Canad. Math. Bull. Vol. **14** (1), 1971

ON D. J. LEWIS'S EQUATION $x^3+117y^3=5$

BY

R. FINKELSTEIN AND H. LONDON

In a recent publication [2], D. J. Lewis stated that the Diophantine equation $x^3+117y^3=5$ has at most 18 integer solutions, but the exact number is unknown. In this paper we shall solve this problem by proving the following

THEOREM. *The equation* $x^3+117y^3=5$ *has no integer solutions.*

Proof. Let $\theta^3=117$, θ real. From Selmer [3], we obtain the following properties of the field $Q(\theta)$:

(1) An integral basis of $Q(\theta)$ is $(1, \theta, \theta^2/3)$.

(2) $[5]=[5, \theta-3][5, \theta^2+3\theta+4]$.

Since $[5, \theta-3]=[8-\theta^2/3]$, we get

$$5 = (8-\theta^2/3)(64+13\theta+8\theta^2/3),$$

where the factors of 5 are primes in $Q(\theta)$, $N(8-\theta^2/3)=5$ and $N(64+13\theta+8\theta^2/3)=25$. By Voronoi's algorithm [1, Ch. 4], we get that the fundamental unit of $Q(\theta)$ is $\varepsilon_0=412-50\theta-7\theta^2$.

We want all the integers of $Q(\theta)$ of norm 5 of the form $a+b\theta$. Setting

$$a+b\theta = (8-\theta^2/3)\varepsilon_0^n, \quad n\in Z,$$

we have an impossible situation, since

$$\varepsilon_0 \equiv 1+\theta-\theta^2, \qquad \varepsilon_0^2 \equiv 1-\theta-\theta^2, \qquad \varepsilon_0^3 \equiv 1 \pmod 3.$$

Thus $x^3+117y^3=5$ has no integer solutions.

BIBLIOGRAPHY

1. B. N. Delone and D. K. Faddeev, *The theory of irrationalities of the third degree*, Amer. Math. Soc., Providence, R.I., 1964.

2. W. J. LeVeque, *Studies in number theory*, Math. Assoc. of America, Washington, D.C., 1969.

3. E. S. Selmer, *The Diophantine Equation* $ax^3+by^3+cz^3=0$, Acta Math. **85** (1951), 203–362.

BOWLING GREEN STATE UNIVERSITY,
BOWLING GREEN, OHIO

MCGILL UNIVERSITY,
MONTREAL, QUEBEC