \pmod{J'})\subset \operatorname{im}(A'' \pmod{J'})$. For this, let $x'' \pmod{J'}\in \ker(B'' \pmod{J'})$ with $x''\in\mathcal{R}^{n''}$. Then $B''x''$ lies in J' and hence we can find $x'\in\mathcal{R}^{n'}$ such that $B''x''=B'x'$. We let $x=(-x',x'')\in\mathcal{R}^n$ so that $Bx=0$. By the exactness of the given complex, we can find $y\in\mathcal{R}^m$ such that $Ay=x$. But then $A''y=x''$ and this implies $x'' \pmod{J'}\in \operatorname{im}(A'' \pmod{J'})$. □

By combining the previous two lemmas, we find.

Corollary 5.10. *As a module over $\widetilde{R}$ the ideal I has a presentation*

$$(\widetilde{R})^8 \xrightarrow{A'} (\widetilde{R})^3 \longrightarrow I \to 0,$$

where A' is the matrix

$$\begin{pmatrix} 0 & \beta & 0 & -b & 0 & -\alpha & 0 & a \\ \beta & 0 & 0 & q-1+a & -\alpha & 0 & 0 & c \\ -\alpha & 0 & q-1+a & 0 & -\gamma & 0 & c & 0 \end{pmatrix}.$$

Corollary 5.11. *We have $\widetilde{\lambda}(\mathrm{Fitt}_0^{\widetilde{R}}(I))=(q-1)t(s,t,q-1)\subset\mathcal{O}$.*

Proof. The ideal $\mathrm{Fitt}_0^{\widetilde{R}}(I)$ is the ideal generated by the 3×3-minors of the matrix A' from Corollary 5.10. Hence, its image under $\widetilde{\lambda}$ is the ideal of $\mathcal{O}$ generated by the 3×3-minors of

$$\widetilde{\lambda}(A')=\begin{pmatrix} 0 & t & 0 & -s & 0 & 0 & 0 & 0 \\ t & 0 & 0 & q-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

This is the ideal $(t^2(q-1),t(q-1)^2,ts(q-1))=t(q-1)(s,t,q-1)$. □

Remark 5.12. Using the matrix A', a Macaulay2 computation shows that $\mathrm{Fitt}_0^{\widetilde{R}}(I)$ equals

$$((q-1+a)^2\beta,(q-1+a)b\beta,(q-1+a)c\beta,(q-1+a)\beta^2,(q-1+a)\beta\gamma,ac\alpha,ac\beta,bc\beta,c^2\alpha,c^2\beta,c\alpha\gamma,c\beta\gamma,c\beta^2).$$

Corollary 5.13. *We have $\mathrm{Hom}_{R_v^{\mathrm{st}}}(I/I^2,E/\mathcal{O})\cong\mathcal{O}/(s,t,q-1)\times\mathcal{O}/(t,q-1)\times(t,q-1))/(t(q-1))$.*

Proof. Note first that

$$\mathrm{Hom}_{R_v^{\mathrm{st}}}(I/I^2,E/\mathcal{O})\cong\mathrm{Hom}_{R_v^{\mathrm{st}}}(I\otimes_{\widetilde{R}}R_v^{\mathrm{st}},E/\mathcal{O})\cong\mathrm{Hom}_{\mathcal{O}}(I\otimes_{\widetilde{R}}^{\widetilde{\lambda}}\mathcal{O},E/\mathcal{O}),$$

where in the second isomorphism, we use that E is regarded as a R_v^{st}-module via the augmentation λ. Tensoring now the presentation of I in subsection 5.10 with $\mathcal{O}$ over $\widetilde{R}$ (via $\widetilde{\lambda}$) gives the right exact sequence of $\mathcal{O}$-modules

$$\mathcal{O}^8 \xrightarrow{\widetilde{\lambda}(A')} \mathcal{O}^3 \longrightarrow I\otimes_{\widetilde{R}}^{\widetilde{\lambda}}\mathcal{O}\to 0. \tag{5.3}$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Using the theory of invariant factors and elementary divisors of matrices over principal ideal domains, for example, [Jac85, Thm. 3.9], the cokernel of this sequence is seen to be isomorphic to $\prod_{i=1}^{3}\mathcal{O}/d_i\mathcal{O}$ where d_1, d_1d_2 and $d_1d_2d_3$ are the gcds of the $i\times i$-minors of $\widetilde{\lambda}(A')$ displayed in Equation (5.2) for $i=1,2,3$. One readily computes

$$\begin{aligned}
d_1 &= \gcd(s,t,q-1),\\
d_2 &= \gcd(t^2,t(q-1),t(q-1),ts,(q-1)s,(q-1)^2) = \gcd(s,t,q-1)\gcd(t,q-1),\\
d_3 &= \gcd(t^2(q-1),t(q-1)s,t(q-1)^2) = t(q-1)\gcd(s,t,q-1),
\end{aligned}$$

and this implies the assertion of the corollary. □

Lemma 5.14. *For the ideals $P=(\alpha,\beta)$, $Q=(q-1+a,c)$ and $I'=((q-1+a)\alpha,(q-1+a)\beta,c\alpha,c\beta)$ of $\widetilde{R}$ the following hold.*

1. *P is a prime ideal and $P=\{x\in\widetilde{R}\mid xr_4^{\mathrm{st}}=0\}$.*
2. *Q is a prime ideal and $Q=\{x\in\widetilde{R}\mid xr_6^{\mathrm{st}}=0\}$.*
3. *One has (a) $P\cap Q=\widetilde{R}[I]$ and (b) $P\cap Q=I'$.*

Proof. 1. Note first that $\widetilde{R}/P$ is isomorphic to $\mathcal{O}[[a,b,c,\beta]]/((q-1+a)a+bc)$. Since this is a domain, P is a prime ideal. Next, observe that α and β annihilate r_4^{st} as follows by considering columns 2 and 6 in the relation matrix A' in Corollary 5.10. It remains to show that $P=(\alpha,\beta)$ contains $\{x\in\widetilde{R}\mid xr_4^{\mathrm{st}}=0\}$. So suppose that $xr_4^{\mathrm{st}}=0$ in $\widetilde{R}$. The main observation is that r_4^{st} mod $P=(q-1+a)\gamma$ is a nonzero element in the domain $\widetilde{R}/P$. Therefore, x mod P is zero and thus $x\in P$, as had to be shown. The proof of 2. is completely parallel to that of 1. and left to the reader.

From the definition of P, Q and I' it is clear that $I'\subset P\cap Q$. It is also straight forward to see from the columns of A' that I' annihilates r_5^{st} (multiply the first column and the fifth column by c or by $(q-1+a)$, and use 1.; alternatively, multiply the forth and the eighth column by α and β, and use 2.). We shall now prove 3(b), and from this and what we already proved, 3(a) will follow.

To see 3(b), let x be in $P\cap Q$. Write $x=f_1\alpha+f_2\beta$. To show that x lies in $I'\subset P\cap Q$, we may subtract from x arbitrary elements in I'. Writing elements in $\mathcal{R}$ as power series over $\mathcal{O}$ in $q-1+a,b,c,\alpha,\beta,\gamma$, we may thus assume $f_1,f_2\in(\alpha,\beta,\gamma,b)$. Shifting multiples of α in f_2 to αf_1, we may further assume $f_2\in(\beta,\gamma,b)$ and using r_1^{st}, we can replace $\beta\gamma$ by α^2, and this finally allows us to assume that f_2 lies in (b,β). We now reduce $x\in P\cap Q$ modulo Q. This yields $f_1\alpha+f_2\beta=0$ in $\mathcal{O}[[\alpha,\beta,\gamma,b]]/(\alpha^2+\beta\gamma)$. In other words, we can find $f_3\in\mathcal{R}':=\mathcal{O}[[\alpha,\beta,\gamma,b]]$ such that

$$f_1\alpha+f_2\beta+f_3(\alpha^2+\beta\gamma)=0 \text{ in } \mathcal{R}'.$$

Reducing modulo α and using $f_2\in(b,\beta)$ it follows that γ had to divide f_2 and hence that $f_2=0$. Since $\mathcal{R}'$ is a UFD it follows that $r_1^{\mathrm{st}}=\alpha^2+\beta\gamma$ divides f_1 and hence that $f_1=0$ in $\widetilde{R}$. Hence, we proved that x lies in I'. □

Corollary 5.15. *Let e be the ramification index of E over $\mathbf{Q}_l$. Then $\widetilde{\lambda}(\widetilde{R}[I])=(q-1)t\subset\mathcal{O}$ and $c_{1,\lambda}(R_v^{\mathrm{st}})=\frac{1}{e}\log_p(\mathcal{O}/(s,t,q-1))=\frac{n_v}{e}$.*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Proof. In Lemma 5.14, we identified $\widetilde{R}[I]$ with I'. The image of I' under λ is simply $((q-1)t)$. Invoking also Corollary 5.11, we deduce

$$c_{1,\lambda}(R_v^{\mathrm{st}}) = \frac{1}{e}\cdot \log_p\left(\#((q-1)t)/((q-1)t(s,t,(q-1)))\right) = \frac{1}{e}\cdot \log_p\left(\#\mathcal{O}/(s,t,q-1)\right).$$

□

To complete the computation of $D_{1,\lambda}(R_v^{\mathrm{st}})$, we still have to compute the size of the cokernel of $\mathrm{Hom}_{R_v^{\mathrm{st}}}(\widehat{\Omega}_{R_v^{\mathrm{st}}}, E/\mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}}, E/\mathcal{O})$. Using the methods of [BKM21, §7.2] and its terminology, we need to compute the lattice $\widetilde{\Lambda} \subset \mathcal{O}^8$ that is the kernel of the natural surjection $\mathcal{O}^8 \cong \widehat{\Omega}_{\mathcal{R}/\mathcal{O}} \otimes^{\widetilde{\lambda}}_{\mathcal{R}} \mathcal{O} \to \widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes^{\widetilde{\lambda}}_{\widetilde{R}} \mathcal{O}$. The lattice $\widetilde{\Lambda}$ is contained in Λ^{st}, and the cardinality wanted is $\#(\Lambda^{\mathrm{st}}/\widetilde{\Lambda})$.

Lemma 5.16. *The lattice $\widetilde{\Lambda} \subset \mathcal{O}^8$ is spanned by the rows of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & s & q-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & q-1 & 0 & 0 & 0 \end{pmatrix}.$$

and the quotient $\Lambda^{\mathrm{st}}/\widetilde{\Lambda}$ as an $\mathcal{O}$-module is isomorphic to $(s,t,q-1)/(t) \times (s,t,q-1)/(q-1)$.

Proof. In the notation of [BKM21, §7.3], the ring $\widetilde{R}$ is given as $\mathcal{O}[[a,b,c,e,\alpha,\beta,\gamma,\delta]]$ modulo the relations $a-e, \alpha+\delta, \alpha\delta-\beta\gamma, (q-1+a)e+bc, (q-1+a)\delta-c\beta$. The spanning vectors of $\widetilde{\Lambda}$ are then the image of the Jacobian matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \delta & -\gamma & -\beta & \alpha \\ e & c & b & q-1+a & 0 & 0 & 0 & 0 \\ \delta & 0 & -\beta & 0 & 0 & -c & 0 & q-1+a \end{pmatrix}$$

under the augmentation $\widetilde{\lambda}$. The matrix displayed in the assertion of the lemma is obtained from this image after some simple row operations. By [BKM21, §7.2], the lattice Λ^{st} is spanned by the rows of

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & (s,t,q-1) & 0 \\ 0 & 0 & t & 0 & q-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (t,q-1) & \frac{s}{t}(t,q-1) & 0 & 0 & 0 \end{pmatrix}$$

if $\mathrm{ord}_\varpi(s) \geq \mathrm{ord}_\varpi(t)$; and in the other case, the last two rows have to be replaced by

$$\begin{pmatrix} 0 & 0 & -s & q-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{t}{s}(s,q-1) & (s,q-1) & 0 & 0 & 0 \end{pmatrix}.$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

In both cases, it is easy to express the basis spanning $\widetilde{\Lambda}$ in terms of the basis spanning Λ^{st}, by an upper triangular transition matrix over $\mathcal{O}$ diagonal entries $1,1,1,\frac{t}{(s,t,q-1)},\frac{(q-1)}{(s,t,q-1)}$. The assertions of the lemma are now clear. $\square$

Corollary 5.17. *We have* $D_{1,\lambda}(R_v^{\mathrm{st}}) = \frac{1}{e}\cdot\log_p \#(\mathcal{O}/(s,t,q-1))^3 = 3\frac{n_v}{e}$.

Proof. From Lemma 5.16, the observations preceding it and from Theorem 3.23, we have

$$\#\ker(\mathrm{Hom}_{R_v^{\mathrm{st}}}(I/I^2,E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{st}},E/\mathcal{O})) = \#\Lambda^{\mathrm{st}}/\widetilde{\Lambda} = \#(s,t,q-1)^2/(t(q-1))$$

In Corollary 5.13 we computed $\#\mathrm{Hom}_{R_v^{\mathrm{st}}}(I/I^2,E/\mathcal{O}) = \#\mathcal{O}/(t(q-1)(s,t,q-1))$. Forming the quotient, the result follows from Theorem 3.23. $\square$

From $\delta_\lambda(R_v^{\mathrm{st}}) = D_{1,\lambda}(R_v^{\mathrm{st}}) - c_{1,\lambda}(R_v^{\mathrm{st}})$ and Corollaries 5.15 and 5.17, we deduce.

Theorem 5.18. *We have* $\delta_\lambda(R_v^{\mathrm{st}}) = 2\frac{n_v}{e}$.

5.3. Unipotent deformations with a choice of Frobenius at trivial primes

In the following, $\underline{s}$ and $\underline{t}$ will denote indeterminates that we shall specialize to s and t, respectively, whenever we pass to $\mathcal{O}$-algebras. Recall the expressions $r_i^{\varphi\text{-uni}}$ from Lemma 5.2, and observe that in the following we regard them as elements in the ring $\mathcal{R}_{\mathbf{Z}} = \mathbf{Z}[\underline{q},a,b,c,X,\alpha,\beta,\gamma] \subset \mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]$, replacing any occurrence of q by the indeterminate $\underline{q}+1$. Set

$$\begin{aligned}
s_1^{\varphi\text{-uni}} &= r_9^{\varphi\text{-uni}} + r_6^{\varphi\text{-uni}},\\
s_2^{\varphi\text{-uni}} &= r_8^{\varphi\text{-uni}} - r_7^{\varphi\text{-uni}} + r_2^{\varphi\text{-uni}},\\
s_3^{\varphi\text{-uni}} &= r_5^{\varphi\text{-uni}},\\
s_4^{\varphi\text{-uni}} &= r_2^{\varphi\text{-uni}} + ar_3^{\varphi\text{-uni}} + r_4^{\varphi\text{-uni}} + ar_6^{\varphi\text{-uni}} - br_7^{\varphi\text{-uni}} - r_3^{\varphi\text{-uni}},\\
s_4'^{\varphi\text{-uni}} &= ar_3^{\varphi\text{-uni}} + r_4^{\varphi\text{-uni}} + ar_6^{\varphi\text{-uni}} - br_7^{\varphi\text{-uni}} - r_3^{\varphi\text{-uni}},
\end{aligned}$$

and also

$$\widetilde{\mathcal{I}} = (s_1^{\varphi\text{-uni}},s_2^{\varphi\text{-uni}},s_3^{\varphi\text{-uni}},s_4^{\varphi\text{-uni}}) \text{ and } \widetilde{\mathcal{I}}' = (s_1^{\varphi\text{-uni}},s_2^{\varphi\text{-uni}},s_3^{\varphi\text{-uni}},s_4'^{\varphi\text{-uni}}). \quad (5.4)$$

The next result summarizes some explicit computations done via Macaulay2.

Lemma 5.19.

1. *The ring*

$$\mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]/(\underline{q},\underline{s},\underline{t},b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\dots,4)$$

is free over $\mathbf{Z}$ *of rank* 16. *The same holds if we replace* $s_4^{\varphi\text{-uni}}$ *by* $s_4'^{\varphi\text{-uni}}$. *A basis is* $1,a,aX,aX\alpha,a\alpha,b,b\alpha,X,X^2,X^2\alpha,X\alpha,X\alpha^2,X\alpha^3,\alpha,\alpha^2,\alpha^3$. *A basis of the socle of the ring modulo any prime is* $X\alpha^3$.

2. *The ring* $\mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]/((\underline{q},\underline{s},\underline{t},b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X)+\mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}})$ *is free over* $\mathbf{Z}$ *of rank* 6. *A basis is* $1,a,b,bX,X,\alpha$. *A basis of the socle of the ring modulo any prime is* $X\alpha$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

3. *Write* $x_1,\dots,x_7$ *for* $a,b,c,X,\alpha,\beta,\gamma$. *Then the ideal in* $\mathbf{Z}[\underline{q},\underline{s},\underline{t}]$ *generated by the* 4×4*-minors of the Jacobian* $(\partial s_i^{\varphi\text{-uni}}/\partial x_j)_{i=1,\dots,4;j=1,\dots,7}$ *evaluated at* $(x_1,\dots,x_7)=(0,\underline{s},0,0,0,\underline{t},0)$ *is* $(\underline{s}-\underline{t})\underline{t}^2(\underline{q},\underline{s},\underline{t})$. *If one replaces* $s_4^{\varphi\text{-uni}}$ *by* $s_4'^{\varphi\text{-uni}}$, *the resulting ideal is* $\underline{st}^2(\underline{q},\underline{s},\underline{t})$.

Remark 5.20. We note that the number 16 in part 1 is optimal. After reducing the number of variables by those relations that are linear, the $s_i^{\varphi\text{-uni}}$ are quadratic relations of a polynomial ring over $\mathbf{Z}$ in 4 variables. Now, the intersection of four quadrics in general position consists of 16 points. Therefore, dimension 16 for the coordinate ring of the corresponding scheme is optimal.

Let $s,t\in\mathfrak{m}$ with $t\neq 0$.

Corollary 5.21.

1. *The ring* $\widetilde{R}=\mathcal{O}[[a,b,c,X,\alpha,\beta,\gamma]]/(s_i^{\varphi\text{-uni}},i=1,\dots,4)$ *is a complete intersection, flat over* $\mathcal{O}$ *and of relative dimension* 3, *and this also holds with* $s_4^{\varphi\text{-uni}}$ *replaced by* $s_4'^{\varphi\text{-uni}}$. *One has a natural surjection* $\widetilde{R}\to R_v^{\varphi\text{-uni}}$ *induced from the inclusion of ideals* $(s_i^{\varphi\text{-uni}},i=1,\dots,4)\subset(r_j^{\varphi\text{-uni}},j=1,\dots,9)$.
2. *Via the ring map* $S=\mathcal{O}[[y_1,y_2,y_3]]\to\widetilde{R}$ *given by* $y_1\mapsto b-s-c, y_2\mapsto\beta-t-c, y_3\mapsto\gamma-X$, *the rings* $\widetilde{R}$ *and* $R_v^{\varphi\text{-uni}}$ *are free* S*-modules of rank* 16 *and* 6, *respectively (for either choice of* $\widetilde{R}$*).*
3. *The augmentation* $\widetilde{\lambda}\colon\widetilde{R}\to\mathcal{O}$ *given by* $a,c,X,\alpha,\gamma\mapsto 0$, $b\mapsto s$ *and* $\beta\mapsto t$ *defines a formally smooth point of* $\operatorname{Spec}\widetilde{R}[\frac{1}{\varpi}]$, *for at least one of the two choices of* $\widetilde{R}$ *from 1, provided that* $t\in\mathcal{O}\setminus\{0\}$.

Proof. The quotient $\widetilde{R}/(\varpi,b-s-c,\beta-t-c,\gamma-X)$ is isomorphic to the ring from Lemma 5.19.1 tensored with k over $\mathbf{Z}$ – since the latter ring is free of rank 16 over $\mathbf{Z}$, no completion is necessary. This implies that $(\varpi,b-s-c,\beta-t-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\dots,4)$ is a regular sequence in $\mathcal{R}$ with quotient a k-algebra of k-dimension 16. We deduce part 1 and the first half of part 2. The second half of part 2 uses Lemma 5.19.2 in an analogous way.

To prove part 3, observe that not both, s and $s+t$ can be zero since otherwise $t=0$ which is ruled out. So we choose $s_4^{\varphi\text{-uni}}$ or $s_4'^{\varphi\text{-uni}}$ accordingly. Then we evaluate the ideal in Lemma 5.19.3 at the made choice. This gives either the nonzero value $(s-t)t^2$ $\gcd(t,s,q-1)$ or $st^2\gcd(t,s,q-1)$ for a generator of the corresponding ideal over $\mathcal{O}$. This implies the stated formal smoothness. $\square$

Our aim is to compute $D_{1,\lambda}(R_v^{\varphi\text{-uni}})$ and $c_{1,\lambda}(R_v^{\varphi\text{-uni}})$. Instead, we shall compute these invariants for the ring $R_v^{\varphi\text{-uni}}\otimes_S\mathcal{O}$, where S is the ring from Corollary 5.21 and where the map $S\to\mathcal{O}$ is the augmentation $\widetilde{\lambda}$ composed with $S\to\widetilde{R}$. This is allowed due to Theorems 3.9 and 3.20.[9] It is probably not strictly necessary to perform this base change. However, it seems easier to work with Gorenstein and complete intersection rings that are finite flat

[9]Our choices $S\to\widetilde{R}\to R_v^{\varphi\text{-uni}}$ are almost certainly unrelated to any choices that arise from the Taylor–Wiles–Kisin patching.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

over $\mathcal{O}$. In particular, this will allow us to (have Macaulay2) compute structural constants of these rings, namely their multiplication tables in a given $\mathcal{O}$-bases. In the remainder of this subsection, we consider the rings

$$\mathcal{O} \longrightarrow \overline{R} = \widetilde{R}/(y_1,y_2,y_3) \xrightarrow{\overline{\pi}} \overline{R}_v^{\varphi\text{-uni}} = R_v^{\varphi\text{-uni}}/(y_1,y_2.y_3),$$

and we let $\overline{I}$ be the kernel of $\overline{\pi}: \overline{R} \to \overline{R}_v^{\varphi\text{-uni}}$.

We first explain the part that for us was the most difficult one, namely the computation of $\overline{R}[\overline{I}]$. Let $(b_i)_{i=1,\ldots,16}$ be an $\mathcal{O}$-basis of $\overline{R}$ such that $(b_i)_{i=7,\ldots,16}$ is a basis of the kernel of $\overline{R} \to \overline{R}_v^{\varphi\text{-uni}}$. Suppose further that b_6 and b_{16} are chosen so that they reduce to a generators of the socle of the finite Gorenstein rings $\overline{R}_v^{\varphi\text{-uni}}/(\varpi)$ and $\overline{R}/(\varpi)$, respectively; this is always possible. Denote by $(b_i^*)_{i=1,\ldots,16}$ the dual basis. It follows from Proposition 5.39 that b_6^* is a generator of $\mathrm{Hom}_{\mathcal{O}}(\overline{R}_v^{\varphi\text{-uni}},\mathcal{O})$ as a free $\overline{R}_v^{\varphi\text{-uni}}$-module, and b_{16}^* of $\mathrm{Hom}_{\mathcal{O}}(\overline{R},\mathcal{O})$ as a free $\overline{R}$-module. Denote by Θ the isomorphism

$$\Theta\colon \overline{R} \to \mathrm{Hom}_{\mathcal{O}}(\overline{R},\mathcal{O}), f \mapsto (b_{16}^*(f\cdot), g \mapsto b_{16}^*(fg)),$$

and consider the chain of isomorphisms

$$\overline{R}[\overline{I}] \cong \mathrm{Hom}_{\overline{R}}(\overline{R}_v^{\varphi\text{-uni}},\overline{R}) \cong \mathrm{Hom}_{\overline{R}}(\overline{R}_v^{\varphi\text{-uni}},\mathrm{Hom}_{\mathcal{O}}(\overline{R},\mathcal{O})) \cong \mathrm{Hom}_{\mathcal{O}}(\overline{R}_v^{\varphi\text{-uni}} \otimes_{\overline{R}} \overline{R},\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\overline{R}_v^{\varphi\text{-uni}},\mathcal{O})$$

from Lemma 3.14. The generator b_6^* on the right is successively mapped to, first $h_1 \otimes h_2 \mapsto b_6^*(h_1 \cdot \overline{\pi}(h_2))$, second $(h_1 \mapsto (h_2 \mapsto b_6^*(h_1 \cdot \overline{\pi}(h_2)))$, third $(h_1 \mapsto \Theta^{-1}(h_2 \mapsto b_6^*(h_1 \cdot \overline{\pi}(h_2)))$, lastly to

$$\Theta^{-1}(h_2 \mapsto b_6^*(\overline{\pi}(h_2))) = \Theta^{-1}(h_2 \mapsto b_6^*(h_2)) = \Theta^{-1} \circ b_6^*.$$

Now, write $\Theta^{-1} \circ b_6^* = \sum_i \mu_i b_i$ with $\mu_i \in \mathcal{O}$. By the definition of Θ, this is equivalent to $b_6^*(f) = b_{16}^*(\sum_i \mu_i b_i f)$ for all $f \in \overline{R}$. Let $c_{ijk} \in \mathcal{O}$ be the structural constants for multiplication in $\overline{R}$ over $\mathcal{O}$ with respect to the basis (b_j) so that $b_i b_j = \sum_k c_{ijk} b_k$. Then substituting for f all basis elements of $\overline{R}$ over $\mathcal{O}$ gives

$$b_6^*(b_j) = b_{16}^*(\sum_i \mu_i b_i b_j) = b_{16}^*(\sum_{i,k} \mu_i c_{ijk} b_k) = \sum_i \mu_i c_{ij16}.$$

Let C be the matrix $(c_{ij16})_{i,j=1,\ldots,16}$. Then the row vector (μ_i) is given as the product $e_6 C^{-1}$ for e_6 the 6-th standard basis vector of the column vector space $\mathcal{O}^{16}$. To obtain C, consider the following commutative diagram

$$\begin{tikzcd}
\dfrac{\mathbf{Q}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]}{(b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\ldots,4)} \arrow[rr] & & \overline{R}[\frac{1}{\varpi}] \\
\dfrac{\mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]}{(b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\ldots,4)} \arrow[u] \arrow[d] \arrow[r, "\underline{s}\mapsto s{,}\underline{t}\mapsto t{,}\underline{q}\mapsto q-1"] & \overline{R} = & \dfrac{\mathcal{O}[[a,b,c,X,\alpha,\beta,\gamma]]}{(b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\ldots,4)} \arrow[u] \arrow[d] \\
\dfrac{\mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]}{(p,\underline{q},\underline{s},\underline{t},b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\ldots,4)} \arrow[rr] & & \overline{R}/(\varpi).
\end{tikzcd}$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Applying Nakayama's lemma to the right column, we see that the basis in Lemma 5.19 is an $\mathcal{O}$-basis of $\overline{R}$, and thus an E-basis of $\overline{R}[\frac{1}{\varpi}]$. The analogous diagram holds for $\overline{R}_v^{\varphi\text{-uni}}$ in place of $\overline{R}$. Macaulay2 computations give us the following lemma:

Lemma 5.22.

1. *The ring*
$$R_1 = \mathbf{Q}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]/(b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X,s_i^{\varphi\text{-uni}},i=1,\dots,4)$$
is free over $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 16 *with the same basis as that given in Lemma 5.19.1. The same holds if we replace* $s_4^{\varphi\text{-uni}}$ *by* $s_4'^{\varphi\text{-uni}}$.
2. *The ring* $R_2 = \mathbf{Q}[\underline{q},\underline{s},\underline{t},,a,b,c,X,\alpha,\beta,\gamma]/((b-\underline{s}-c,\beta-\underline{t}-c,\gamma-X)+\mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}})$ *is free over* $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 6 *with the same basis as that given in Lemma 5.19.2.*
3. *The kernel of the surjective ring homomorphism* $R_1 \to R_2$ *is free over* $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 10*.*

Thus, we can compute C as a matrix with entries in $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$, that is, before specialization. For this, we computed new basis elements $b_7,\dots,b_{16}$ that span $\ker(R_1 \to R_2)$. To our surprise, we found $\det C = 1$, and inverting C posed no problem. This allowed us to compute the tuples of μ_i and then the $\overline{R}$-generator $\Theta^{-1}(b_6^*)$ of $\overline{R}[\overline{I}]$. Under our augmentation, Macaulay2 evaluated it to $(\underline{s}-\underline{t})\underline{t}$ in $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$. This shows:

Corollary 5.23. *We have*
$$\overline{\lambda}(\overline{R}[\overline{I}]) = \begin{cases} ((s-t)t) \subset \mathcal{O}, & \text{if we work with } s_4^{\varphi\text{-uni}}, \text{ and} \\ (st) \subset \mathcal{O}, & \text{if we work with } s_4'^{\varphi\text{-uni}}. \end{cases}$$

The next steps are the computation of $\widetilde{\lambda}(\mathrm{Fitt}_0^{\widetilde{R}}(I))$ and of $\mathrm{Hom}_{R_v^{\varphi\text{-uni}}}(I/I^2,E/\mathcal{O})$. For this, we proceed essentially as in the Steinberg case, cf. Corollaries 5.11 and 5.13, except that we rely on Macaulay2. Namely, we compute the first two steps of a resolution of $\mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}}$, considered as an ideal of $\mathcal{R}_{\mathbf{Z}}[\underline{s},\underline{t}] = \mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]$. This results in a right exact sequence
$$\mathcal{R}_{\mathbf{Z}}[\underline{s},\underline{t}]^{26} \xrightarrow{A} \mathcal{R}_{\mathbf{Z}}[\underline{s},\underline{t}]^{9} \longrightarrow \mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}} \longrightarrow 0,$$
for some matrix A in $M_{9\times 26}(\mathcal{R}_{\mathbf{Z}}[\underline{s},\underline{t}])$ (with rather simple entries). We tensor the sequence over $\mathcal{R}_{\mathbf{Z}}[\underline{q},\underline{s}]$ with $R_3 = \mathcal{R}_{\mathbf{Z}}[\underline{q},\underline{s}]/\widetilde{\mathcal{I}}$, where $\widetilde{\mathcal{I}}$ (as well as $\widetilde{\mathcal{I}}'$) is defined in Equation (5.4). Now, observe that over R_3, the ideal $I_3 = \mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}} \otimes_{\mathcal{R}_{\mathbf{Z}}[\underline{q},\underline{s}]} R_3$ is generated by the elements $r_1^{\varphi\text{-uni}},r_2^{\varphi\text{-uni}},r_3^{\varphi\text{-uni}},r_6^{\varphi\text{-uni}},r_7^{\varphi\text{-uni}}$; because these five elements together with our generators of $\widetilde{\mathcal{I}}$ generate $\mathcal{I}_{\mathbf{Z}}^{\varphi\text{-uni}}$. So we extract a matrix $A' \in M_{5\times 26}(R_3)$, from the specialization of A under $\mathcal{R}_{\mathbf{Z}}[\underline{s},\underline{t}] \to R_3$, that gives a short exact sequence
$$R_3^{26} \xrightarrow{A'} R_3^{5} \longrightarrow I_3 \longrightarrow 0.$$
Specializing under $R_3 \to \mathbf{Z}[\underline{q},\underline{s},\underline{t}]$ via $a,c,\alpha,\gamma,X \mapsto 0$, $b \mapsto \underline{s}$, $\beta \mapsto \underline{t}$, and computing the ideal of the resulting 5×5-minors gives the ideal $(\underline{s},\underline{t},\underline{q})^3 \cdot (\underline{s}-\underline{t})\underline{t}$. If we work with $\widetilde{\mathcal{I}}'$ in place of $\widetilde{\mathcal{I}}$, the answer is $(\underline{s},\underline{t},\underline{q})^3 \cdot \underline{s}\underline{t}$. Continuing with the natural map $\mathbf{Z}[\underline{q},\underline{s},\underline{t}] \to \mathcal{O}$, and observing the computations in Corollaries 5.11 and 5.13, we find:

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Corollary 5.24. *We have*

$$\#\mathcal{O}/\widetilde{\lambda}(\mathrm{Fitt}_0^{\widetilde{R}}(I))$$
$$= \#\mathrm{Hom}_{R_v^{\varphi\text{-uni}}}(I/I^2, E/\mathcal{O}) = \begin{cases} \#\mathcal{O}/((s-t)t(s,t,q-1)^3), & \text{if we work with } \widetilde{\mathcal{I}}, \text{ and} \\ \#\mathcal{O}/(st(s,t,q-1)^3), & \text{if we work with } \widetilde{\mathcal{I}}'. \end{cases}$$

Finally, we had Macaulay2 work out the analog of Lemma 5.16 to determine the lattice $\Lambda^{\varphi\text{-uni}}$, which, as to be expected, is rather easy. Following the proof of Corollary 5.17, one finds.

Corollary 5.25. *We have*

$$\#\ker(\mathrm{Hom}_{R_v^{\varphi\text{-uni}}}(I/I^2, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\varphi\text{-uni}}, E/\mathcal{O})) = \#\Lambda^{\varphi\text{-uni}}/\widetilde{\Lambda}$$
$$= \begin{cases} \#(s,t,q-1)^3/((s-t)t), & \text{if we work with } \widetilde{\mathcal{I}}, \\ \#(s,t,q-1)^3/(st), & \text{if we work with } \widetilde{\mathcal{I}}'. \end{cases}$$

As in the Steinberg case, the following result is now an immediate consequence. It is independent of whether we use $\widetilde{\mathcal{I}}$ or $\widetilde{\mathcal{I}}'$.

Theorem 5.26. *Let e be the ramification index of E over $\mathbf{Q}_l$. Then we have*

1. $D_{1,\lambda}(R_v^{\varphi\text{-uni}}) = 6\frac{n_v}{e}$.
2. $c_{1,\lambda}(R_v^{\varphi\text{-uni}}) = 3\frac{n_v}{e}$.
3. $\delta_\lambda(R_v^{\varphi\text{-uni}}) = 3\frac{n_v}{e}$.

5.4. Unipotent deformations

This case we handled in the same way as the previous one via the use of Macaulay2 code. For this, we found a model over $\mathbf{Z}$ of a complete intersection cover of the (model of the) Gorenstein ring that we are interested in. We only indicate outcomes of some intermediate steps but give no further details. The steps are completely parallel to those in Subsection 5.3. We define $\widetilde{\mathcal{I}} = (s_1^{\mathrm{uni}}, s_2^{\mathrm{uni}}, s_3^{\mathrm{uni}}, s_4^{\mathrm{uni}})$, where

$$s_1^{\mathrm{uni}} = r_7^{\mathrm{uni}} + r_2^{\mathrm{uni}},\ s_2^{\mathrm{uni}} = r_4^{\mathrm{uni}} - r_2^{\mathrm{uni}},\ s_3^{\mathrm{uni}} = r_1^{\mathrm{uni}} + (q-1)(r_6^{\mathrm{uni}} - r_2^{\mathrm{uni}}),\ s_4^{\mathrm{uni}} = r_9^{\mathrm{uni}} - r_7^{\mathrm{uni}} - r_2^{\mathrm{uni}}.$$

Note that these elements of $\mathcal{I}^{\mathrm{uni}}$ have simple expressions modulo $\varpi, b-c, b-\beta, X-\gamma$, namely $(s_1^{\mathrm{uni}}, \dots, s_4^{\mathrm{uni}}) \equiv (b^2, \alpha^2, X^2 + a\alpha, X(1+a) + a^2)$. We chose them as lifts of reductions that are well understood if, for instance, one follows the proof of Lemma 5.4. The challenge is to find lifts so that in particular the properties stated in Lemma 5.27 and 5.30 hold. Once suitable candidates for the lifts are guessed, we use Macaulay2 to verify these properties and to compute the quantities collected in Proposition 5.31.

Lemma 5.27.

1. *The ring*

$$\mathbf{Z}[\underline{q}, \underline{s}, \underline{t}, a, b, c, X, \alpha, \beta, \gamma]/(\underline{q}, \underline{s}, \underline{t}, b - \underline{s} - \beta + \underline{t}, X - \gamma, b - \underline{s} - c, s_i^{\mathrm{uni}}, i = 1, \dots, 4)$$

   is free over $\mathbf{Z}$ of rank 16. A basis is $1, a, ab, abX, abX\alpha, ab\alpha, aX, aX\alpha, a\alpha, b, bX, bX\alpha, b\alpha, X, X\alpha, \alpha$. A basis of its socle over $\mathbf{Z}$ is $abX\alpha$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

2. *The ring*

$$\mathbf{Z}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]/((\underline{q},\underline{s},\underline{t},b-\underline{s}-\beta+\underline{t},X-\gamma,b-\underline{s}-c)+\mathcal{I}_{\mathbf{Z}}^{\mathrm{uni}})$$

is free over $\mathbf{Z}$ *of rank* 6*. A basis is* $1,a,b,b\alpha,X,\alpha$*. A basis of its socle over* $\mathbf{Z}$ *is* $b\alpha$*.*

3. *Write* $x_1,\dots,x_7$ *for* $a,b,c,X,\alpha,\beta,\gamma$*. Then the ideal in* $\mathbf{Z}[\underline{q},\underline{s},\underline{t}]$ *generated by the* 4×4*-minors of the Jacobian* $(\partial s_i^{\mathrm{uni}}/\partial x_j)_{i=1,\dots,4;j=1,\dots,7}$ *evaluated at* $(x_1,\dots,x_7)=(0,\underline{s},0,0,0,\underline{t},0)$ *is* $\underline{q}^2(\underline{s}-\underline{t})\underline{t}(\underline{q},\underline{s},\underline{t})$*.*

Remark 5.28. As the observant reader will have noted, the evaluation in Lemma 5.27.3 may lead to zero if $s=t$ under our standard hypothesis $t\neq 0$. We will explain in Remark 5.32 on how to modify $\widetilde{\mathcal{I}}$ (by changing s_3^{uni}) so that our computations are also valid in the case $s=t$.

Let $s,t\in\mathfrak{m}$ with $t\neq 0$ and $s\neq t$.

Corollary 5.29.

1. *The ring* $\widetilde{R}=\mathcal{O}[[a,b,c,X,\alpha,\beta,\gamma]]/(s_i^{\mathrm{uni}},i=1,\dots,4)$ *is a complete intersection, flat over* $\mathcal{O}$ *and of relative dimension* 3*. One has a natural surjection* $\widetilde{R}\to R_v^{\mathrm{uni}}$ *induced from* $(s_i^{\mathrm{uni}},i=1,\dots,4)\subset(r_j^{\mathrm{uni}},j=1,\dots,9)$*.*
2. *Via the ring map* $S=\mathcal{O}[[y_1,y_2,y_3]]\to\widetilde{R}$ *given by* $y_1\mapsto b-\underline{s}-\beta+\underline{t}$*,* $y_2\mapsto X-\gamma$*,* $y_3\mapsto b-\underline{s}-c$*, the rings* $\widetilde{R}$ *and* R_v^{uni} *are free* S*-modules of rank* 16 *and* 6*, respectively.*
3. *The augmentation* $\widetilde{\lambda}\colon\widetilde{R}\to\mathcal{O}$ *given by* $a,c,X,\alpha,\gamma\mapsto 0$*,* $b\mapsto s$ *and* $\beta\mapsto t$ *defines a formally smooth point of* $\operatorname{Spec}\widetilde{R}[\frac{1}{\varpi}]$*.*

A further Macaulay2 shows the following:

Lemma 5.30.

1. *The ring*

$$R_1=\mathbf{Q}[\underline{q},\underline{s},\underline{t},a,b,c,X,\alpha,\beta,\gamma]/(b-\underline{s}-\beta+\underline{t},X-\gamma,b-\underline{s}-c,s_i^{\mathrm{uni}},i=1,\dots,4)$$

is free over $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 16 *with the same basis as that given in Lemma 5.27.1.*

2. *The ring*

$$R_2=\mathbf{Q}[\underline{q},\underline{s},\underline{t},,a,b,c,X,\alpha,\beta,\gamma]/((b-\underline{s}-\beta+\underline{t},X-\gamma,b-\underline{s}-c)+\mathcal{I}_{\mathbf{Z}}^{\mathrm{uni}})$$

is free over $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 6 *with the same basis as that given in Lemma 5.27.2.*

3. *The kernel of the surjective ring homomorphism* $R_1\to R_2$ *is free over* $\mathbf{Q}[\underline{q},\underline{s},\underline{t}]$ *of rank* 10*.*

Computations as for Corollaries 5.23, 5.24 and 5.25 give the following result:

Proposition 5.31. *We have*

1. $\widetilde{\lambda}(\widetilde{R}[\widetilde{I}])=(s-t)(q-1)^2$*.*
2. $\#\mathcal{O}/\widetilde{\lambda}(\mathrm{Fitt}_0^{\widetilde{R}}(I))=\#\mathrm{Hom}_{R_v^{\mathrm{un}}}(I/I^2,E/\mathcal{O})=\#\mathcal{O}/((q-1)^2(s-t)\gcd(s,t,q-1))$*.*
3. $\#\ker(\mathrm{Hom}_{R_v^{\mathrm{uni}}}(I/I^2,E/\mathcal{O})\to\widehat{\mathrm{Der}}_{\mathcal{O}}^1(R_v^{\mathrm{uni}},E/\mathcal{O}))=\#\Lambda^{\mathrm{uni}}/\widetilde{\Lambda}=\#(\gcd(s,t,q-1)/(s-t)(q-1)^2)$*.*

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Remark 5.32. In the case $s = t$ (but $t \neq 0$), one can replace s_3^{uni} by $s_3^{\mathrm{uni}} + r_2^{\mathrm{uni}}$ and work with the modified ideal $\widetilde{\mathcal{I}}$. Then the results in Subsection 5.4 hold with the following modifications (where in all calculations we used $s = t$ and the modified $\widetilde{\mathcal{I}}$):

1. In Lemma 5.27, in part 1, the basis is $1, a, ab, ab\alpha, aX, aX^2, aX^2\alpha, aX\alpha, a\alpha, b, b\alpha, X, X^2, X^2\alpha, X\alpha, \alpha$ and the socle is $aX^2\alpha$, and in part 3, the evaluation gives the value $(q-1)t^2 \gcd(q-1,t)$.
2. In Proposition 5.31, one has
   (a) $\widetilde{\lambda}(\widetilde{R}[\widetilde{I}]) = (t(q-1))$.
   (b) $\#\mathcal{O}/\widetilde{\lambda}(\mathrm{Fitt}_0^{\widetilde{R}}(I)) = \#\mathrm{Hom}_{R_v^{\mathrm{un}}}(I/I^2, E/\mathcal{O}) = \#\mathcal{O}/((q-1)t\gcd(t,q-1))$.
   (c) $\#\ker(\mathrm{Hom}_{R_v^{\mathrm{uni}}}(I/I^2, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{uni}}, E/\mathcal{O})) = \#\Lambda^{\mathrm{uni}}/\widetilde{\Lambda} = \#(\gcd(t, q-1)/(t(q-1))$.

Theorem 5.33. *Let e be the ramification index of E over $\mathbf{Q}_l$, and let $t \in \mathfrak{m} \setminus \{0\}$. Then we have*

1. $D_{1,\lambda}(R_v^{\mathrm{uni}}) = 2\frac{n_v}{e}$.
2. $c_{1,\lambda}(R_v^{\mathrm{uni}}) = \frac{n_v}{e}$.
3. $\delta_\lambda(R_v^{\mathrm{uni}}) = \frac{n_v}{e}$.

5.5. Recollections about Cohen–Macaulay and Gorenstein rings

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. In this subsection, we want to briefly recall some results on Cohen–Macaulay and Gorenstein rings that occur repeatedly in our arguments or, more importantly, in our computations. We also present a result on generating sets of dual modules that was useful in our computations. For basic notions such as depth, R-sequence, Cohen–Macaulay and Gorenstein rings, we refer to [BH93, §§1.2, 2.1, 3.1],

Definition 5.34. The socle of R is defined as $\mathrm{socle}\, R = R[\mathfrak{m}] = \{x \in R \mid \mathfrak{m}x = 0\}$.

Proposition 5.35 [Mat80, Thm. 17.4 and p. 136], [BH93, 2.1.3, 2.1.8, 3.1.19].

1. *Any local Artin ring R is Cohen–Macaulay. It is Gorenstein if in addition it satisfies* $\mathrm{socle}\, R \cong k$.
2. *If R is Noetherian local, and if $(x_1 \ldots, x_n)$ is an R-sequence in $\mathfrak{m}$, then R is Cohen–Macaulay or Gorenstein, respectively, if and only if $R/(x_1, \ldots, x_n)$ has this property. In particular, if $R/(x_1 \ldots, x_n)$ is Artinian, then R is Cohen–Macaulay, and if moreover* $\mathrm{socle}\, R/(x_1 \ldots, x_n) \cong k$, *then R is Gorenstein.*
3. *If R is a local Cohen–Macaulay ring, then any system of parameters is a regular R-sequence.*

Let now $(A, \mathfrak{m})$ be a local Artin ring. In this case, $I \cap \mathrm{socle}\, A \supsetneq 0$ for any nonzero ideal I of A: To see this, consider $n \in \mathbf{Z}_{\geq 0}$ such that $\mathfrak{m}^{n-1} I \neq 0$ and $\mathfrak{m}^n I = 0$. Then $\mathfrak{m}^{n-1} I \subset I \cap \mathrm{socle}\, A$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Lemma 5.36. *Let $(\bar{\psi}_i)_{i\in B}$ be a finite tuple in* $\mathrm{Hom}_k(A,k)$ *such that* $(\bar{\psi}_i)_{i\in B}\colon \mathrm{socle}(A)\to k^B$ *is injective. Then $(\bar{\psi}_i)_{i\in B}$ is a set of generators of* $\mathrm{Hom}_k(A,k)$ *as an A-module. In particular, if A is Gorenstein and if $B=\{0\}$ is a singleton, then $\bar{\psi}_0$ is an A-basis of* $\mathrm{Hom}_k(A,k)$.

The proof relies on the following result from linear algebra.

Lemma 5.37. *Let V be a finite-dimensional k-vector space. Let $(V_j)_{j\in J}$ be a finite tuple of sub vector spaces such that $\bigcap_{j\in J}V_j=0$. Then for any $\bar{\psi}\in\mathrm{Hom}_k(V,k)$, there exist $\bar{\psi}_j\in\mathrm{Hom}_k(V,k)$ with $V_j\subset\ker\bar{\psi}_j$ for $j\in J$ such that $\bar{\psi}=\sum_{j\in J}\bar{\psi}_j$.*

Proof. We may assume $J=\{1,\dots,t\}$ for some $t\in\mathbf{Z}_{\geq 1}$. We induct over t, noting that the case $t=1$ is trivial, since then $V_1=0$. For the induction step suppose $t\geq 2$, and let $W=\bigcap_{j=2}^{t}V_j$. Then $V_1\cap W=0$, and so we can choose a basis for W and one for V_1 and then extend the one for V_1 to a complementary basis to that of W. Then one can find $\bar{\psi}_1$ and $\bar{\phi}$ in $\mathrm{Hom}_k(V,k)$ such that $\ker\bar{\psi}_1\supseteq V_1$ and $\ker\bar{\phi}\supseteq W$, and $\bar{\psi}=\bar{\psi}_1+\bar{\phi}$. Now, apply the induction hypothesis to V/V_1 and $(V_j/V_1)_{j=2,\dots,t}$ and $\bar{\psi}_1$ considered as a map in $\mathrm{Hom}_k(V/V_1,k)$. $\square$

Proof of Lemma 5.36. Let $N=\sum_{i\in B}A\bar{\psi}_i$. We shall show that $\mathrm{Hom}_k(A,k)\subseteq N+\mathfrak{m}\,\mathrm{Hom}_k(A,k)$. Then the lemma will follow from Nakayama's lemma.

Let $\bar{\psi}$ be in $\mathrm{Hom}_k(A,k)$. By our hypothesis, there is a k-linear map $\alpha\colon k^B\to k$ such that the restriction $\bar{\psi}|_{\mathrm{socle}\,A}$ agree with $\alpha\circ(\bar{\psi}_i)_{i\in B}$. In other words, the map

$$\bar{\phi}:=\bar{\psi}-\sum_{i\in B}\alpha(\bar{e}_i))\bar{\psi}_i$$

vanishes on $\mathrm{socle}\,A$.

Next, let $x_1,\dots,x_t$ be a set of A-module generators of $\mathfrak{m}$, and let $V_i=\{r\in A\mid x_ir=0\}$. Then

$$\mathrm{socle}\,A=\bigcap_{i=1,\dots,t}V_i$$

By Lemma 5.37 applied to $A/\mathrm{socle}\,A$, there exist $\bar{\phi}_i\in\mathrm{Hom}_k(A,k)$ with $\ker\bar{\phi}_i\supset V_i$, and $\bar{\phi}=\sum_{i\in B}\bar{\phi}_i$.

Now, consider the short exact sequence $0\to V_i\to A\xrightarrow{x_i\cdot}x_iA\to 0$. Then the $\bar{\psi}_i$ induce k-linear maps $x_iA\to k$. The latter can be extended to k-linear maps $\bar{\xi}_i\colon A\to k$ under $x_iA\subset A$. In other words $\bar{\phi}_i=x_i\bar{\xi}_i$, and this gives

$$\bar{\psi}-\sum_{i\in B}\bar{\psi}(e_i)\bar{\psi}_i=\sum_{j=1,\dots,t}x_j\bar{\xi}_j,$$

proving the claim from the first line and hence the lemma. $\square$

Let now $(R,\mathfrak{m})$ be a local complete Noetherian Cohen–Macaulay ring that is an $\mathcal{O}$-algebra, and suppose that $\mathbf{r}=(\varpi,r_1,\dots,r_n)$ is a system of parameters. Let $(\bar{e}_i)_{i\in B}$ be a k-basis of $A=R/\mathbf{r}R$, let $(e_i)_{i\in B}$ be a tuple of preimages in R, and consider the $\mathcal{O}$-algebra homomorphism $S=\mathcal{O}[[x_1,\dots,x_n]]\to R,x_i\mapsto e_i$.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Lemma 5.38. *As an S-module, R is free with basis $(e_i)_{i=1,\dots,n}$.*

Proof. The ring S is regular local and thus of finite global dimension. Hence, R has finite projective dimension over S. By Nakayama's lemma R is also finitely generated as an S-module because $\dim_k R/\mathbf{r}R$ is finite for the system of parameters $\mathbf{r}$. The sequence $\mathbf{r}$ is in fact regular as R is Cohen–Macaulay. It follows that $\operatorname{depth}_S R = 1+n = \dim S$, so that by the Auslander–Buchsbaum theorem R is a finite free S-module. One finds that $\psi\colon S^B \to R, (s_i)_{i\in B} \mapsto \sum_i s_i e_i$ is an isomorphism because S is local and ψ mod $\mathbf{r}$ is bijective. □

The following result gives a generating set (or a basis) over R of the free S-module $\operatorname{Hom}_S(R,S)$.

Proposition 5.39. *Let $\psi_i \in \operatorname{Hom}_S(R,S)$, $i\in B$, be a tuple of elements such that the elements $\bar{\psi}_i := \psi_i \otimes_R A\colon A \to k$ satisfy the condition of Lemma 5.36. Then $(\psi_i)_{i\in B}$ is a set of R-module generators of $\operatorname{Hom}_S(R,S)$. If moreover R is Gorenstein and $B=\{0\}$ is a singleton, then ψ_0 is an R-basis of $\operatorname{Hom}_S(R,S)$.*

Proof. This is immediate from Nakayama's lemma and Lemma 5.36. □

6. Wiles defect of Hecke algebras and global deformation rings

In this section, we'll describe how the commutative algebra results from Sections 4 and 5 can be applied to Galois deformation rings, in the setup of Taylor–Wiles–Kisin patching. For ease of exposition we'll restrict our attention to the case of two-dimensional Galois representations over a totally real number field, and moreover ones that are modular of parallel weight 2, as all of the computations and applications we give in this paper will be concerned with this case. This is not a fundamental limitation on our methods, and indeed everything we describe in this section will generalize automatically to any '$\ell_0 = 0$' patching setup (such as the definite unitary groups considered by [CHT08] and others).

Let F be a totally real number field. Fix a finite set Σ of finite places of F. For each $v\in\Sigma$, fix a $\tau_v \in \{\text{min, st, uni, } \varphi\text{-uni, } \square\}$, let $\tau = (\tau_v)_{v\in\Sigma}$ and for $\sigma \in \{\text{min, st, uni, } \varphi\text{-uni, } \square\}$ write $\Sigma^\sigma = \{v\in\Sigma | \tau_v = \sigma\}$.

Pick a prime $p>2$ which is not ramified in F and is not divisible by any prime in Σ. Let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k. Let $\varepsilon_p : G_F \to \mathcal{O}^\times$ be the cyclotomic character. Let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be a Galois representation for which:

- ρ corresponds to a Hilbert modular form of parallel weight 2;
- $\det\rho = \varepsilon_p$;
- For every $v \notin \Sigma$ and $v \nmid p$, ρ is unramified at v;
- For every $v|p$, $\overline{\rho}|_{G_v}$ is finite flat;
- If $v\in\Sigma^{\min}$, then either $|\mathcal{O}/v| \not\equiv -1 \pmod{\ell}$, $\overline{\rho}|_{I_v}$ is irreducible or $\overline{\rho}|_{G_v}$ is absolutely reducible;
- If $v\in \Sigma^{\mathrm{st}}\cup\Sigma^{\mathrm{un}}\cup\Sigma^{\varphi\text{-uni}}$, then $\rho|_{G_v}$ is Steinberg (i.e., $\rho|_{G_v} \sim \begin{pmatrix} \chi\varepsilon_p & * \\ 0 & \chi \end{pmatrix}$ for some unramified quadratic character);

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

- The residual representation $\overline{\rho}: G_F \to \mathrm{GL}_2(k)$ is absolutely irreducible, and moreover that it satisfies the *Taylor–Wiles conditions:* $\overline{\rho}|_{G_{F(\zeta_p)}}$ is still absolutely irreducible, and in the case when $p = 5$, $\sqrt{5} \in F$ and the projective image $\mathrm{proj}\,\overline{\rho}: G_F \to \mathrm{PGL}_2(\overline{\mathbf{F}}_5)$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, that $\ker \mathrm{proj}\,\overline{\rho} \not\subseteq G_{F(\zeta_5)}$.

Let $Q = \Sigma^{\mathrm{st}}$, and let D be a quaternion algebra over F ramified at the primes in Q (and no other finite primes) and at either all, or all but one infinite place of F (depending on whether $|Q| + [F : \mathbf{Q}]$ is even or odd). Define a compact open subgroup $K^\tau = \prod_v K_v^\tau \subset (D \otimes \mathbf{A}_{F,f})^\times$ by:

- $K_v^\tau = \mathrm{GL}_2(\mathcal{O}_{F,v})$ if $v \notin \Sigma$;
- K_v^τ is a maximal compact subgroup of $(D \otimes F_v)^\times$ if $v \in \Sigma^{\mathrm{st}} = Q$;
- $K_v^\tau = U_0(v)$ if $v \in \Sigma^{\mathrm{un}} \cup \Sigma^{\varphi\text{-}\mathrm{uni}}$;
- $K_v^\tau = U_0(v^{a_v})$ if $v \in \Sigma^{\mathrm{min}}$, where a_v is the Artin conductor of $\overline{\rho}|_{G_v}$;
- $K_v^\tau = U_0(v^{a_v+2})$ if $v \in \Sigma^{\square}$.

For convenience, we will simply write $K = K^\tau$ and $K_v = K_v^\tau$.

When D is ramified at all but one infinite places (resp. all infinite places) let X_K be the Shimura curve (resp. Shimura set) associated to K. Let $\mathbf{T}^D(K)$ be the Hecke algebra acting on $H^1(X_K,\mathcal{O})$ in the Shimura curve case and on $H^0(X_K,\mathcal{O})$ in the Shimura set case, generated (as an $\mathcal{O}$-algebra) by the Hecke operators T_v and S_v for all finite primes $v \notin \Sigma$, and let $\overline{\mathbf{T}}^D(K) = \mathbf{T}^D(K)[U_v | v \in \Sigma^{\varphi\text{-}\mathrm{uni}}]$. Note that $\mathbf{T}^D(K)$ and $\overline{\mathbf{T}}^D(K)$ are finite $\mathcal{O}$-algebras.

Let $\mathbf{T}^D(K)^\varepsilon = \mathbf{T}^D(K)/(S_v - \varepsilon_p(\mathrm{Frob}_v) | v \notin \Sigma)$ and $\overline{\mathbf{T}}^D(K)^\varepsilon = \overline{\mathbf{T}}^D(K)/(S_v - \varepsilon_p(\mathrm{Frob}_v) | v \notin \Sigma)$ be the fixed determinant Hecke algebras.

The assumption that ρ corresponds to a Hilbert modular form of parallel weight 2 gives the following:

Proposition 6.1. *There is an augmentation $\lambda : \overline{\mathbf{T}}^D(K)^\varepsilon \twoheadrightarrow \mathcal{O}$ with the property that for any $v \notin \Sigma \cup \Sigma_p$, $\rho(\mathrm{Frob}_v)$ has characteristic polynomial $x^2 - \lambda(T_v)x + \lambda(S_v)$. Moreover, $\Phi_\lambda(\overline{\mathbf{T}}^D(K)^\varepsilon)$ is finite.*

Let $\mathfrak{m} = \lambda^{-1}(\varpi\mathcal{O}) \subseteq \overline{\mathbf{T}}^D(K)^\varepsilon$ be the maximal ideal of $\overline{\mathbf{T}}^D(K)^\varepsilon$ corresponding to $\overline{\rho}$. By slight abuse of notation, also write $\mathfrak{m} = \mathfrak{m} \cap \mathbf{T}^D(K)$ for the maximal ideal of $\mathbf{T}^D(K)$ corresponding to $\overline{\rho}$.

Write $\mathbf{T}^\tau = \mathbf{T}^D(K)^\varepsilon_\mathfrak{m}$ and $\overline{\mathbf{T}}^\tau = \overline{\mathbf{T}}^D(K)^\varepsilon_\mathfrak{m}$ for the localizations at $\mathfrak{m}$ (and note that we are suppressing ε from our notation).

Note that any $x : \overline{\mathbf{T}}^\tau \to \overline{\mathbf{Q}}_p$ corresponds to a Galois representation $\rho_x : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ lifting $\overline{\rho}$ with $\det \rho_x = \varepsilon_p = \det \rho$ and $\mathrm{tr}\,\rho_x(\mathrm{Frob}_v) = x(T_v)$ for all $v \notin \Sigma$ (so that $\rho = \rho_\lambda$).

Define $H^\tau = H^1(X_K,\mathcal{O})^*$ if D is indefinite and $H^\tau = H^0(X_K,\mathcal{O})^*$ if D is definite (where for any $\mathcal{O}$-module M, $M^* = \mathrm{Hom}_\mathcal{O}(M,\mathcal{O})$), viewed as a $\overline{\mathbf{T}}^D(K)$-module, and hence as a $\mathbf{T}^D(K)$-module. Define

$$M^\tau = \overline{\mathbf{T}}^\tau \otimes_{\overline{\mathbf{T}}^D(K)} H^\tau = \mathbf{T}^\tau \otimes_{\mathbf{T}^D(K)} H^\tau = H^\tau / ((S_v - \varepsilon_p(\mathrm{Frob}_v))x | v \notin \Sigma, x \in H^\tau). \quad (6.1)$$

For the convenience of the reader, we recall some notation and results from Sections 4 and 5. For each prime v of F, the universal (fixed determinant) ring,

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

parameterizing framed deformations of $\overline{\rho}|_{G_{F_v}}$ with determinant ε_p is $R_v^\square$. For $v \nmid p$ and $\tau_v \in \{\min, \mathrm{st}, \mathrm{uni}, \varphi\text{-uni}, \square\}$, let $R_v^{\tau_v}$ be the deformation ring defined in Section 5, provided it exists (which is does for $v \in \Sigma$ and $\tau = \tau_v$, by assumption). The ring $R_v^{\tau_v}$ is naturally an $R_v^\square$-algebra, and unless $\tau_v = \varphi$-uni it is a quotient of $R_v^\square$. Summarizing the results of Proposition 4.6 we have:

Proposition 6.2. *For each $v \in \Sigma$, the ring $R_v^{\tau_v}$ is a complete, Noetherian $\mathcal{O}$-algebra which is flat and equidimensional over $\mathcal{O}$ of relative dimension 3. Moreover, $R_v^{\tau_v}$ is Cohen–Macaulay and is a complete intersection whenever $\tau_v = \min$ or $\square$ or whenever $\overline{\rho}|_{G_v}$ is not a scalar.*

As in Section 4, let:

$$R_{\mathrm{loc}} = \left(\widehat{\bigotimes_{v\in\Sigma}} R_v^\square\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes_{v|p}} R_v^{\mathrm{fl}}\right), \qquad \text{and} \qquad R_{\mathrm{loc}}^\tau = \left(\widehat{\bigotimes_{v\in\Sigma}} R_v^{\tau_v}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes_{v|p}} R_v^{\mathrm{fl}}\right)$$

so that R_{loc}^τ is naturally a R_{loc}-algebra. By Propositions 4.6 and 4.7, R_{loc} is flat over $\mathcal{O}$ and Cohen–Macaulay.

By R (resp. $R^\square$), we denote the (global) unframed (resp. framed) deformation ring parameterizing lifts of $\overline{\rho}$ with determinant ε_p which are flat at every prime $v|p$. One may noncanonically fix an isomorphism $R^\square = R[[X_1,\dots,X_{4j-1}]]$ for some j, and thereby treat R as a quotient of $R^\square$. Using the natural map $R_{\mathrm{loc}} \to R^\square$ (and $R_{\mathrm{loc}} \to R$), one defines $R^{\square,\tau} = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}} R^\square$ and $R^\tau = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}} R$.

Lemma 6.3. *There is a surjective map $R^\tau \twoheadrightarrow \overline{\mathbf{T}}^\tau$ inducing a representation $\rho^\tau : G_F \to \mathrm{GL}_2(R^\tau) \twoheadrightarrow \mathrm{GL}_2(\mathbf{T}^\tau)$ such that for all $v \notin \Sigma \cup \Sigma_p$, $\rho^\tau(\mathrm{Frob}_v)$ has characteristic polynomial $t^2 - T_v t + S_v$, and for all $v \in \Sigma^{\mathrm{un}} \cup \Sigma^{\varphi\text{-uni}}$, $\rho^\tau|_{G_{F_v}}$ is unipotent and if $\mathrm{Frob}_v \in G_{F_v}$ is any lift of Frobenius, then $\rho^\tau(\mathrm{Frob}_v)$ again has characteristic polynomial $t^2 - T_v t + S_v$.*

Proof. If $\Sigma^{\varphi\text{-uni}} = \varnothing$, this is just [Man21, Lemma 2.4].

In general, for each for each $v \in \Sigma$, set $\sigma_v = \tau_v$ if $\tau_v \in \{\min, \mathrm{st}, \mathrm{uni}, \square\}$ and $\sigma_v = \mathrm{uni}$ if $\tau_v = \varphi$-uni. Note that under this definition, $K^\sigma = K^\tau = K$ and $\overline{\mathbf{T}}^\sigma = \mathbf{T}^\sigma = \mathbf{T}^\tau$.

It follows that there is a surjection $R^\sigma \twoheadrightarrow \overline{\mathbf{T}}^\sigma = \mathbf{T}^\sigma = \mathbf{T}^\tau$ satisfying the desired conditions on ρ^σ. By definition, $\overline{\mathbf{T}}^\tau = \mathbf{T}^\tau[U_v | v \in \Sigma^{\varphi\text{-uni}}]$. From the identity $U_v^2 - T_v U_v + S_v = 0$ in $\overline{\mathbf{T}}^D(K)$ and the definition of modified global deformation rings given in Section 4, it follows that $R^\sigma \twoheadrightarrow \mathbf{T}^\tau \to \overline{\mathbf{T}}^\tau$ induces a map $R^\tau \twoheadrightarrow \overline{\mathbf{T}}^\tau$ sending α_v to U_v for $v \in \Sigma^{\varphi\text{-uni}}$, which is therefore surjective, and hence is the desired map. $\square$

Now, similarly to [BKM21, Theorem 6.3], the Taylor–Wiles–Kisin patching method gives the following:

Theorem 6.4. *There exist integers $g,d \geq 0$ and rings*

$$R_\infty^\tau = R_{\mathrm{loc}}^\tau[[x_1,\dots,x_g]]$$
$$S_\infty = \mathcal{O}[[y_1,\dots,y_d]]$$

satisfying the following:

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

1. $\dim S_\infty = \dim R^\tau_\infty$.
2. *There exists a continuous $\mathcal{O}$-algebra morphism $i : S_\infty \to R^\tau_\infty$ making R^τ_∞ into a finite free S_∞-module.*
3. *There is an isomorphism $R^\tau_\infty \otimes_{S_\infty} \mathcal{O} \cong R^\tau$ of R^τ_{loc}-algebras, and R^τ is finite free over $\mathcal{O}$.*
4. *The map $R^\tau \twoheadrightarrow \overline{\mathbf{T}}^\tau$ from Lemma 6.3 is an isomorphism. These rings are reduced if $\Sigma^{\varphi\text{-uni}} = \varnothing$.*
5. *If λ is the induced map $R^\tau_\infty \twoheadrightarrow R^\tau \xrightarrow{\sim} \overline{\mathbf{T}}^\tau \xrightarrow{\lambda} \mathcal{O}$, then $\operatorname{Spec} R^\tau_\infty[1/\varpi]$ is formally smooth at the point corresponding to λ.*

Proof. This is proved similarly to Theorem 6.3 in [BKM21].

First, we will consider the case when $\Sigma^{\varphi\text{-uni}} = \varnothing$, and so $\overline{\mathbf{T}}^D(K) = \mathbf{T}^D(K)$. More precisely, as in the proof of Lemma 6.3, for each $v \in \Sigma$, define $\sigma_v = \tau_v$ if $\tau_v \in \{\mathrm{min}, \mathrm{st}, \mathrm{uni}, \square\}$ and define $\sigma_v = \mathrm{uni}$ if $\tau_v = \varphi\text{-uni}$. Note that under this definition, $K^\sigma = K^\tau = K$, $M^\sigma = M^\tau$ and $\overline{\mathbf{T}}^\sigma = \mathbf{T}^\sigma = \mathbf{T}^\tau$.

By assumption, $\overline{\rho}$ satisfies the Taylor–Wiles conditions, and so we may apply the Taylor–Wiles–Kisin patching method (as summarized in [Man21, Section 4]) to the rings R^σ and $\mathbf{T}^\sigma$ and the module M^σ.

First, as in [Man21, Section 4.2], we may add auxiliary level structure at a carefully chosen prime not in Σ to remove any isotropy issues, without affecting any of the objects considered considered in this theorem.

Now, exactly as in the proof of [BKM21, Theorem 6.3] (and the method outlined in [Man21, Section 4.3]), there exist integers $g,d \geq 0$, satisfying $d+1 = \dim R_{\mathrm{loc}} + g = \dim R^\sigma_{\mathrm{loc}} + g$ (see [Man21, Lemma 2.5] and [Kis09, Proposition (3.2.5)]) such that for each $n \geq 1$, there is a unframed global deformation ring R^σ_n and a framed global deformation ring $R^{\sigma,\square}_n$ (with fixed determinant, the same deformation conditions as R^σ at each $v \in \Sigma$, and relaxed deformation conditions at a carefully selected set Q_n of 'Taylor–Wiles' primes) such that $R^{\sigma,\square}_n$ has the structure of a S_∞-algebra and there is a surjective map $R_\infty \twoheadrightarrow R^{\sigma,\square}_n$ and an isomorphism $R^{\sigma,\square}_n \otimes_{S_\infty} \mathcal{O} \cong R^\sigma$, where S_∞ is as in the theorem statement, and R^σ_∞ satisfies the properties of R^τ_∞ from the theorem statement.

Moreover, for each $n \geq 1$ the construction in [Man21, Section 4.2] also constructs a compact open subgroup $K_n = \prod_v K_{n,v} \subseteq (D \otimes \mathbf{A}_{F,f})^\times$ (with $K_{n,v} = K_v$ for all $v \notin Q_n$), and a Hecke algebra $\mathbf{T}^\sigma_n$ and Hecke module M^σ_n at level K_n (defined analogously to $\mathbf{T}^\sigma$ and M^σ above, by localizing at a particular maximal ideal, and fixing determinants by taking a quotient). One then has a surjection $R^\sigma_n \twoheadrightarrow \mathbf{T}^\sigma_n$, making M^σ_n into a R^σ_n-module. Using this surjection, we may define framed versions of these objects: $\mathbf{T}^{\sigma,\square}_n = \mathbf{T}^\sigma_n \otimes_{R^\sigma_n} R^{\sigma,\square}_n$ and $M^{\sigma,\square}_n = M^\sigma_n \otimes_{R^\sigma_n} R^{\sigma,\square}_n$.

Applying the 'ultrapatching' construction described in [Man21, Section 4.1] (as well as in the proof of Lemma 4.8) then produces an S_∞-algebra $\mathcal{R}^\sigma_\infty$ as well as an $\mathcal{R}_\infty$-module M^σ_∞ (which would be called $\mathscr{P}(\{R^{\sigma,\square}_n\})$ and $\mathscr{P}(\{M^{\sigma,\square}_n\})$ in the notation of that paper), for which:

- M^σ_∞ is finite free over S_∞;
- $\mathcal{R}^\sigma_\infty \otimes_{S_\infty} \mathcal{O} \cong R^\sigma$ and $M^\sigma_\infty \otimes_{S_\infty} \mathcal{O} \cong M^\sigma$;

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

- There is a surjection $R^\sigma_\infty \twoheadrightarrow \mathcal{R}^\sigma_\infty$ such that the composition

$$R^\sigma_{\mathrm{loc}} \hookrightarrow R^\sigma_\infty \twoheadrightarrow \mathcal{R}^\sigma_\infty \twoheadrightarrow R^\sigma$$

  is the map $R^\sigma_{\mathrm{loc}} \to R^\sigma$ from above.

Just as in the proof of [BKM21, Theorem 6.3], we may lift the structure map $S_\infty \to \mathcal{R}^\sigma_\infty$ to a map $i : S_\infty \to R^\sigma_\infty$ making $\pi_\infty : R^\sigma_\infty \to \mathcal{R}^\sigma_\infty$ into an S_∞-module surjection, and so it follows that M^σ_∞ is a maximal Cohen–Macaulay R^σ_∞-module.

But now by standard properties of maximal Cohen–Macaulay modules, the support of M^σ_∞ is a union of irreducible components of $\operatorname{Spec} R^\sigma_\infty$. As $R^\sigma_\infty = R^\sigma_{\mathrm{loc}}[[x_1,\dots,x_g]]$, the irreducible components of $\operatorname{Spec} R^\sigma_\infty$ are in bijection with those of $\operatorname{Spec} R^\sigma_{\mathrm{loc}}$.

By an analogous result to Lemma 6.2 from [BKM21] (using Corollary 3.1.7 of [Gee11] instead of the results of [DT94] that are used there), it follows that each irreducible component of $\operatorname{Spec} R^\sigma_\infty$ contains a point in the support of $M^\sigma_\infty/(i(y_1),\dots,i(y_d)) \otimes_{\mathcal{O}} E = M^\sigma \otimes_{\mathcal{O}} E$, which is not contained in any other component. Then as in the proof of [BKM21, Theorem 6.3], as R^σ_∞ is reduced, it follows that R^σ_∞ acts faithfully on M_∞ and so $\mathcal{R}^\sigma_\infty = R^\sigma_\infty$, and so we indeed have an isomorphism $R^\sigma_\infty \otimes_{S_\infty} \mathcal{O} \cong R^\sigma$, proving the first part of (3).

By Proposition 6.2, R^τ_∞ is Cohen–Macaulay. As in the proof of [BKM21, Theorem 6.3] this, combined with the fact that M^σ_∞ is free over S_∞, implies that R^τ_∞ is free over S_∞, proving (2). As in [BKM21, Theorem 6.3], this also implies that $R^\sigma = R^\sigma_\infty \otimes_{S_\infty} \mathcal{O}$ is finite free over $\mathcal{O}$, proving the second part of (3). In particular (as $\mathbf{T}^\sigma$ is finite free over $\mathcal{O}$ by definition) to show that $R^\sigma \twoheadrightarrow \mathbf{T}^\sigma$ is an isomorphism, it will suffice to show that the induced map $R^\sigma[1/\varpi] \twoheadrightarrow \mathbf{T}^\sigma[1/\varpi]$ is.

Now, as in the proof of [BKM21, Theorem 6.3], $\operatorname{Spec} R^\sigma_\infty[1/\varpi]$ is formally smooth at every point in the support of $\operatorname{Spec} M^\sigma \otimes_{\mathcal{O}} E$, and so in particular at the point corresponding to $\lambda : R^\sigma_\infty \twoheadrightarrow \mathcal{O}$, proving (5). This is proved as in [BKM21, Lemma 6.1] by using the fact that Galois representations arising from cohomological Hilbert modular forms are known to be generic in the sense of [All16, Lemma 1.1.5], which follows from the genericity of the corresponding automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ at all finite places and local-global compatibility as recorded in [All16, Theorem 2.1.2].

The argument of [BKM21, Theorem 6.3] now proves that $R^\sigma[1/\varphi] \twoheadrightarrow \mathbf{T}^\sigma[1/\varpi]$ is an isomorphism, and hence $R^\sigma \twoheadrightarrow \mathbf{T}^\sigma$ is an isomorphism. This proves (4) in the case when $\Sigma^{\varphi\text{-uni}} = \varnothing$ (the last claim in (4), that the rings are reduced, is a consequence of the standard fact that the Hecke operators T_v and S_v for $v \notin \Sigma$ are all simultaneously diagonalizable as operators on H^σ).

In the case when $\Sigma^{\varphi\text{-uni}} = \varnothing$, and hence $\sigma = \tau$, this completes the proof. In the case when $\Sigma^{\varphi\text{-uni}} \neq \varnothing$ and so $\sigma \neq \tau$, it remains to deduce the statement of the theorem for τ from the one for σ.

First, by the definition of modified global deformation rings given in Equation 4.2, we have that

$$R^\tau = R^\tau_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R = R^\tau_{\mathrm{loc}} \otimes_{R^\sigma_{\mathrm{loc}}} R^\sigma$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

and similarly $R_n^\tau = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R_n^\sigma$ and $R_n^{\tau,\square} = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R_n^{\sigma,\square}$ for all $n \geq 1$. The S_∞-algebra structure on $R_n^{\sigma,\square}$ then induces an S_∞-algebra structure on $R_n^{\sigma,\square}$, and we have

$$R_n^{\tau,\square} \otimes_{S_\infty} \mathcal{O} = (R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R_n^{\sigma,\square}) \otimes_{S_\infty} \mathcal{O} = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} (R_n^{\sigma,\square} \otimes_{S_\infty} \mathcal{O}) = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R^\sigma = R^\tau.$$

Also, as $R_n^{\sigma,\square}$ is a quotient of R_∞^σ (as a R_{loc}^σ-algebra), if we let

$$R_\infty^\tau = R_{\mathrm{loc}}^\tau[[x_1,\dots,x_g]] = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R_\infty^\sigma,$$

then $R_n^{\tau,\square}$ is a quotient of $R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}^\sigma} R_\infty^\sigma = R_\infty^\tau$ (as a R_{loc}^τ-algebra).

Now, just as in the proof of Lemma 6.3, the map $R_n^\sigma \twoheadrightarrow \mathbf{T}_n^\sigma$ induces a map $R_n^\tau \twoheadrightarrow \overline{\mathbf{T}}_n^\tau$ making the diagram

$$\begin{tikzcd}
R_n^\tau \arrow[r, two heads] & \overline{\mathbf{T}}_n^\tau \\
R_n^\sigma \arrow[u] \arrow[r, two heads] & \mathbf{T}_n^\sigma \arrow[u]
\end{tikzcd}$$

commute. As the $\mathbf{T}_n^\sigma$-action on M_n^σ extends to a $\overline{\mathbf{T}}_n^\tau$-action (since the U_v operators naturally act on M^σ), the R_n^σ-action on M_n^σ also extends to a R_n^τ-action on M_n^σ. Passing to the framed versions (by applying $-\otimes_{R_n^\sigma} R_n^{\sigma,\square}$), it follows that the action of $R_n^{\sigma,\square}$ on $M_n^{\sigma,\square}$ extends to an action of $R_n^{\tau,\square}$. Moreover, it's easy to check that the isomorphism $M_n^{\sigma,\square} \otimes_{S_\infty} \mathcal{O} \cong M^\sigma$ is compatible with the action of the U_v-operators, and so it is an isomorphism of $\overline{\mathbf{T}}^\tau$-modules and hence of R^τ-modules.

Combining all of this, we can again use the 'ultrapatching' construction of [Man21, Section 4.1], with $\{R_n^{\tau,\square}\}$ in place of $\{R_n^{\sigma,\square}\}$ and R_∞^τ in place of R_∞^σ. This produces a S_∞-algebra $\mathcal{R}_\infty^\tau$ together with a surjection $R_\infty^\tau \twoheadrightarrow \mathcal{R}_\infty^\tau$ and an isomorphism $\mathcal{R}_\infty^\tau \otimes_{S_\infty} \mathcal{O} \cong R^\tau$ such that the composition

$$R_{\mathrm{loc}}^\tau \hookrightarrow R_\infty^\tau \twoheadrightarrow \mathcal{R}_\infty^\tau \twoheadrightarrow R^\tau$$

is the map $R_{\mathrm{loc}}^\tau \to R^\tau$.

By the functorality of the ultrapatching construction, the maps $R_n^{\sigma,\square} \to R_n^{\tau,\square}$ induce an S_∞-algebra homomorphism $R_\infty^\sigma = \mathcal{R}_\infty^\sigma \to \mathcal{R}_\infty^\tau$. Moreover, the action of $R_n^{\tau,\square}$ on $M_n^{\sigma,\square}$ induces an action of $\mathcal{R}_\infty^\tau$ on M_∞^σ, extending the action of $\mathcal{R}_\infty^\sigma$. In particular, we may treat M_∞^σ as a R_∞^τ-module.

We can now finish the proof. First, we have $R_v^\sigma = R_v^\tau$ for $v \notin \Sigma^{\varphi\text{-uni}}$ and $\dim R_v^\sigma = \dim R_v^\tau = 3+1$ for $v \in \Sigma^{\varphi\text{-uni}}$, so $\dim R_\infty^\tau = \dim R_\infty^\sigma = \dim S_\infty$, proving (1).

We shall now show (5). First, for $v \in \Sigma \setminus \Sigma^{\varphi\text{-uni}}$, we have $R_v^{\tau_v} = R_v^{\sigma_v}$ and $\operatorname{Spec} R_v^{\sigma_v}[1/\varpi]$ is formally smooth at the point corresponding to $\lambda : R_v^{\sigma_v} \hookrightarrow R_\infty^\sigma \xrightarrow{\lambda} \mathcal{O}$ by the above. Thus, to show (5), it suffices to show that for each $v \in \Sigma^{\varphi\text{-uni}}$, $\operatorname{Spec} R_v^{\varphi\text{-uni}}[1/\varpi]$ is also formally smooth at the point corresponding to $\lambda : R_v^{\varphi\text{-uni}} \hookrightarrow R_\infty^\tau \xrightarrow{\lambda} \mathcal{O}$.

Take any such $v \in \Sigma^{\varphi\text{-uni}}$. Recall that by assumption the representation $\rho|_{G_v}$ is Steinberg. Thus, the point of $\operatorname{Spec} R_v^{\mathrm{un}}[1/\varpi]$ corresponding to $\lambda : R_v^{\mathrm{un}} \hookrightarrow R_\infty^\sigma \xrightarrow{\lambda} \mathcal{O}$ is

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

in the Steinberg component and not in the unramified component (it can't lie on both components, as it corresponds to a formally smooth point of $\operatorname{Spec} R^{\sigma}_{\infty}[1/\varpi]$, by the above argument). But now by the explicit descriptions of the rings R_v^{un} and $R_v^{\varphi\text{-uni}}$ given in Lemmas 5.3 and 5.4, it follows that the natural map $R_v^{\mathrm{un}} \to R_v^{\varphi\text{-uni}}$ induces an isomorphism $R_v^{\mathrm{un}}/\mathcal{I}^{\mathrm{st}} \cong R_v^{\varphi\text{-uni}}/I_2$ between the Steinberg quotients constructed in Lemmas 5.3 and 5.4. It follows that the point of $\operatorname{Spec} R_v^{\varphi\text{-uni}}[1/\varpi]$ corresponding to $\lambda : R_v^{\varphi\text{-uni}} \hookrightarrow R^{\tau}_{\infty} \xrightarrow{\lambda} \mathcal{O}$ is also contained in the Steinberg component and that $\operatorname{Spec} R_v^{\varphi\text{-uni}}[1/\varpi]$ is formally smooth at this point (which again implies that this point does not lie on any other components). This proves (5).

As M^{σ}_{∞} is maximal Cohen–Macaulay over R^{σ}_{∞}, it follows that it is also maximal Cohen–Macaulay over R^{τ}_{∞}, and so the support of M^{σ}_{∞} as an R^{τ}_{∞}-module is again a union of irreducible components of $\operatorname{Spec} R^{\tau}_{\infty}$. But now for each $v \in \Sigma$, the irreducible components of R_v^{σ} are in bijection with those of R_v^{τ} (this is trivial for $v \notin \Sigma^{\varphi\text{-uni}}$ and for $v \in \Sigma^{\varphi\text{-uni}}$ follows from the description of the minimal primes of R_v^{un} and $R_v^{\varphi\text{-uni}}$ given in Lemmas 5.3 and 5.4). By Proposition 4.7, it follows that the irreducible components of $\operatorname{Spec} R^{\sigma}_{\infty}$ are in bijection with those of $\operatorname{Spec} R^{\tau}_{\infty}$. Since M^{σ}_{∞} is supported on all of $\operatorname{Spec} R^{\sigma}_{\infty}$, it follows that M^{σ}_{∞} is supported on all of $\operatorname{Spec} R^{\tau}_{\infty}$ as well. Since R^{τ}_{∞} is reduced, it follows that R^{τ}_{∞} acts faithfully on M^{σ}_{∞}. Since the action of R^{τ}_{∞} on M^{σ}_{∞} factors through $R^{\tau}_{\infty} \twoheadrightarrow \mathcal{R}^{\tau}_{\infty}$, it follows that $R^{\tau}_{\infty} \cong \mathcal{R}^{\tau}_{\infty}$.

Just as before, (2) and (3) follow from this, and again, the second part of (3) implies that to show that $R^{\tau} \twoheadrightarrow \overline{\mathbf{T}}^{\tau}$ is an isomorphism, it will suffice to show that the induced map $R^{\tau}[1/\varpi] \twoheadrightarrow \overline{\mathbf{T}}^{\tau}[1/\varpi]$ is.

To prove (4), consider the commutative diagram

$$\begin{tikzcd}
R^{\tau}[1/\varpi] \arrow[r, two heads] & \overline{\mathbf{T}}^{\tau}[1/\varpi] \\
R^{\sigma}[1/\varpi] \arrow[r, two heads] \arrow[u] & \mathbf{T}^{\sigma}[1/\varpi] \arrow[u]
\end{tikzcd}$$

As the bottom map is an isomorphism of finite free reduced E-algebras, to show that the top map is an isomorphism, it will suffice to show that for any $\overline{\mathbf{Q}}_p$ point $\eta : \mathbf{T}^{\sigma}[1/\varpi] \to \overline{\mathbf{Q}}_p$ of $\operatorname{Spec} \mathbf{T}^{\sigma} \cong \operatorname{Spec} R^{\sigma}$ the induced map $R^{\tau} \otimes_{\eta} \overline{\mathbf{Q}}_p \twoheadrightarrow \overline{\mathbf{T}}^{\tau} \otimes_{\eta} \overline{\mathbf{Q}}_p$ is an isomorphism.

Fix any such $\eta : \mathbf{T}^{\sigma}[1/\varpi] \to \overline{\mathbf{Q}}_p$. Then η corresponds to a modular Galois representation $\rho_{\eta} : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ lifting $\overline{\rho}$. For each $v \in \Sigma^{\varphi\text{-uni}}$, $\overline{\rho}|_{G_{F_v}}$ must be either Steinberg or unramified. Let $S_{\eta} \subseteq \Sigma^{\varphi\text{-uni}}$ be the set of $v \in \Sigma^{\varphi\text{-uni}}$ for which $\rho_{\eta}|_{G_{F_v}}$ is unramified.

By Equation 4.2 and Definition 4.2, we have $R^{\tau} = R^{\sigma}\left[a_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right]$ as subrings of R^{τ}, where for each $v \in \Sigma$, a_v is the chosen root of the characteristic polynomial of $\rho_{\eta}(\mathrm{Frob}_v)$. Hence,

$$R^{\tau} \otimes_{\eta} \overline{\mathbf{Q}}_p = (R^{\sigma} \otimes_{\eta} \overline{\mathbf{Q}}_p)\left[a_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right] = \overline{\mathbf{Q}}_p\left[a_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right]$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

For $v \in \Sigma^{\varphi\text{-uni}} \setminus S_\eta$ (so that $\rho_\eta|_{G_{F_v}}$ is Steinberg) the definition of $R_v^{\varphi\text{-uni}}$ implies that $a_v = \pm 1 \in \overline{\mathbf{Q}}_p$, so in fact, $R^\tau \otimes_\eta \overline{\mathbf{Q}}_p = \overline{\mathbf{Q}}_p[a_v|v \in S_\eta]$, and so $R^\tau \otimes_\eta \overline{\mathbf{Q}}_p$ is a quotient of

$$\overline{\mathbf{Q}}_p[x_v|v \in S_\eta]/(x_v^2 - x_v \operatorname{tr} \rho_v^\square(\mathrm{Frob}_v) + \det \rho_v^\square(\mathrm{Frob}_v)).$$

In particular, we have $\dim_{\overline{\mathbf{Q}}_p} R^\tau \otimes_\eta \overline{\mathbf{Q}}_p \le 2^{|S_\eta|}$.

On the other hand, $\overline{\mathbf{T}}^\tau = \mathbf{T}^\sigma\left[U_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right]$ is a subalgebra of $\mathrm{End}_{\mathcal{O}}(M^\tau)$, and so

$$\overline{\mathbf{T}}^\tau \otimes_\eta \overline{\mathbf{Q}}_p = (\mathbf{T}^\sigma \otimes_\eta \overline{\mathbf{Q}}_p)\left[U_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right] = \overline{\mathbf{Q}}_p\left[U_v \middle| v \in \Sigma^{\varphi\text{-uni}}\right] = \overline{\mathbf{Q}}_p[U_v|v \in S_\eta]$$

is a subalgebra of $\mathrm{End}_{\overline{\mathbf{Q}}_p}(M^\tau \otimes_\eta \overline{\mathbf{Q}}_p)$ (where the last inequality comes from the fact that U_v acts as a scalar on $M^\tau \otimes_\eta \overline{\mathbf{Q}}_p$ if $\rho_\eta|_{G_{F_v}}$ is Steinberg). But now as ρ_η is unramified at each $v \in S_\eta$, it corresponds to a Hilbert modular form f_η of level not divisible by any $v \in S_\eta$. Standard properties of Hilbert modular forms now imply that $\dim_{\overline{\mathbf{Q}}_p} \overline{\mathbf{T}}^\tau \otimes_\eta \overline{\mathbf{Q}}_p = \dim_{\overline{\mathbf{Q}}_p} \overline{\mathbf{Q}}_p[U_v|v \in S_\eta] = 2^{|S_\eta|}$; we are using here that the U_v for $v \in S_\eta$ act as independent nonscalar endomorphisms on the $2^{|S_\eta|}$ dimensional ($\overline{\mathbf{Q}}_p$-) vector space generated by the image of f_η under the standard degeneracy maps arising from the places $v \in S_\eta$. Thus, $\dim_{\overline{\mathbf{Q}}_p} \overline{\mathbf{T}}^\tau \otimes_\eta \overline{\mathbf{Q}}_p = 2^{|S_\eta|} \ge \dim_{\overline{\mathbf{Q}}_p} R^\tau \otimes_\eta$, and so as the map $R^\tau \otimes_\eta \overline{\mathbf{Q}}_p \twoheadrightarrow \overline{\mathbf{T}}^\tau \otimes_\eta \overline{\mathbf{Q}}_p$ is surjective, it must be an isomorphism. This completes the proof of (4), and thus of the theorem. $\square$

Combining this with Proposition 3.32 and the computations in Section 5 gives the following generalization of [BKM21, Theorem 10.1]:

Theorem 6.5. *In setting described in this section, we have:*

$$\delta(R^\tau) = \delta(\mathbf{T}^\tau) = \sum_{v \in \Sigma^{\mathrm{st}}} \frac{2n_v}{e} + \sum_{v \in \Sigma^{\varphi\text{-uni}}} \frac{3n_v}{e} + \sum_{v \in \Sigma^{\mathrm{uni}}} \frac{n_v}{e},$$

where n_v is as above, and e is the ramification index of $E/\mathbf{Q}_p$.

Proof. Theorem 6.4 implies that the map $\theta : S_\infty \to R_\infty^\tau$ satisfies property (P), and so Theorem 3.25 implies that implies that

$$\delta_\lambda(\mathbf{T}^\tau) = \delta_\lambda(R^\tau) = \delta_\lambda(R_\infty^\tau \otimes_{S_\infty} \mathcal{O}) = \delta_\lambda(R_\infty^\tau).$$

Now, by Proposition 3.32 and Proposition 3.28 we get

$$\begin{aligned}
\delta_\lambda(R_\infty^\tau) &= \delta_\lambda(R_{\mathrm{loc}}^\tau[[x_1,\dots,x_g]]) = \delta_\lambda(R_{\mathrm{loc}}^\tau) + \delta_\lambda(\mathcal{O}[[x_1,\dots,x_g]]) = \delta_\lambda(R_{\mathrm{loc}}^\tau) \\
&= \delta_\lambda\left(\left(\widehat{\bigotimes_{v\in\Sigma}} R_v^{\tau_v}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes_{v|p}} R_v^{\mathrm{fl}}\right)\right) = \sum_{v\in\Sigma} \delta_\lambda(R_v^{\tau_v}) + \sum_{v|p} \delta_\lambda(R_v^{\mathrm{fl}}) \\
&= \sum_{v\in\Sigma} \delta_\lambda(R_v^{\tau_v}) + \sum_{v|p} \delta_\lambda(\mathcal{O}[[x_1,\dots,x_{3+[F_v:\mathbf{Q}_p]}]]) = \sum_{v\in\Sigma} \delta_\lambda(R_v^{\tau_v}) \\
&= \sum_{v\in\Sigma^{\min}} \delta_\lambda(R_v^{\min}) + \sum_{v\in\Sigma^{\mathrm{st}}} \delta_\lambda(R_v^{\mathrm{st}}) + \sum_{v\in\Sigma^{\varphi\text{-uni}}} \delta_\lambda(R_v^{\varphi\text{-uni}}) + \sum_{v\in\Sigma^{\mathrm{uni}}} \delta_\lambda(R_v^{\mathrm{uni}}) + \sum_{v\in\Sigma^\square} \delta_\lambda(R_v^\square).
\end{aligned}$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Now, Proposition 4.6 implies that $R_v^{\min}$ and $R_v^{\square}$ are complete intersections, and so Proposition 3.28 gives $\delta_\lambda(R_v^{\min}) = 0 = \delta_\lambda(R_v^{\square})$. Thus, the claim follows by the computations in Theorems 5.18, 5.26 and 5.33. $\square$

Remark 6.6. While Theorem 6.5 only computes the 'noncohomological' Wiles defect, and [BKM21, Theorem 10.1] computes both the cohomological and noncohomological defects, we still have these defects are equal in the minimal level case (i.e., $\Sigma^{\mathrm{un}} = \Sigma^{\varphi\text{-uni}} = \Sigma^{\square} = \varnothing$) by [Man21, Theorem 1.2] and [BKM21, Theorem 3.12].

In the next section, we show that in fact our work here, which determines the defect of Hecke algebras and deformation rings, can be used to show an equality of cohomological and noncohomological defects in many situations.

7. Cohomological Wiles defects and degrees of parametrizations by Shimura curves

The main theorem of this paper, Theorem 6.5, that we have proven above computes Wiles defects of Hecke algebras acting on the cohomology of modular curves and Shimura curves. We use this to compute in the present section the Wiles defect of the modules of the Hecke algebras of Theorem 6.5 that are given by the cohomology of the Shimura curve on which the respective Hecke algebras acts faithfully; Theorem 7.5 and Proposition 7.7 below.

Our methods here also allow us to improve on the results of [RT97] about degrees of optimal parametrizations of elliptic curves over $\mathbf{Q}$ by Shimura curves: See Corollaries 7.9 and 7.10 below. (By optimal we mean as usual that the induced maps on the Jacobian of the Shimura curve has connected kernel.) Our approach diverges considerably from the one of [RT97]. Our proofs are rather indirect but fill in a lacuna caused by the basic problem that one does not know in generality surjectivity of maps on p-parts of component groups at primes q (of multiplicative reduction), induced by optimal parametrization of an elliptic curve E over $\mathbf{Q}$ by a Shimura curve which has multiplicative reduction at q (the prime q divides the discriminant of the quaternion algebra from which the Shimura curve arises). The difficulty of proving the surjectivity alluded to above is specially vexing when considering component groups at a prime q that is trivial for $E[p]$ (and thus in particular q is not 1 mod p). Both corollaries are deduced from Theorem 7.5 and Proposition 7.7. We only consider non-Eisenstein primes, namely primes p such that $E[p]$ is irreducible. The arguments in [RT97, page 11113] rely on auxiliary hypotheses: for instance, that there is a prime q such that the image of an inertia group I_q at q acting on $E[p]$ has image of order p. This hypothesis is fulfilled when E is a semistable elliptic curve over $\mathbf{Q}$ and $E[p]$ is irreducible, our methods allow one to consider all elliptic curves over $\mathbf{Q}$ provided $E[p]$ is irreducible as a $G_{\mathbf{Q}(\zeta_p)}$-module. We work with the setup in [BKM21, Section 5] and thus operate (mainly for simplicity) at less generality than the work in the previous sections (for instance, we will assume $F = \mathbf{Q}$.) There are slight differences between the setup here and that of [BKM21, Section 5] that we begin by highlighting.

7.1. Cohomological Wiles defects

Fix Q a finite set of primes, and let D_Q be the quaternion algebra over $\mathbf{Q}$ considered in [BKM21, §5]: It is definite if Q has odd cardinality and indefinite if Q is of even

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

cardinality. (By abuse of notation, we will also frequently use Q to denote the product of all the primes in the set Q. The context will make clear which meaning is intended.) We assume here that Q has even cardinality and thus D_Q is an indefinite quaternion algebra. For a positive integer N with $(N,Q)=1$ let $\Gamma_0^Q(N)$ be the congruence subgroup for $D_Q^\times$, which is maximal compact at primes in Q, and upper triangular mod ℓ for all $\ell|N$. We consider also the usual congruence subgroups $\Gamma_0(NQ)$ and $\Gamma_0(N^2Q^2)$ of $\mathrm{SL}_2(\mathbf{Z})$. Let $K_0(N^2Q^2)\subseteq \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q},f})$ and $K_0^Q(NQ)\subseteq D_Q^\times(\mathbf{A}_{\mathbf{Q},f})$ be the corresponding compact open subgroups. Let N' be the squarefree part of N.

We consider $X_0^Q(N)$ the (compact) Riemann surface

$$D_Q^\times(\mathbf{Q})\backslash\left(D_Q^\times(\mathbf{A}_{\mathbf{Q},f})\times\mathcal{H}\right)/K_0^Q(N)$$

(where $\mathcal{H}$ is the complex upper half plane). Give $X_0^Q(N)$ its canonical structure as an algebraic curve over $\mathbf{Q}$. Let as before p be a prime not dividing $2NQ$, and we fix a finite extension $E/\mathbf{Q}_p$, with $\mathcal{O}$ the ring of integers in E, ϖ a uniformizer, $k=\mathcal{O}/\varpi$ the residue field, and e the ramification index of $E/\mathbf{Q}_p$. We will assume below that E is sufficiently large so that $\mathcal{O}$ contains the Fourier coefficients of all newforms in $S_2(\Gamma_0(N^2Q^2))$. Consider the finite free $\mathcal{O}$-modules $S^Q(\Gamma_0^Q(N))=H^1(X_0^Q(N),\mathcal{O})$, $S(N^2Q^2)=H^1(X_0(N^2Q^2),\mathcal{O})$ and $S(NQ)=H^1(X_0(NQ),\mathcal{O})$. Let $\mathbf{T}(N^2Q^2)$, $\mathbf{T}(NQ)$ and $\mathbf{T}^Q(N)$ be the $\mathcal{O}$-algebras at level $\Gamma_0(NQ^2)$, $\Gamma_0(NQ)$ and $\Gamma_0^Q(N)$, respectively, generated by the Hecke operators T_r for primes r coprime to NQ acting on $S(N^2Q^2)$, $S(NQ)$ and $S^Q(\Gamma_0^Q(N))$. (We call such Hecke algebras deprived of operators U_r for dividing the level anemic Hecke algebras.) Note that by the Jacquet–Langlands correspondence, $\mathbf{T}^Q(N)$ is a quotient of $\mathbf{T}(N^2Q^2)$, and this quotient factors through $\mathbf{T}(NQ)$.

Let $f\in S_2(\Gamma_0(NQ))$ be a newform of level NQ such that all its Fourier coefficients lie in E, and consider the corresponding $\mathcal{O}$-algebra homomorphisms $\lambda_f:\mathbf{T}(N^2Q^2)\to\mathcal{O}$ and (abusing notation slightly) $\lambda_f:\mathbf{T}(NQ)\to\mathcal{O}$. We will fix this newform and our main results will be in relation to f. By the Jacquet–Langlands correspondence, this also gives a related homomorphism $\mathbf{T}^Q(N)\to\mathcal{O}$ that we again denote by the same symbol λ_f. We denote the corresponding maximal ideals which contain the prime ideal $\ker(\lambda_f)$ by the same symbol $\mathfrak{m}$. Let $\rho_f:G_{\mathbf{Q}}\to\mathrm{GL}_2(\mathcal{O})$ be the Galois representation associated by Eichler and Shimura to f and assume that the corresponding residual Galois representation $\overline{\rho}_f=\overline{\rho}:G_{\mathbf{Q}}\to\mathrm{GL}_2(k)$ is absolutely irreducible. By enlarging $\mathcal{O}$ if necessary, we may assume that k contains all eigenvalues of $\overline{\rho}(\sigma)$ for all $\sigma\in G_{\mathbf{Q}}$. The Galois representation $\rho_f: G_{\mathbf{Q}}\to\mathrm{GL}_2(E)$, with irreducible residual representation $\overline{\rho}$, is locally at primes $q\in Q$ of the form

$$\begin{pmatrix}\varepsilon_p & *\\ 0 & 1\end{pmatrix},$$

up to twist by an unramified character χ of order dividing 2. The $\beta_q\in\{\pm1\}$ of Section 4 (see discussion after Equation (4.1)) will be chosen so that $\rho_f|_{G_q}$ gives rise to a point of $\operatorname{Spec} R_q^{\mathrm{st}}$ in what follows (and thus depends on whether χ is trivial or not). Let $\mathcal{A}_f$ stand for the isogeny class of the abelian variety A_f (which is an optimal quotient of $J_0(NQ)$). The residual representations arising from the class $\mathcal{A}_f$ with respect to the fixed embedding $K_f \hookrightarrow \overline{\mathbf{Q}}_p$ are all isomorphic to our fixed absolutely irreducible $\overline{\rho}$. Consider the representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ arising from A_f. As in [BKM21, Definition 7.7], we define the invariants m_q, n_q for $q \in Q$ as follows. The representation $\rho_f|_{G_q} : G_q \to \mathrm{GL}_2(\mathcal{O})$ is of the form

$$\begin{pmatrix} \varepsilon_p & * \\ 0 & 1 \end{pmatrix}$$

up to twist by an unramified character of order dividing 2 and is ramified (i.e., generic). We define the local invariants m_q (respectively, n_q) to be the largest integer n such that $\rho_\lambda(I_q)$ (respectively, $\rho_\lambda(G_q)$) mod ϖ^n has trivial projective image.

There is an oldform f^{NQ} in $S_2(\Gamma_0(N^2Q^2))$ with corresponding newform f which is characterized by the property that it is an eigenform for the Hecke operators T_ℓ for ℓ prime with $(\ell, NQ) = 1$ and U_ℓ for $\ell|NQ$ and such that $a_\ell(f^{NQ}) = 0$, that is, $f^Q|U_\ell = 0$, for $\ell|NQ$. Let $\lambda_{f^{NQ}} : \mathbf{T}^{\mathrm{full}}(N^2Q^2) \to \mathcal{O}$ be the induced homomorphism of the full Hecke algebra $\mathbf{T}^{\mathrm{full}}(N^2Q^2)$ acting on $H^1(X_0(N^2Q^2), \mathcal{O})$ which is generated as an $\mathcal{O}$-algebra by the action of the Hecke operators T_ℓ for $(\ell, NQ^2) = 1$ and U_ℓ for $\ell|NQ$ on $S(N^2Q^2) = H^1(X_0(N^2Q^2), \mathcal{O})$. We denote by $\mathfrak{m}_Q$ the maximal ideal of $\mathbf{T}^{\mathrm{full}}(N^2Q^2)$ that contains the kernel of $\lambda_{f^{NQ}}$.

The homomorphism $\lambda_f : \mathbf{T}^Q(N) \to \mathcal{O}$ extends to the full Hecke algebra $\mathbf{T}^Q(N)^{\mathrm{full}}$ (which has operators U_r for r dividing NQ) acting on $S^Q(\Gamma_0^Q(N))$, and we denote by $\mathfrak{m}_Q$ again the maximal ideal of $\mathbf{T}^Q(N)^{\mathrm{full}}$ which contains the kernel of the extended homomorphism. We define $\mathbf{T}$, $\mathbf{T}^{\mathrm{uni}}$ (resp. $\mathbf{T}^{\mathrm{st},Q}$) to be the image of $\mathbf{T}(NQ^2)$ (resp. $\mathbf{T}^Q(N)$) in the endomorphisms of the finitely generated $\mathcal{O}$-modules $S(N^2Q^2) = H^1(X_0(N^2Q^2), \mathcal{O})_{\mathfrak{m}_Q}$, $S(NQ) = H^1(X_0(NQ), \mathcal{O})_{\mathfrak{m}}$ (resp. $S^Q(\Gamma_0^Q(N))_{\mathfrak{m}_Q}$).

We denote by $R, R^{\mathrm{uni}}, R^{\mathrm{st},Q}$ the corresponding universal deformation rings and thus we have surjective maps $R \twoheadrightarrow \mathbf{T}$, $R^{\mathrm{uni}} \twoheadrightarrow \mathbf{T}^{\mathrm{uni}}$ and $R^{\mathrm{st},Q} \twoheadrightarrow \mathbf{T}^{\mathrm{st},Q}$ of $\mathcal{O}$-algebras. (Thus, in each of these cases the type $\tau = (\tau_v)$ for $v|N'Q$ is such that τ_v is unrestricted, or unipotent, or unipotent at $v|N$ and Steinberg at $v|Q$.) We have the corresponding universal modular deformation $\rho^{\mathrm{mod}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T})$ by results of Carayol [Car94] which is a specialization of a universal representation $G_{\mathbf{Q}} \to \mathrm{GL}_2(R)$.

Define

$$M(N^2Q^2) = \mathrm{Hom}_{\mathbf{T}[G_{\mathbf{Q}}]}(\rho^{\mathrm{mod}}, S(N^2Q^2)^*_{\mathfrak{m}_Q}),$$
$$M(NQ) = \mathrm{Hom}_{\mathbf{T}[G_{\mathbf{Q}}]}(\rho^{\mathrm{mod}}, S(NQ)^*_{\mathfrak{m}}),$$
$$M^{\mathrm{st},Q}(N) = \mathrm{Hom}_{\mathbf{T}[G_{\mathbf{Q}}]}(\rho^{\mathrm{mod}}, S^Q(\Gamma_0^Q(N))^*_{\mathfrak{m}_Q}).$$

As in Lemma 5.1 of [BKM21], we have using [Car94] that the evaluation map $M(N^2Q^2) \otimes_{\mathbf{T}} \rho^{\mathrm{mod}} \to S(N^2Q^2)^*_{\mathfrak{m}_Q}$ is an isomorphism, as is $M^{\mathrm{st},Q}(N) \otimes_{\mathbf{T}} \rho^{\mathrm{mod}} \to S^Q(\Gamma_0^Q(N))^*_{\mathfrak{m}_Q}$. In particular, as $\mathbf{T}$-modules we have $S(N^2Q^2)^*_{\mathfrak{m}_Q} = M(N^2Q^2)^{\oplus 2}$ and $S^Q(\Gamma_0^Q(N))^*_{\mathfrak{m}_Q} = M^{\mathrm{st},Q}(N)^{\oplus 2}$.

We have the following lemma proved using Proposition 4.7 of [DDT97] (see proof of Theorem 5.2 of [BKM21]).

Lemma 7.1.

(i) *The Hecke module* $M(N^2Q^2)[\frac{1}{p}]$ *is free of rank one over* $\mathbf{T}[\frac{1}{p}]$.

(ii) *The* $\mathbf{T}$*-modules*

$$M(N^2Q^2), M(NQ), M^{\mathrm{st},Q}(N)$$

are self-dual.

(iii) *The* $\mathcal{O}$*-modules*

$$M(N^2Q^2)[\ker(\lambda_{f^{NQ}})], M(NQ)[\ker(\lambda_f)], M^{\mathrm{st},Q}(N)[\ker(\lambda_f)]$$

are each free of rank 1 over $\mathcal{O}$.

Proof. The first part follows from the arguments in Proposition 4.7 of [DDT97] (see proof of Theorem 5.2 of [BKM21]). For the second part, we use that f is a newform of level NQ and the explicit description of f^{NQ} and the corresponding maximal ideal $\mathfrak{m}_Q$ that is used to define $M(N^2Q^2)$. □

Remark 7.2. In general the modules $M(NQ), M^{\mathrm{st},Q}(N)$, because of the presence of oldforms, are not generically free over the anemic Hecke algebras acting on them that do not have the operators U_v for $v|NQ$ in them. This generic freeness holds for $M^{\mathrm{st},Q}(N)$ if $N|N(\overline{\rho})$ which was the assumption in [BKM21]. They are generically free over the full Hecke algebras acting on them that have the operators U_v for $v|NQ$ in them.

Remark 7.3. The definition of the modules $M(N^2Q^2)$, $M(NQ)$ and $M^{\mathrm{st},Q}(N)$ differs slightly from the definition of the modules M^τ; see (6.1) from Section 6. In particular, we do not quotient by the elements $S_v - \varepsilon_p(\mathrm{Frob}_v)$ (or even explicitly use the Hecke operators S_v). The definition of M^τ from Section 6 is needed when $F \neq \mathbf{Q}$ in order to make the patching argument work (for subtle reasons involving the unit group $\mathcal{O}_F^\times$). In this section, we are only considering the case $F = \mathbf{Q}$ for convenience, and so we are still able to use patching arguments with the simpler definitions of the modules given in this section.

Also, here we 'factor out' the Galois representation ρ^{mod} as above, while we do not do so in Section 6. This also does not significantly affect the patching argument. See [BKM21, Theorem 6.3] or [Man21, Section 4] for more details on patching arguments in which the Galois representation is factored out.

One can prove completely analogous versions of Theorems 6.4 and 6.5 for the modules defined in this section by applying the patching arguments applied there to the modules M^τ instead to the modules $M(N^2Q^2)$, $M(NQ)$ and $M^{\mathrm{st},Q}(N)$. We will leave the details of this to the interested reader, and for the remainder of the section we will simply cite the results of Section 6 as if they literally applied to the modules considered in this section.

We denote by $\langle\ ,\ \rangle$ certain $\mathcal{O}$-valued, perfect $\mathbf{T}$-equivariant pairings on the $\mathbf{T}$-modules

$$M(N^2Q^2), M(NQ), M^{\mathrm{st},Q}(N)$$

that are induced by Poincare duality (see [BKM21, §9]). We then recall from [BKM21, §3, Lemma 3.5], that if X, Y, Z are generators of the rank one $\mathcal{O}$-modules

$$M(N^2Q^2)[\ker(\lambda_{f^{NQ}})], M(NQ)[\ker(\lambda_f)], M^{\mathrm{st},Q}(N)[\ker(\lambda_f)],$$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

we have the following relationship:

$$\Psi_\lambda(M(N^2Q^2)) = \mathcal{O}/(\langle X,X\rangle), \Psi_\lambda(M(NQ)) = \mathcal{O}/(\langle Y,Y\rangle), \Psi_\lambda(M^{\mathrm{st},Q}(N)) = \mathcal{O}/(\langle Z,Z\rangle).$$

Here, we are abbreviating all the augmentations arising from the newform f to λ.

We recall the definition of the Wiles defect for modules from [BKM21]. (As we did not consider defects of modules till now we have deferred the definition till this section.)

Definition 7.4. Let R denote a finite, local $\mathcal{O}$-algebra, which is ϖ-torsion free and reduced. Let M be a R-module, that is finite free over $\mathcal{O}$ and with $\mathrm{rank}_\lambda M = d > 0$.

The *Wiles defect* of M is the quantity

$$\delta_{\lambda,R}(M) = \frac{d\log|\Phi_{\lambda,R}| - \log|\Psi_\lambda(M)|}{d\log|\mathcal{O}/p|},$$

which we will denote by $\delta_\lambda(M)$ when R is clear from context.

Recall from [BKM21, Definition 3.3] that the congruence module $\Psi_\lambda(M)$ is the cokernel of the composition

$$M[\ker\lambda] \to M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M^*,\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(M^*[\ker\lambda],\mathcal{O}).$$

We should remark that the Wiles defect is normalized differently in [BIK23]; the definitions differ by a factor of $d\log|\mathcal{O}/p|$. The interest of studying defects of the modules considered in Theorem 7.5 is that, besides the intrinsic interest, this is directly responsible for our improvements to the result of Ribet–Takahashi about changes of degrees of optimal parametrizations when we switch between Shimura curves. We have recalled above the definition [BKM21, Definition 7.7] of the inertial invariants m_q for $q \in Q$.

Theorem 7.5. *Let N' be the squarefree part of N. We have the equality of lengths of $\mathcal{O}$-modules:*

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(N^2Q^2))) = \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{\ell|N'} \mathrm{ord}_{\mathcal{O}}(\ell^2-1) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1)),$$

and

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) = \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{q\in Q} m_q.$$

We have equality of defects $\delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}}(M^{\mathrm{st},Q}(N)) = \delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}}(\mathbf{T}^{\mathrm{st},Q}) = \sum_{\ell|N'}\frac{n_\ell}{e} + \sum_{q\in Q}\frac{2n_q}{e}$.

Proof. The proof follows from the following facts:

1. We use the exact computation of the length of a relative cotangent space, namely

$$\ell_{\mathcal{O}}(\Phi_{R/R^{\mathrm{st},Q}}) = \ell_{\mathcal{O}}(\Phi_{\mathbf{T}/\mathbf{T}^{\mathrm{st},Q}}) = \sum_{\ell|N'}(\mathrm{ord}_{\mathcal{O}}(\ell^2-1) - n_\ell) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1) - 2n_q)$$

by a slight variant of the arguments in the proof of [BKM21, Corollary 7.15] using as key input Theorem 6.4 (there the level considered when we relax ramification conditions is NQ^2 rather than N^2Q^2, and it is assumed that $N|N(\overline{\rho})$, but the arguments carry over to our slightly different situation *mutatis mutandis*);

2. $\delta_{\lambda,\mathbf{T}}(\mathbf{T}) = \delta_{\lambda,\mathbf{T}}(M(N^2Q^2)) = 0$. This follows from the arguments in [BKM21, Theorem 5.2] (see also [BKM21, Remark 5.3, 5.4]) which is proved using the arguments of [Dia97, Theorem 3.4].
3. The inequality

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(N^2Q^2))) \leq \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{\ell|N'}(\mathrm{ord}_{\mathcal{O}}(\ell^2-1)) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1))$$

that follows from the following two inequalities:

- $$\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) \leq \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{q\in Q} m_q$$

  which follows from [RT97, Theorem 2]. To justify this, as noted above as a consequence of [BKM21, §3, Lemma 3.5], we have $\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) = \mathrm{ord}_{\mathcal{O}}(\langle Y,Y\rangle)$ and $\ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) = \mathrm{ord}_{\mathcal{O}}(\langle Z,Z\rangle)$. Further, the ideals generated by the inner products $(\langle Y,Y\rangle)$ and $(\langle Z,Z\rangle)$ can be read off from the optimal quotients ξ and ξ' of the isogeny class of abelian varieties $\mathcal{A}_f$ by the Jacobians of $X_0(NQ)$ and $X_0^Q(N)$ as follows. The composition $\xi_*\xi^*$ of the pullback ξ^* and pushforward of the maps induced by ξ on the $\mathrm{Ta}_\wp(A)_\mathfrak{m} = \mathcal{O}^2$ is identified with multiplication by a scalar in $\mathcal{O}$. We denote the ideal of $\mathcal{O}$ generated by this scalar by $(\xi_*\xi^*)$. Then $(\langle Y,Y\rangle) = (\xi_*\xi^*)$. Similarly, $(\langle Z,Z\rangle) = (\xi'_*\xi'^*)$. Then using [RT97, Theorem 2] in the case when $\mathcal{A}_f$ is an isogeny class of elliptic curves, and its generalization to optimal abelian variety quotients in [Kha03] we deduce that the ideal $(\xi_*\xi^*)(\xi'_*\xi'^*)^{-1}$ divides the ideal $(\Pi_{q\in Q}\omega^{m_q})$ of $\mathcal{O}$ which justifies our claim.
- $$\ell_{\mathcal{O}}(\Psi_\lambda(M(N^2Q^2))) \leq \ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) + \sum_{\ell|N'Q} \mathrm{ord}_{\mathcal{O}}(\ell^2-1)$$

  This statement, in the stronger form of an equality follows easily from the arguments in Step 2 of proof of [BKM21, Proposition 9.1].

4. The inequality

$$\ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q})) \leq \ell_{\mathcal{O}}(\Psi_\lambda(\mathbf{T}^{\mathrm{st},Q})),$$

which is equivalent to the inequality

$$\delta_\lambda(M^{\mathrm{st},Q}(N)) \geq \delta_\lambda(\mathbf{T}^{\mathrm{st},Q}).$$

This follows from [BKM21, Theorem 3.12].

5. The equality $\delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}} = \sum_{\ell|N'} \frac{n_\ell}{e} + \sum_{q|Q} \frac{2n_q}{e}$ which is a consequence of our main theorem, Theorem 6.5. (To deduce this from our main theorem, we use for $\ell|N'$ the local deformation condition described by R_ℓ^{uni} and for $q\in Q$ that described by R_q^{st}.)

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Using the first three points (1), (2) and (3), we conclude that $\delta_\lambda(M^{\mathrm{st},Q}(N)) \leq \sum_{\ell|N'} \frac{n_\ell}{e} + \sum_{q|Q} \frac{2n_q}{e}$. Using (4) and (5) we deduce the series of (in)equalities

$$\sum_{\ell|N'} \frac{n_\ell}{e} + \sum_{q\in Q} \frac{2n_q}{e} = \delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}}(\mathbf{T}^{\mathrm{st},Q}) \leq \delta_\lambda(M^{\mathrm{st},Q}(N)) \leq \sum_{\ell|N'} \frac{n_\ell}{e} + \sum_{q\in Q} \frac{2n_q}{e}$$

and hence

$$\delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}}(M^{\mathrm{st},Q}(N)) = \delta_{\lambda,\mathbf{T}^{\mathrm{st},Q}}(\mathbf{T}^{\mathrm{st},Q}) = \sum_{\ell|N'} \frac{n_\ell}{e} + \sum_{q\in Q} \frac{2n_q}{e}.$$

From this, using (1) and (2) we conclude that

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(N^2Q^2))) = \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{\ell|N'} \mathrm{ord}_{\mathcal{O}}(\ell^2-1) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1)).$$

Finally, using the two inequalities that occurred in proof of (3) above we deduce that

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) = \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{q\in Q} m_q,$$

finishing the proof of all parts of the theorem. □

Remark 7.6.

- The first part of Theorem 7.5 was proved in [BKM21, Proposition 9.1], using the methods of [RT97], in particular [RT97, Theorem 1]. We have reverse engineered the arguments of [BKM21, Proposition 9.1] and are able to deduce [RT97, Theorem 1] below by a different method which is more robust. We still use [RT97, Theorem 2] to prove upper bounds on change of congruence modules (or equivalently degrees of parametrizations)

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) \leq \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{q\in Q} m_q$$

  but not the less robust and delicate methods of the proof of the second part of [RT97, Theorem 1, see also page 11113], which show that these upper bounds in fact give exactly the change of lengths of the congruence modules. We view the correct upper bounds on change of congruence modules, when we relax deformation conditions at primes in Q (from Steinberg to unrestricted with fixed determinant) as 'easier' than the corresponding correct lower bounds (correctness lying in the fact that the bounds are expected to turn into equalities). In the analogous case of lengths of relative cotangent spaces, the inequality

$$\ell_{\mathcal{O}}(\Phi_{R/R^{\mathrm{st}}}) \leq \sum_{\ell|N'} (\mathrm{ord}_{\mathcal{O}}(\ell^2-1) - n_\ell) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1) - 2n_q)$$

  follows purely from local arguments: See [BKM21, Proposition 7.9] for the local computation, and also note that the surjectivity of the map $\Phi_{\lambda,R_\infty/R_\infty^{\mathrm{st}}} \to \Phi_{\lambda,R/R^{\mathrm{st}}}$ of [BKM21, Theorem 7.14] is elementary. The injectivity of this map which is proved in [BKM21, Theorem 7.14] lies deeper and uses patching arguments. Thus,

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

the heuristic that we justify by our work here is that (correct) upper bounds on change of congruence modules, or change of cotangent spaces, are 'easy' and our methods allow one to convert these upper bounds to equalities using the methods of this paper.

- Using (a straightforward modification) of [BKM21, Theorem 5.2] and [BKM21, Theorem 8.1, Cor. 8.3] (which considered $M(NQ^2)$ rather than $M(N^2Q^2)$), and under the assumption that $N|N(\overline{\rho})$ of [BKM21, §2] we know from [BKM21] that

$$\Psi_\lambda(M(N^2Q^2)) = \Psi_\lambda(\mathbf{T}), \Psi_\lambda(M^{\mathrm{st},Q}(N)) = \Psi_\lambda(\mathbf{T}^{\mathrm{st},Q}).$$

On the other hand using Theorem 6.5, together with [BKM21, Proposition 7.9, Corollary 7.15] we know that

$$\ell_{\mathcal{O}}(\Psi_\lambda(\mathbf{T})) = \ell_{\mathcal{O}}(\Psi_\lambda(\mathbf{T}^{\mathrm{st},Q})) + \sum_{\ell|N'} \mathrm{ord}_{\mathcal{O}}(\ell^2-1) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1)).$$

Combining this we can deduce the first part

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(NQ^2))) = \ell_{\mathcal{O}}(\Psi_\lambda(M^{\mathrm{st},Q}(N))) + \sum_{\ell|N'} \mathrm{ord}_{\mathcal{O}}(\ell^2-1) + \sum_{q\in Q}(m_q + \mathrm{ord}_{\mathcal{O}}(q^2-1))$$

of the theorem above. The arguments given in Theorem 7.5 use Theorem 6.5 to deduce *numerically* the equality of cohomological and ring theoretic defects or equivalently of lengths as $\mathcal{O}$-modules of ring theoretic and cohomological congruence modules seem more versatile and apply in cases where the arguments of [BKM21, Corollary 8.3] do not apply and do not use the assumption that $N|N(\overline{\rho})$.

- We assumed in this section that f was a newform of level NQ, and so in particular ρ_f ramifies at each prime dividing N. It it possible to prove the equality of cohomological and ring theoretic defects somewhat more generally by using the arguments of [Dia97].

  Specifically, assume that f is a newform of level $N_\varnothing Q$ for some integer $N_\varnothing$. Then Theorem 7.5 gives an equality $\delta_\lambda(M^{\mathrm{st},Q}(N_\varnothing)) = \delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\varnothing))$.

  Now, let Σ be a finite set of primes not containing any primes dividing $N_\varnothing Q$, and let N_Σ be the level considered in [Dia97, Section 3.2]. The inequalities given in the proof of [Dia97, Theorem 3.4] (which in our case rely on Ihara's Lemma for the Shimura curves $X_0^Q(N)$) then show that $\delta_\lambda(M^{\mathrm{st},Q}(N_\Sigma)) \le \delta_\lambda(M^{\mathrm{st},Q}(N_\varnothing))$.

  But now for each prime $q \in \Sigma$, one has that $R_q^{\min}$ and $R_q^\square$ are both complete intersections. Theorems 6.4 and 6.5 of Section 6 (that express defects of global deformation rings as sums of local defects) give that $\delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\varnothing)) = \delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\Sigma))$. One then deduces that

$$\delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\Sigma)) \le \delta_\lambda(M^{\mathrm{st},Q}(N_\Sigma)) \le \delta_\lambda(M^{\mathrm{st},Q}(N_\varnothing)) = \delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\varnothing)) = \delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\Sigma))$$

  and so $\delta_\lambda(\mathbf{T}^{\mathrm{st},Q}(N_\Sigma)) = \delta_\lambda(M^{\mathrm{st},Q}(N_\Sigma))$ for all Σ, generalizing Theorem 7.5. By a similar argument, one can also generalize Proposition 7.7.

We note a variant of the result above which computes defects for the module $M(NQ)$ when considered as a module for an anemic Hecke algebra and a full Hecke algebra. (We assume for simplicity for the result below that $N' = 1$.) The module $M(NQ)$ is a module for the (anemic) Hecke algebra $\mathbf{T}^{\mathrm{uni}}$, and it is also a module for the (full) Hecke algebra

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$\overline{\mathbf{T}}^{\mathrm{uni}}$ (and thus $U_v \in \overline{\mathbf{T}}^{\mathrm{uni}}$ for all primes v dividing NQ) that acts faithfully on $M(NQ)$. The augmentation $\lambda : \mathbf{T}^{\mathrm{uni}} \to \mathcal{O}$ extends uniquely to $\lambda' : \overline{\mathbf{T}}^{\mathrm{uni}} \to \mathcal{O}$, and $\lambda'(U_v) = \pm 1$ for $v|NQ$. We determine next the defects $\delta_{\lambda', \overline{\mathbf{T}}^{\mathrm{uni}}}(M(NQ))$ and $\delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(M(NQ))$.

Proposition 7.7. *Assume that* $N' = 1$.

(i) $\delta_{\lambda, \overline{\mathbf{T}}^{\mathrm{uni}}}(M(NQ)) = \delta_\lambda(\overline{\mathbf{T}}^{\mathrm{uni}}) = \sum_{v|N'Q} \frac{3n_v}{e}$.

(ii) $\delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(M(NQ)) = \delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(\mathbf{T}^{\mathrm{uni}}) = \sum_{v|N'Q} \frac{n_v}{e}$.

Proof. (i) By Theorem 6.5, $\delta_\lambda(\overline{\mathbf{T}}^{\mathrm{uni}}) = \sum_{v|N'Q} \frac{3n_v}{e}$. Using arguments pioneered by Mazur to prove mod p multiplicity one statements (see, for instance, [Wil95, Theorem 2.1] for an example of this type of argument, note that under our hypothesis $(p, NQ) = 1$), one sees that $M(NQ)$ is a free $\overline{\mathbf{T}}^{\mathrm{uni}}$-module, and thus $\delta_{\lambda, \overline{\mathbf{T}}^{\mathrm{uni}}}(M(NQ)) = \delta_\lambda(\overline{\mathbf{T}}^{\mathrm{uni}})$.

(ii) In this case, we argue as in the proof of Theorem 7.5 except that the proof is easier. Namely, we first observe that

$$\ell_{\mathcal{O}}(\Phi_{R/R^{\mathrm{uni}}}) = \sum_{\ell|N'Q} (\mathrm{ord}_{\mathcal{O}}(\ell^2 - 1) - n_\ell)$$

by a slight variant of the arguments in the proof of [BKM21, Corollary 7.15]. Further,

$$\ell_{\mathcal{O}}(\Psi_\lambda(M(N^2Q^2))) = \ell_{\mathcal{O}}(\Psi_\lambda(M(NQ))) + \sum_{\ell|N'Q} \mathrm{ord}_{\mathcal{O}}(\ell^2 - 1).$$

This together with $\delta_{\lambda, \mathbf{T}}(M(N^2Q^2)) = 0$, proves that $\delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(M(NQ)) = \sum_{v|N'Q} \frac{n_v}{e}$. Theorem 6.5 gives that $\delta_\lambda(\mathbf{T}) = \sum_{v|N'Q} \frac{n_v}{e}$, and thus altogether we get that $\delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(M(NQ)) = \delta_{\lambda, \mathbf{T}^{\mathrm{uni}}}(\mathbf{T}^{\mathrm{uni}}) = \sum_{v|N'Q} \frac{n_v}{e}$. □

Remark 7.8. We could prove Proposition 7.7(i) by a different method that exploits the equality of congruence modules $\Psi_{\lambda, \overline{\mathbf{T}}^{\mathrm{uni}}}(M(NQ)) = \Psi_{\lambda, \mathbf{T}^{\mathrm{uni}}}(M(NQ))$. This should follow from [BKM21, Lemma 3.4] (see also [BIK23, Lemma 3.7]) on using the fact that $M(NQ)[\ker(\lambda)] = M(NQ)[\ker(\lambda')] = \mathcal{O}$. Then we have to compute the change of the local cotangent space at v when we consider the induced augmentations of the map of local deformation rings $R_v^{\mathrm{uni}} \to \overline{R}_v^{\mathrm{uni}}$. We have not done this computation, but one can make the educated guess that the difference of the lengths of the respective cotangent spaces is $2n_v$. This would also compute the defects when we consider $M(NQ)$ as a module for Hecke algebras that have U_v in them for only a subset Σ of places that divide $N'Q$, and our educated guess for this defect is

$$\sum_{v \in \Sigma} \frac{3n_v}{e} + \sum_{v|N'Q, v \notin \Sigma} \frac{n_v}{e}.$$

7.2. Change of degrees of parametrizations by Shimura curves

From Theorem 7.5, it is easy to deduce the formula for the change of degrees of optimal parametrizations of elliptic curves by Shimura curves which may be summarized in the following formula (compare to [RT97, Theorem 1]).

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Corollary 7.9. *Let $\mathcal{E}$ be an isogeny class of elliptic curves over $\mathbf{Q}$ of conductor N and p be a prime such that the mod p representation arising from $\mathcal{E}$ is irreducible as a $G_{\mathbf{Q}(\zeta_p)}$-module. We also assume that p is prime to N. Consider a factorisation $N = D \cdot (N/D)$ with D a positive squarefree integer with an an even number of prime factors, and an optimal parametrization $X_0^D(N/D) \to E$ with $E \in \mathcal{E}$, and let δ_D be its degree. Then for primes q,r such that $qr|D$, the p-part of*

$$\frac{\delta_{D/qr}}{\delta_D}$$

and the p-part of $c_q c_r$ are equal where c_q,c_r are the orders of the component groups of any $E \in \mathcal{E}$ at the primes q and r.

Proof. The result follows from the first part of Theorem 7.5 and the well-known relation between congruence modules and degrees. For instance, $\mathrm{ord}_p(\delta_D)$ is the same as $\mathrm{ord}_p(\langle X,Y \rangle)$, where X,Y is a $\mathcal{O}$-basis of $H^1(X_0^D(N/D),\mathcal{O})[\ker\lambda]$, where λ is the augmentation of the Hecke algebra acting on $H^1(X_0^D(N/D),\mathcal{O})$ arising from E. We leave the details to the interested reader. □

We get results about the surjectivity of maps on component groups at primes q of multiplicative reduction of elliptic curves E that are induced by parametrizations of E by Shimura curves whose Jacobians have purely toric reduction at q (compare to the the arguments on [RT97, page 11113]).

Corollary 7.10. *With the notation of the previous corollary, for a prime $q|D$, the map induced by an optimal parametrization $X_0^D(N/D) \to E$ on the p-parts of the component groups $\phi_q(J_0^D(N/D)) \to \phi_q(E)$ is surjective.*

Proof. This follows from the corollary above and [RT97, Proposition 2]. □

Remark 7.11.

- The proof of [RT97, Theorem 1, part 2] on page 11113 depends on the hypothesis that N/D is not prime (that is used to 'permute' primes around there) and uses the hypothesis that E is semistable to ensure the hypothesis:

  (*): There is a prime q dividing the conductor of E (of semistable bad reduction) at which the order of the group of components at q is not divisible by p. Equivalently the mod p representation $\overline{\rho}$ arising from E is such that $\overline{\rho}(I_q)$ is either not finite flat (in the case $q = p$), and ramified (in the case $q \neq p$), with I_q an inertia group at q.

  We can dispense with these hypotheses in Corollary 7.9.
- The results of this section should in principle generalize to the cases of totally real fields F. (The main theorems of this paper, for instance Theorem 6.5, on which our results depend are written in the setting of such F.)

  Theorem 7.5 should generalize without too much difficulty to the case of newforms of weight $k > 2$. There are some related results in [KO23]; they only consider situations where the Hecke algebras are complete intersections and hence of defect 0. The results given here are more illustrative than exhaustive.

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

Appendix. A formula of Venkatesh

By N. Fakhruddin and C. Khare

The results of this section are inspired by unpublished notes of A. Venkatesh [Ven16]. Venkatesh's formula was stated (as a conjecture, but it was checked in many cases) for certain derived commutative rings, but we prove a version in the context of ordinary commutative algebra; we briefly explain the connection in Section A.1. The invariants c_0 and c_1 are essentially the same as those defined in [Ven16], but our method of proof is different from the approach taken there. The main result is Proposition A.6. This is used in the main text to compute the Wiles defect for certain Hecke algebras that are not complete intersections.

Let $\mathcal{O}$ be a complete discrete valvation ring (DVR), and let B be a complete local Noetherian $\mathcal{O}$-algebra with $\dim(B)=1$ with an augmentation $\pi_B : B \to \mathcal{O}$. Let E be the quotient field of $\mathcal{O}$ which we view as a module over any augmented ring using the augmentation. We assume that the augmentation has a finite cotangent space, by which we mean that $\ker(\pi_B)/\ker(\pi_B)^2$ is a finite length $\mathcal{O}$-module. Let C be the largest Cohen–Macaulay quotient of B – if B is finite over $\mathcal{O}$, then this is simply the quotient of B by its $\mathcal{O}$-torsion (which is an ideal) – and let $\pi_C : C \to \mathcal{O}$ be the augmentation of C induced by π_B.

Definition A.1. $c_0(B) := \ell(\mathcal{O}/\pi_C(\mathrm{Ann}(\ker(\pi_C))))$.

Since B is complete, we may write it as a quotient of $S = \mathcal{O}[[x_1, x_2, \ldots, x_n]]$ for some $n \geq 0$. Then by the prime avoidance lemma ([BH93, Lemma 1.2.2]), we may find a quotient A of S through which the map to B factors and such that A is a complete intersection ring with $\dim(A)=1$. Denote this map $A \to B$ by ϕ_B and the induced map $A \to \mathcal{O}$ by π_A. We may (and do) choose A such that $\ker(\pi_A)/\ker(\pi_A)^2$ is a finite length $\mathcal{O}$-module. Furthermore, if B is finite over $\mathcal{O}$ the lemma also allows us to choose A finite over $\mathcal{O}$.

Let $\mathbf{x}$ be a sequence of generators of $\ker(\phi_B)$ of length δ and consider the Koszul complex[10] $K_A(\mathbf{x})$. It is a graded-commutative differential graded A-algebra whose homology modules are B-modules. Let $H_\delta(K_A(\mathbf{x}))_1$ be the submodule of $H_\delta(K_A(\mathbf{x}))$ generated by products of elements of $H_1(K_A(\mathbf{x}))$. The Koszul complex is functorial for ring homomorphisms, so we have a map

$$\pi_{A,*} : H_*(K_A(\mathbf{x})) \to H_*(K_{\mathcal{O}}(\overline{\mathbf{x}})),$$

where $\overline{\mathbf{x}}$ denotes the image of the sequence $\mathbf{x}$ in $\mathcal{O}$. However, all terms of this sequence are 0, so $H_*(K_{\mathcal{O}}(\overline{\mathbf{x}}))$ is the exterior algebra in δ generators (in homological degree 1). In particular, $H_\delta(K_{\mathcal{O}}(\overline{\mathbf{x}})) \cong \mathcal{O}$.

Definition A.2. $c_1(B) := \ell(\pi_{A,*}(H_\delta(K_A(\mathbf{x})))/\pi_{A,*}(H_\delta(K_A(\mathbf{x}))_1))$.

We see that this is finite by localizing at the prime ideal corresponding to the kernel of π_A and observing that this localization map factors through π_A.

From the definition of the Koszul complex, it follows that $H_\delta(K_A(\mathbf{x}))$ is the annihilator of the ideal I generated by the sequence $\mathbf{x}$. The A-submodule of $H_\delta(K_A(\mathbf{x}))$ generated

[10] We use the notation and standard properties of the Koszul complex as in [BH93, §1.6].

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

by products of elements of $H_1(K_A(\mathbf{x}))$ is precisely the Fitting ideal of I (sitting inside its annihilator). It follows that

$$c_1(B) = \ell(\pi_A(\mathrm{Ann}(\ker(\phi_B)))/\pi_A(\mathrm{Fitt}(\ker(\phi_B)))). \tag{A.1}$$

We show below in Lemma A.5 that $c_1(B)$ is independent of all choices. For a fixed A as above, the Koszul complex only depends on the minimal number of generators of the kernel. Moreover, adding more elements in the kernel to the sequence of generators has the effect of tensoring the Koszul complex with an exterior algebra ([BH93, Proposition 1.6.21]) in which case it is easy to see that c_1 does not change.

To show that it is independent of the choice of $\phi_B : A \to B$, we will need the following elementary lemma.

Lemma A.3. *Let $\mathcal{O}$ be any commutative ring, A_1, A_2, B be local Noetherian $\mathcal{O}$-algebras and $\phi_i : A_i \to B$, $i = 1,2$ surjections of $\mathcal{O}$-algebras. Then*

1. *$A := A_1 \times_B A_2$ is also a local Noetherian $\mathcal{O}$-algebra and* $\dim(A) = \max\{\dim(A_1), \dim(A_2)\}$.
2. *If A_1 and A_2 are complete, then so is A.*
3. *Let P be any prime ideal in B, $P_i = \phi_i^{-1}(P)$ the corresponding prime ideals of A_i and $P_A = \phi^{-1}(P)$ that of A (where $\phi : A \to B$ is the surjection induced by ϕ_i). Then $A_{P_A} = (A_1)_{P_1} \times_{B_P} (A_1)_{P_2}$.*

Proof. We have $A = \{(a_1, a_2) \in A_1 \times A_2 : \phi_1(a_1) = \phi_2(a_2)\}$. The ideal m_A of A consisting of all pairs (a_1, a_2) with $a_i \in m_{A_i}$ is the unique maximal ideal of A since the surjectivity of ϕ_1, ϕ_2 implies that the complement consists of invertible elements, so A is local. The two projections induce surjections $p_i : A \to A_i$. If I is an ideal of A, then $p_1(I)$ is an ideal of A_1. The kernel of the map $I \to p_1(I)$ is naturally an ideal of A_2. Since A_1 and A_2 are Noetherian, this implies that A is Noetherian.

Now, since A is a subring of $A_1 \times A_2$ which is finite as an A-module (it is generated by $(1,0)$ and $(0,1)$), it follows from the going-up theorem [Mat80, Theorem 5, (i), (ii) and (iii)] that $\dim(A) = \dim(A_1 \times A_2) = \max\{\dim(A_1), \dim(A_2)\}$.

Suppose A_1 and A_2 are complete. To show that A is complete it suffices to prove that the m_A-adic topology on A is the same as the topology induced from the inclusion of A in $A_1 \times A_2$. Since $m_A^n \subset m_{A_1}^n \times m_{A_2}^n$ for all $n > 0$, we only need to show that given any $n' > 0$,

$$(m_{A_1}^n \times m_{A_2}^n) \cap A \subset m_A^{n'} \text{ for all } n \gg 0.$$

This follows immediately by applying the Artin–Rees lemma [Mat80, Theorem 15], with $I = m_A$, $M = A_1 \times A_2$ and $N = A$ since $I^n M = m_{A_1}^n \times m_{A_2}^n$.

We will use the following elementary fact whose simple proof we skip: If A is any commutative ring, S any multiplicative subset of A, M_1, M_2 and N any A-modules with maps $M_i \to N$, $i = 1,2$, then the natural map $M_1 \times_N M_2 \to (M_2)_S \times_{N_S} (M_2)_S$ of A-modules induces an isomorphism $(M_1 \times_N M_2)_S \to (M_1)_S \times_{N_S} (M_2)_S$. The statement (3) follows from this by taking M_i to be A_i, N to be B and $S = A \setminus P_A$ and by observing that $A_i \otimes_A A_{P_A} = (A_i)_{P_i}$, $i = 1,2$ and $B \otimes_A A_{P_A} = B_P$. $\square$

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

It follows from Lemma A.3 and the prime avoidance lemma already used earlier, that if A_i are complete intersections of the same dimension with surjections to B, then both of them may be dominated by a complete intersection A' of the same dimension. The condition on the finiteness of the cotangent space can also be preserved by (3) of Lemma A.3. For the independence of the choice of A in the definition of $c_1(B)$ we will also need:

Lemma A.4. *Let $f: A' \to A$ be a surjection of (complete) complete intersection local rings and let $\phi_B: A \to B$ be any surjection of rings. Let $\mathbf{z}$ be any finite sequence of generators of $\ker(f)$, $\mathbf{x}$ any sequence of generators of $\ker(\phi_B)$, and $\mathbf{x}'$ a lift of $\mathbf{x}$ to A'. Then $H_*(K_{A'}((\mathbf{z},\mathbf{x}')))$ is isomorphic to $H_*(K_A(\mathbf{x}))$ tensored with an exterior algebra over A with $|\mathbf{z}| + \dim(A) - \dim(A')$ free generators.*

Proof. Let $g: S \to A'$ be a surjection from a regular local ring S (which exists because A' is complete), so both $\ker(g)$ and $\ker(fg)$ are generated by regular sequences. Choose a sequence of generators $\mathbf{y}$ of $\ker(\phi_B fg)$ by first choosing a regular sequence of generators $\mathbf{w}$ of $\ker(g)$ and then adding lifts $\tilde{\mathbf{z}}$ of elements of $\mathbf{z}$ and lifts $\tilde{\mathbf{x}}'$ of lifts $\mathbf{x}'$ in A' of elements of $\mathbf{x}$. We then set $\mathbf{y} = (\mathbf{w},\tilde{\mathbf{x}}',\tilde{\mathbf{z}})$ and consider $K_S(\mathbf{y})$. Since the Koszul complex of a regular sequence is a resolution of the corresponding quotient ring by applying this to $\mathbf{w}$ we see that $K_S(\mathbf{y})$ is quasi-isomorphic (as a differential graded S-algebra) to $K_S((\tilde{\mathbf{z}},\tilde{\mathbf{x}}')) \otimes_S A'$, that is, $K_{A'}((\mathbf{z},\mathbf{x}'))$. On the other hand, since A is also a complete intersection ring, by choosing a minimal generating set of $\ker(fg)$ from among the elements of $(\mathbf{w},\tilde{\mathbf{z}})$, one sees that $K_S(\mathbf{y})$ is quasi-isomorphic to $K_A(\mathbf{x})$ tensored with an exterior algebra (since $gf(\tilde{\mathbf{x}}') = \mathbf{x}$ and the remaining elements of $(\mathbf{w},\tilde{\mathbf{z}})$ become 0 in A). On taking homology, we see that $H_*(K_{A'}((\mathbf{z},\mathbf{x}')))$ is isomorphic to $H_*(K_A(\mathbf{x}))$ tensored with an exterior algebra. The minimal number of generators of this exterior algebra is easily seen to be $|\mathbf{z}| + \dim(A) - \dim(A')$ since $|\mathbf{w}| = \dim(S) - \dim(A')$. □

Lemma A.5. *The invariant $c_1(B)$ is well defined.*

Proof. By Lemma A.3 and the remarks following it, it suffices to show that if $\phi_B: A \to B$ is as above and we have a surjection $f: A' \to A$ such that $\phi'_B := \phi_B f$ also satisfies the conditions analogous to those imposed on ϕ_B, then the number $c_1(B)$ defined using ϕ_B is equal to the one defined using ϕ'_B.

Let $\mathbf{x}$ be a sequence of generators of $\ker(\phi_B)$, $\mathbf{x}'$ a lift of this sequence to A' and $\mathbf{z}$ a sequence of generators of $\ker(f)$. Let $\mathbf{w} = (\mathbf{z},\mathbf{x}')$, so $\mathbf{w}$ is a sequence of generators of $\ker(\phi'_B)$. Thus, $\delta = |\mathbf{x}|$ and δ', the corresponding number of generators for $\ker(\phi'_B)$, equals $|\mathbf{w}| = \delta + |\mathbf{z}|$. Note that $H_{\delta'}(K_{A'}((\mathbf{z},\mathbf{x}')))$ is canonically isomorphic to $\mathrm{Ann}(\ker(\phi'_B))$ and $H_\delta(K_A(\mathbf{x}))$ is canonically isomorphic to $\mathrm{Ann}(\ker(\phi_B))$. From the result of Lemma A.4 (specialized to the case $\dim(A) = \dim(A')$) that $H_*(K_{A'}((\mathbf{z},\mathbf{x}')))$ is isomorphic to $H_*(K_A(\mathbf{x}))$ tensored with an exterior algebra over A with $|\mathbf{z}|$ free generators, it follows that there is an isomorphism of A-modules $\alpha: \mathrm{Ann}(\ker(\phi'_B)) \to \mathrm{Ann}(\ker(\phi_B))$ such that $\alpha(\mathrm{Fitt}(\ker(\phi'_B))) = \mathrm{Fitt}(\ker(\phi_B))$.

Now, we use the finite cotangent space assumption on A and A'. This implies that the ideal $\pi_A(\mathrm{Ann}(\ker(\phi_B))) \subset \mathcal{O}$ is nonzero and equal to the image of $\mathrm{Ann}(\ker(\phi_B)) \otimes_A \mathcal{O}$ in $A \otimes_A \mathcal{O} = \mathcal{O}$ (and similarly for A' and also for the fitting ideals). The $\mathcal{O}$-module

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

$\mathrm{Ann}(\ker(\phi_B)) \otimes_A \mathcal{O}$ modulo its torsion is free of rank one (and similarly for A') so the lemma follows from Equation (A.1) and the above by using the isomorphism $\alpha \otimes_A \mathcal{O}$. $\square$

For any map of rings $R_1 \to R_2$, an R_2-module M and $i \geq 0$, we denote by $\mathrm{Der}^i_{R_1}(R_2, M)$ the i-th André–Quillen cohomology group ([Iye07, Def. 5.8] or [And74, III a), Def. 11 and 12]) of R_2 with coefficients in M. Let E denote the quotient field of $\mathcal{O}$ viewed as a B-module via π_B.

The invariants $c_0(B)$ and $c_1(B)$ defined above are linked by the following proposition, which may be viewed as a derived version of Wiles's formula for complete intersections [Wil95], [Len95], [FKR21, §A]; a variant of this formula was first discovered by A. Venkatesh [Ven16].

Proposition A.6. *Let $\mathcal{O}$, E and $\pi_B : B \to \mathcal{O}$ be as in the beginning of Appendix A, and let $c_0(B)$ (resp. $c_1(B)$) be the invariant of B defined in Definition A.1 (resp. A.2). Then*

$$c_0(B) - c_1(B) = \ell(\mathrm{Der}^0_{\mathcal{O}}(B, E/\mathcal{O})) - \ell(\mathrm{Der}^1_{\mathcal{O}}(B, E/\mathcal{O})). \tag{A.2}$$

Proof. We denote by J the ideal $\ker(\phi_B)$ with $\phi_B : A \to B$ as above. The sequence of maps $\mathcal{O} \to A \to B$ gives rise to an exact sequence of André–Quillen cohomology

$$0 \to \mathrm{Der}^0_{\mathcal{O}}(B, E/\mathcal{O}) \to \mathrm{Der}^0_{\mathcal{O}}(A, E/\mathcal{O}) \to \mathrm{Hom}_A(J/J^2, E/\mathcal{O}) \to \mathrm{Der}^1_{\mathcal{O}}(B, E/\mathcal{O}) \to 0. \tag{A.3}$$

The 0 on the left comes from the fact that $\mathrm{Der}^0_A(B, E/\mathcal{O}) = 0$ since ϕ_B is surjective (which also gives that $\mathrm{Hom}_A(J/J^2, E/\mathcal{O})$ is equal to $\mathrm{Der}^1_A(B, E/\mathcal{O})$). The 0 on the right comes from the fact that $\mathrm{Der}^1_{\mathcal{O}}(A, E/\mathcal{O}) = \mathrm{Der}^2_{\mathcal{O}}(A, \mathcal{O}) = 0$, where the first equality is because $\mathrm{Der}^i_{\mathcal{O}}(A, E) = 0$ for all i (a consequence of the finite tangent space condition on π_A) and the second follows from [Avr99, (1.2) Theorem] because A is a complete intersection, $\mathcal{O}$ is regular and we have a surjection from S onto A.

We claim that $\mathrm{Hom}_A(J/J^2, E/\mathcal{O})$ and $\mathrm{Der}^0_{\mathcal{O}}(A, E/\mathcal{O})$ are finite length $\mathcal{O}$-modules and that we have equalities $\ell(\mathrm{Hom}_A(J/J^2, E/\mathcal{O})) = \ell(\mathcal{O}/\pi_A(\mathrm{Fitt}(J)))$ and $\ell(\mathrm{Der}^0_{\mathcal{O}}(A, E/\mathcal{O})) = \ell(\mathcal{O}/\pi_A(\mathrm{Fitt}(\ker(\pi_A))))$. Assuming the claim, from sequence (A.3) we deduce

$$\ell(\mathrm{Der}^0_{\mathcal{O}}(B, E/\mathcal{O})) - \ell(\mathrm{Der}^1_{\mathcal{O}}(B, E/\mathcal{O})) = \ell(\mathcal{O}/\pi_A(\mathrm{Fitt}(\ker(\pi_A)))) - \ell(\mathcal{O}/\pi_A(\mathrm{Fitt}(J))). \tag{A.4}$$

By definition $c_0(B) = \ell(\mathcal{O}/\pi_C(\mathrm{Ann}(\ker(\pi_C))))$, $c_1(B) = \ell(\pi_A(\mathrm{Ann}(\ker(\phi_B)))/\pi_A(\mathrm{Fitt}(\ker(\phi_B))))$ by Equation (A.1), and Lemma A.9 below implies that

$$\ell(\mathcal{O}/\pi_A(\mathrm{Ann}_A(\ker(\pi_A)))) = \ell(\mathcal{O}/\pi_A(\mathrm{Ann}_A(\ker(\phi_B)))) + \ell(\mathcal{O}/\pi_C(\mathrm{Ann}_C(\ker(\pi_C)))).$$

Recalling that $J = \ker(\phi_B)$ and $\mathrm{Fitt}(\ker(\pi_A)) = \mathrm{Ann}_A(\ker(\pi_A))$ (since A is a complete intersection), the proposition follows by inserting these three equalities in Equation (A.4).

We now prove the claim made above: For the first part, note that $E/\mathcal{O}$ is an A-module via π_A, so that $\mathrm{Hom}_A(J/J^2, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(J/J^2 \otimes_A \mathcal{O}, E/\mathcal{O})$, where $\mathcal{O}$ is an A-module via π_A, and it suffices to show that $J/J^2 \otimes_A \mathcal{O}$ is a finite length $\mathcal{O}$-module. The module $J/J^2 \otimes_A \mathcal{O}$ is of finite type over $\mathcal{O}$ because J/J^2 is of finite type over A, and so we need to show that $J/J^2[1/\varpi] \otimes_{A[1/\varpi]} E$ vanishes. Now, the map $\phi[1/\varpi] : A[1/\varpi] \to B[1/\varpi]$ is a

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

map of finite-dimensional E-algebras and the (compatible) augmentations to $\pi_A[1/\varpi]$ and $\pi_B[1/\varpi]$ give rise to isomorphisms of a single factor with E, that is, $\pi_A[1/\varpi]\otimes_{A[1/\varpi]}E$ is an isomorphism and $J[1/\varpi]\otimes_{A[1/\varpi]}E=0$ because $J[1/\varpi]$ must then be supported on the other factors, and hence $J/J^2[1/\varpi]\otimes_{A[1/\varpi]}E=0$. For the second part, we apply the conormal sequence to $\mathcal{O}\to A\to\mathcal{O}$ which gives the isomorphism $\ker(\pi_A)/\ker(\pi_A)^2\cong\Omega_{A/\mathcal{O}}\otimes_A\mathcal{O}$ due to the splitting of $A\to\mathcal{O}$. By construction the right-hand term in the isomorphism is of finite $\mathcal{O}$-length, and the second part now follows from

$$\mathrm{Der}^0_{\mathcal{O}}(A,E/\mathcal{O})\cong\mathrm{Hom}_A(\Omega_{A/\mathcal{O}},E/\mathcal{O})\cong\mathrm{Hom}_{\mathcal{O}}(\Omega_{A/\mathcal{O}}\otimes_A\mathcal{O},E/\mathcal{O}).$$

For the first assertion on lengths, we need to show that $J/J^2\otimes_A\mathcal{O}$ and $\mathcal{O}/\pi_A(\mathrm{Fitt}(J))$ have the same lengths. Because $\pi_A(J)=0$, the image of $J^2\otimes_A\mathcal{O}$ in $J\otimes_A\mathcal{O}$ is zero, and hence $J/J^2\otimes_A\mathcal{O}\cong J\otimes_A\mathcal{O}$. Next, observe that $\pi_A(\mathrm{Fitt}(J))=\mathrm{Fitt}(J\otimes_A\mathcal{O})$, as follows from the definition of the fitting ideal. The equality of length now follows because for a finite length $\mathcal{O}$-module over the DVR $\mathcal{O}$ the theory of elementary divisors gives $\ell(M)=\ell(\mathcal{O}/\mathrm{Fitt}(M))$. The argument for the second length equality proceeds in the same way. One reduces the equality to showing that $\ker(\pi_A)/\ker(\pi_A)^2\cong\ker(\pi_A)\otimes_A\mathcal{O}$ and $\mathcal{O}/\mathrm{Fitt}(\ker(\pi_A)\otimes_A\mathcal{O})$ have the same length. □

Remark A.7. The above proof shows in particular that the terms $\mathrm{Der}^1_{\mathcal{O}}(B,E/\mathcal{O})$ and $\mathcal{O}/\pi_A(\mathrm{Fitt}(J))$ are of finite $\mathcal{O}$-length.

Remark A.8. If B is a complete intersection in Proposition A.6, we may take $A=B$, so $c_1(B)=0$, $c_0(B)=\ell(\mathcal{O}/\eta_B)$ and Equation (A.4) shows that Proposition A.6 reduces to Wiles's formula. The proposition shows once again that $c_1(B)$ is independent of all choices since all the other terms in the formula are clearly so.

The following lemma was used in the proof of Proposition A.6.

Lemma A.9. *Let A be a Gorenstein local ring with an augmentation $\pi_A: A\to\mathcal{O}$ such that the length of $\ker(\pi_A)/\ker(\pi_A)^2$ is finite. Assume that π_A factors through a surjective ring homomorphism $\phi_B: A\to B$, and let C be the largest quotient of B which is Cohen–Macaulay, so there are surjections $\phi_C: A\to C$, $\pi_B: B\to\mathcal{O}$ and $\pi_C: C\to\mathcal{O}$. Then*

$$\pi_A(\mathrm{Ann}_A(\ker(\pi_A)))=\pi_A(\mathrm{Ann}_A(\ker(\phi_B)))\,\pi_C(\mathrm{Ann}_C(\ker(\pi_C))).$$

Proof. We may apply Lemma A.10 of [FKR21] to the map ϕ_C since C is Cohen–Macaulay, to deduce that

$$\pi_A(\mathrm{Ann}_A(\ker(\pi_A)))=\pi_A(\mathrm{Ann}_A(\ker(\phi_C)))\,\pi_C(\mathrm{Ann}_C(\ker(\pi_C))),$$

so it suffices to to prove that $\mathrm{Ann}_A(\ker(\phi_C))=\mathrm{Ann}_A(\ker(\phi_B))$. We have $\ker(\phi_B)\subset\ker(\phi_C)$ and the quotient is a finite length A-module by the definition of C. The quotient map

$$\mathrm{Ann}_A(\ker(\phi_B))\ker(\phi_C)\to\mathrm{Ann}_A(\ker(\phi_B))(\ker(\phi_C)/\ker(\phi_B))$$

is an isomorphism since $\mathrm{Ann}_A(\ker(\phi_B))\ker(\phi_B)=(0)$, so $\mathrm{Ann}_A(\ker(\phi_B))\ker(\phi_C)$, being a submodule of a finite length A-module, is also of finite length. On the other hand,

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

it is a submodule of A and $\operatorname{depth}(A) = 1$, so it must be (0). Thus, $\operatorname{Ann}_A(\ker(\phi_C)) = \operatorname{Ann}_A(\ker(\phi_B))$. $\square$

A.1.

We briefly explain how the formula (A.2) can be viewed as a derived version of Wiles's formula:

Suppose we have a presentation $B = \mathcal{O}[[x_1, x_2, \ldots, x_n]]/(f_1, f_2, \ldots, f_{n+\delta})$ with $\delta \geq 0$. We may use this to construct a 'derived' ring

$$\mathcal{B} = \mathcal{O}[[x_1, x_2, \ldots, x_n]] \otimes_{\mathcal{O}[[y_1, y_2, \ldots, y_n, \ldots, y_{n+\delta}]]} \mathcal{O},$$

where the tensor product is defined as in [GV18, Definition 3.3]. Here, the x_i, y_j are in 'degree 0' and the map from $\mathcal{O}[[y_1, y_2, \ldots, y_n, \ldots, y_{n+\delta}]]$ to $\mathcal{O}[[x_1, x_2, \ldots, x_n]]$ is given by $y_j \mapsto f_j$ and to $\mathcal{O}$ by $y_j \mapsto 0$.

If we assume that $A = \mathcal{O}[[x_1, \ldots, x_n]]/(f_1, f_2, \ldots, f_n)$ is a one-dimensional complete intersection, then the derived ring has 'defect' equal to δ. The invariant $c_1(B)$ may then be viewed as coming from $\pi_*(\mathcal{B})$, since this may be computed in terms of a Koszul complex. Venkatesh views Equation (A.2) as an analogue of Wiles's formula for the derived ring $\mathcal{B}$, which is a 'derived complete intersection'. (However, as we have shown, all the terms in the formula only depend on $B = \pi_0(\mathcal{B})$, so it may also be viewed as a generalization of Wiles's formula to rings which are not necessarily complete intersections.)

Acknowledgements. We would like to thank Najmuddin Fakhruddin, Tony Feng, Michael Harris, Srikanth Iyengar and Akshay Venkatesh for helpful discussions related to this paper. G.B. acknowledges support by Deutsche Forschungsgemeinschaft (DFG) through CRC-TR 326 'Geometry and Arithmetic of Uniformized Structures', project number 444845124. We thank the referee for a careful reading of the paper and many helpful suggestions.

Competing interests. The authors have no competing interest to declare.

References

[All16] P. B. Allen, 'Deformations of polarized automorphic Galois representations and adjoint Selmer groups', *Duke Math. J.* **165**(13) (2016), 2407–2460. MR 3546966

[And74] M. André, *Homologie des algèbres commutatives* (Springer-Verlag, Berlin-New York, 1974). MR 352220

[Avr99] L. L. Avramov, 'Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology', *Ann. of Math. (2)* **150**(2) (1999), 455–487. MR 1726700

[BH93] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics, vol. 39 (Cambridge University Press, Cambridge, 1993). MR 1251956

[BIK23] S. Brochard, S. B. Iyengar and C. B. Khare, 'Wiles defect for modules and criteria for freeness', *Int. Math. Res. Not. IMRN* (**8**) (2023), 6901–6923. MR 4574391

[BKM21] G. Böckle, C. B. Khare and J. Manning, 'Wiles defect for Hecke algebras that are not complete intersections', *Compos. Math.* **157**(9) (2021), 2046–2088. MR 4301563

https://doi.org/10.1017/S1474748024000021 Published online by Cambridge University Press

[BLGHT11] T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, 'A family of Calabi-Yau varieties and potential automatically II', *Publ. Res. Inst. Math. Sci.* **47**(1) (2011), 29–98.

[Böc23] G. Böckle, *Wiles defect*, GitHub repository https://github.com/GebhardBoeckle/Wiles-Defect, 2023.

[Cal18] F. Calegari, 'Non-minimal modularity lifting in weight one', *J. Reine Angew. Math.* **740** (2018), 41–62. MR 3824782

[Car94] H. Carayol, 'Formes modulaires et représentations galoisiennes à valeurs dans un anneau local comple't, in *p-adic Monodromy and the Birch and Swinnerton-Dyer Conjecture (Boston, MA, 1991)*, Contemp. Math., vol. 165 (Amer. Math. Soc., Providence, RI, 1994), 213–237. MR 1279611

[CHT08] L. Clozel, M. Harris and R. Taylor, 'Automorphy for some l-adic lifts of automorphic mod l Galois representations', *Publ. Math. Inst. Hautes Études Sci.* (108) (2008), 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687

[DDT97] H. Darmon, F. Diamond and R. Taylor, 'Fermat's last theorem', in *Elliptic Curves, Modular Forms & Fermat's Last Theorem (Hong Kong, 1993)* (Int. Press, Cambridge, MA, 1997), 2–140. MR 1605752

[Dia97] F. Diamond, 'The Taylor-Wiles construction and multiplicity one', *Invent. Math.* **128**(2) (1997), 379–391. MR 1440309

[DT94] F. Diamond and R. Taylor, 'Lifting modular mod l representations', *Duke Math. J.* **74**(2) (1994), 253–269. MR 1272977

[Eis95] D. Eisenbud, *Commutative Algebra*, Graduate Texts in Mathematics, vol. 150 (Springer-Verlag, New York, 1995). With a view toward algebraic geometry. MR 1322960

[Eis05] D. Eisenbud, *The Geometry of Syzygies*, Graduate Texts in Mathematics, vol. 229 (Springer-Verlag, New York, 2005). A second course in commutative algebra and algebraic geometry. MR 2103875

[FKR21] N. Fakhruddin, C. Khare and R. Ramakrishna, 'Quantitative level lowering for Galois representations', *J. Lond. Math. Soc. (2)* **103**(1) (2021), 250–287. MR 4203049

[Gee11] T. Gee, 'Automorphic lifts of prescribed types', *Math. Ann.* **350**(1) (2011), 107–144. MR 2785764

[GR03] O. Gabber and L. Ramero, *Almost Ring Theory*, Lecture Notes in Mathematics, vol. 1800 (Springer-Verlag, Berlin, 2003). MR 2004652

[GV18] S. Galatius and A. Venkatesh, 'Derived Galois deformation rings', *Adv. Math.* **327** (2018), 470–623. MR 3762000

[Hid81] H. Hida, 'Congruence of cusp forms and special values of their zeta functions', *Invent. Math.* **63**(2) (1981), 225–261. MR 610538

[Iye07] S. Iyengar, 'André–Quillen homology of commutative algebras', *Interactions between Homotopy Theory and Algebra*, Contemp. Math., vol. 436 (Amer. Math. Soc., Providence, RI, 2007), 203–234. MR 2355775

[Jac85] N. Jacobson, *Basic Algebra. I*, second edn. (W. H. Freeman and Company, New York, 1985). MR 780184

[Kha03] C. Khare, 'On isomorphisms between deformation rings and Hecke rings', *Invent. Math.* **154**(1) (2003), 199–222, With an appendix by Gebhard Böckle. MR 2004460

[Kis09] M. Kisin, 'Moduli of finite flat group schemes, and modularity', *Ann. of Math. (2)* **170**(3) (2009), 1085–1180. MR 2600871

[KO23] C.-H. Kim and K. Ota, 'On the quantitative variation of congruence ideals and integral periods of modular forms', *Res. Math. Sci.* **10**(2) (2023), Paper No. 22, 34. MR 4588196

[KW09] C. Khare and J.-P. Wintenberger, 'On Serre's conjecture for 2-dimensional mod p representations of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$', *Ann. of Math. (2)* **169**(1) (2009), 229–253. MR 2480604

[Len95] H. W. Lenstra, Jr., 'Complete intersections and Gorenstein rings', in *Elliptic Curves, Modular Forms, & Fermat's Last Theorem (Hong Kong, 1993)*, Ser. Number Theory, I (Int. Press, Cambridge, MA, 1995), 99–109. MR 1363497

[Man21] J. Manning, 'Patching and multiplicity 2^k for Shimura curves', *Algebra Number Theory* **15**(2) (2021), 387–434. MR 4243652

[Mat80] H. Matsumura, *Commutative Algebra*, second edn., Mathematics Lecture Note Series, vol. 56 (Benjamin/Cummings Publishing Co., Inc., Reading, MA, 1980). MR 575344

[Pas24] H. Pasten, 'Shimura curves and the abc conjecture', *J. Number Theory* **254** (2024), 214–335. MR 4636759

[Pra06] K. Prasanna, 'Integrality of a ratio of Petersson norms and level-lowering congruences', *Ann. of Math. (2)* **163**(3) (2006), 901–967. MR 2215136

[RT97] K. A. Ribet and S. Takahashi, 'Parametrizations of elliptic curves by Shimura curves and by classical modular curves', *Proc. Nat. Acad. Sci. U.S.A.* 94(21) (1997), 11110–11114, Elliptic curves and modular forms (Washington, DC, 1996). MR 1491967

[Sho16] J. Shotton, 'Local deformation rings for $_2$ and a Breuil–Mézard conjecture when $\ell \neq p$', *Algebra Number Theory* **10**(7) (2016), 1437–1475. MR 3554238

[Sho18] J. Shotton, 'The Breuil–Mézard conjecture when $l \neq p$', *Duke Math. J.* **167**(4) (2018), 603–678. MR 3769675

[Sno18] A. Snowden, 'Singularities of ordinary deformation rings', *Math. Z.* **288**(3-4) (2018), 759–781. MR 3778977

[Sta19] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2019.

[Tho16] J. A. Thorne, 'Automorphy of some residually dihedral Galois representations', *Math. Ann.* **364**(1-2) (2016), 589–648. MR 3451399

[TU22] J. Tilouine and E. Urban, 'Integral period relations and congruences', *Algebra Number Theory* **16**(3) (2022), 647–695. MR 4449395

[TW95] R. Taylor and A. Wiles, 'Ring-theoretic properties of certain Hecke algebras', *Ann. of Math. (2)* **141**(3) (1995), 553–572. MR 1333036

[Ven16] A. Venkatesh, 'Derived version of Wiles's equality', Unpublished, 2016.

[Ven20] A. Venkatesh, 'Heights of automorphic forms and motives', Submitted to *Proceedings of the International Colloquium on Arithmetic Geometry*, TIFR, 2020.

[Wil95] A. Wiles, 'Modular elliptic curves and Fermat's last theorem', *Ann. of Math. (2)* **141**(3) (1995), 443–551. MR 1333035