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NUMERICAL INDEX AND THE DAUGAVET PROPERTY FOR $L_\infty(\mu, X)$

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Abstract We prove that the space $L_\infty(\mu, X)$ has the same numerical index as the Banach space X for every σ-finite measure μ. We also show that $L_\infty(\mu, X)$ has the Daugavet property if and only if X has or μ is atomless.

Keywords: Daugavet property; numerical index; numerical radius

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1. Introduction

The concept of numerical index was first suggested by Lumer in 1968. Since then a lot of attention has been paid to this quantitative characteristic of a Banach space. Classical references here are [**2**, **3**]. For recent results we refer the reader to [**7**–**9**].

Here and subsequently, for a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $L(X)$. The *numerical range* of $T \in L(X)$ is

$$V(T) = \{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

The *numerical radius* is the seminorm defined on $L(X)$ by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$$

for each $T \in L(X)$. The *numerical index* of the space X is defined by

$$n(X) = \inf\{v(T) : T \in S_{L(X)}\}.$$

In this paper we prove that the numerical index of $L_\infty(\mu, X)$ coincides with the numerical index of X whenever μ is a σ-finite measure and X is an arbitrary Banach space. It should be pointed out that this result is analogous to those given in [**9**] for $C(K, X)$, $L_1(\mu, X)$ and $l_\infty(X)$.

The numerical index is related to the so-called Daugavet property (see [**9**]). The remarkable fact that every compact operator T on $C[0,1]$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{DE}$$

where Id stands for the identity, goes back to Daugavet [**4**] and this equality has currently become known as the *Daugavet equation*. We follow [**6**] in saying that a Banach space X has the *Daugavet property* if every rank-one operator $T \in L(X)$ satisfies (DE). In such a case, it is known that every weakly compact operator on X also satisfies the Daugavet equation. Consequently, this definition is equivalent to that given in [**1**]. For recent results we refer the reader to [**6**, **11**, **12**] and the references therein.

It is known that $C(K)$ has the Daugavet property for every perfect compact space K, and $L_1(\mu)$, $L_\infty(\mu)$ have the Daugavet property for every atomless positive measure μ (see [**12**] for a detailed account of these facts). The non-commutative versions have recently been obtained in [**10**]. It is also known that, for every Banach space X, $C(K,X)$ (respectively, $L_1(\mu,X)$) has the Daugavet property if and only if X has or K is perfect (respectively, μ is atomless) (see [**9**]).

In this paper, we show that $L_\infty(\mu,X)$ has the Daugavet property if and only if X has or the σ-finite measure μ is atomless. This extends an analogous result for $l_\infty(X)$ given in [**13**].

Throughout the paper, (Ω,Σ,μ) stands for a σ-finite measure space and X stands for an arbitrary Banach space. We write $L_\infty(\mu,X)$ for the Banach space of all equivalence classes of essentially bounded (Bochner) measurable functions from Ω into X, endowed with its natural norm

$$\|f\| = \inf\{\lambda \geqslant 0 : \|f(t)\| \leqslant \lambda \text{ a.e.}\}$$

for each $f \in L_\infty(\mu,X)$. To shorten the notation, we use the same letter to denote both a measurable function and its equivalence class. We refer to [**5**] for background on this topic.

2. The results

To generalize the fact given in [**9**] that $n(l_\infty(X)) = n(X)$, we require two preliminary results. The first one is well known for scalar-valued functions.

Lemma 2.1. *Let $f \in L_\infty(\mu,X)$ with $\|f(t)\| > \lambda$ a.e. Then there exists $B \in \Sigma$ with $0 < \mu(B) < \infty$ such that*

$$\left\| \frac{1}{\mu(B)} \int_B f(t)\,\mathrm{d}\mu(t) \right\| > \lambda.$$

Proof. Since $f(\Omega)$ is essentially separable, we can certainly assume that X is separable. Hence we can write

$$X \setminus \lambda B_X = \bigcup_{n\in\mathbb{N}} B_n,$$

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where B_n are closed balls. Therefore, there exists $n \in \mathbb{N}$ such that $A = f^{-1}(B_n)$ has positive measure. Let $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$. By convexity (see [**5**, Corollary II.2.8]),

$$\frac{1}{\mu(B)} \int_B f(t)\,\mathrm{d}\mu(t)$$

is contained in B_n, and the result follows. □

Again according to the fact that every function in $L_\infty(\mu, X)$ is essentially separably valued, the following result, which we shall use throughout the proof of Theorem 2.3, follows immediately.

Lemma 2.2. *Let $f \in L_\infty(\mu, X)$, $C \in \Sigma$ with positive measure, and $\varepsilon > 0$. Then there exist $x \in X$ and $A \subseteq C$ with $0 < \mu(A) < \infty$ such that $\|x\| = \|f\chi_C\|$ and $\|(f-x)\chi_A\| < \varepsilon$. Accordingly, the set*

$$\{x\chi_A + f\chi_{\Omega\setminus A} : x \in S_X,\ f \in B_{L_\infty(\mu,X)},\ A \in \Sigma \text{ with } 0 < \mu(A) < \infty\}$$

is dense in $S_{L_\infty(\mu,X)}$.

We can now state our main result.

Theorem 2.3. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. Then*

$$n(L_\infty(\mu, X)) = n(X).$$

Proof. In order to show that $n(L_\infty(\mu, X)) \geqslant n(X)$, we fix $T \in L(L_\infty(\mu, X))$ with $\|T\| = 1$. The procedure is to prove that $v(T) \geqslant n(X)$. Given $\varepsilon > 0$, we may find $f \in S_{L_\infty(\mu,X)}$, $x_0 \in S_X$, and $A, B \in \Sigma$ with $0 < \mu(B) < \infty$, such that

$$B \subseteq A \quad \text{and} \quad \left\|\frac{1}{\mu(B)} \int_B T(x_0\chi_A + f\chi_{\Omega\setminus A})\,\mathrm{d}\mu\right\| > 1 - \varepsilon. \tag{2.1}$$

Indeed, take $f \in S_{L_\infty(\mu,X)}$ and $C \subseteq \Omega$ with $\mu(C) > 0$ such that

$$\|[Tf](t)\| > 1 - \tfrac{1}{2}\varepsilon \quad (t \in C). \tag{2.2}$$

On account of Lemma 2.2, there exist $y_0 \in B_X$ and $A \subseteq C$ with $\mu(A) > 0$ such that $\|(f - y_0)\chi_A\| < \frac{1}{2}\varepsilon$. Now, write $y_0 = \lambda x_1 + (1-\lambda)x_2$ with $0 \leqslant \lambda \leqslant 1$, $x_1, x_2 \in S_X$, and consider the functions

$$f_j = x_j\chi_A + f\chi_{\Omega\setminus A} \in L_\infty(\mu, X) \quad (j = 1, 2),$$

which clearly satisfy $\|f - (\lambda f_1 + (1-\lambda)f_2)\| < \frac{1}{2}\varepsilon$. Since $A \subseteq C$, by using (2.2), we have

$$\|[Tf_1](t)\| > 1 - \varepsilon \quad \text{or} \quad \|[Tf_2](t)\| > 1 - \varepsilon$$

for every $t \in A$. Now, we choose $i \in \{1, 2\}$ such that

$$A_i = \{t \in A : \|[Tf_i](t)\| > 1 - \varepsilon\}$$

has positive measure, we write $x_0 = x_i$, and finally we use Lemma 2.1 to get $B \subseteq A_i \subseteq A$, satisfying our requirements.

Next we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, we write

$$\Phi(x) = x\chi_A + x_0^*(x) f \chi_{\Omega \setminus A} \in L_\infty(\mu, X) \quad (x \in X),$$

and we consider the operator $S \in L(X)$ given by

$$Sx = \frac{1}{\mu(B)} \int_B T(\Phi(x)) \, \mathrm{d}\mu \quad (x \in X).$$

According to (2.1), we have $\|S\| \geqslant \|Sx_0\| > 1 - \varepsilon$. So we may find $x \in S_X$ and $x^* \in S_{X^*}$ such that

$$x^*(x) = 1 \quad \text{and} \quad |x^*(Sx)| \geqslant n(X)[1 - \varepsilon].$$

Set $g = \Phi(x) \in S_{L_\infty(\mu,X)}$ and define the functional $g^* \in S_{L_\infty(\mu,X)^*}$ by

$$g^*(h) = x^*\biggl(\frac{1}{\mu(B)} \int_B h \, \mathrm{d}\mu\biggr) \quad (h \in L_\infty(\mu, X)).$$

Since $B \subseteq A$, we have $g^*(g) = 1$ and

$$|g^*(Tg)| = |x^*(Sx)| \geqslant n(X)[1 - \varepsilon].$$

Hence $v(T) \geqslant n(X)$, as required.

For the reverse inequality, we fix $S \in L(X)$ with $\|S\| = 1$ and define $T \in L(L_\infty(\mu, X))$ by

$$[T(f)](t) = S(f(t)) \quad (t \in \Omega,\ f \in L_\infty(\mu, X)).$$

Then $\|T\| = 1$ and so $v(T) \geqslant n(L_\infty(\mu, X))$. According to Lemma 2.2 together with [**2**, Theorem 9.3], given $\varepsilon > 0$ there exist $x \in S_X$, $f \in B_{L_\infty(\mu,X)}$, $A \in \Sigma$ with $0 < \mu(A) < \infty$, and $x^* \in S_{X^*}$ with $x^*(x) = 1$ such that

$$v(T) - \varepsilon < \biggl| x^*\biggl(\frac{1}{\mu(A)} \int_A T(x\chi_A + f\chi_{\Omega \setminus A}) \, \mathrm{d}\mu\biggr)\biggr|.$$

On the other hand,

$$\frac{1}{\mu(A)} \int_A T(x\chi_A + f\chi_{\Omega \setminus A}) \, \mathrm{d}\mu = S\biggl(\frac{1}{\mu(A)} \int_A (x\chi_A + f\chi_{\Omega \setminus A}) \, \mathrm{d}\mu\biggr) = Sx.$$

Therefore,

$$n(L_\infty(\mu, X)) - \varepsilon \leqslant v(T) - \varepsilon < |x^*(Sx)| \leqslant v(S)$$

and so $n(X) \geqslant n(L_\infty(\mu, X))$. □

The last part of the paper is dedicated to the study of the Daugavet property for $L_\infty(\mu, X)$. To this end, we need a characterization of this property given in [**12**, Corollary 2.3].

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Lemma 2.4. *X has the Daugavet property if and only if for every $x \in S_X$ and every $\varepsilon > 0$,*

$$B_X = \overline{\text{co}}\{y \in B_X : \|x - y\| \geqslant 2 - \varepsilon\}.$$

Since the proof of the non-easy part of the following result is analogous to that given in [**12**] for $C(K, X)$, it should be known to experts. However, we could not find it in the journal literature.

Theorem 2.5. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a Banach space. Then $L_\infty(\mu, X)$ has the Daugavet property if and only if X has or μ is atomless.*

Proof. Let us first suppose that μ is atomless. Set $f \in S_{L_\infty(\mu,X)}$, $\varepsilon > 0$ and $B \in \Sigma$ with

$$\mu(B) > 0 \quad \text{and} \quad \|f(t)\| > 1 - \tfrac{1}{2}\varepsilon \quad (t \in B).$$

Given $h \in S_{L_\infty(\mu,X)}$ and $n \in \mathbb{N}$, we take $B_1, \dots, B_n$ pairwise disjoint subsets of B with positive measure and we consider the function

$$g_j = h\chi_{\Omega \setminus B_j} - f\chi_{B_j} \in B_{L_\infty(\mu,X)}$$

for each $j \in \{1, \dots, n\}$. For every $t \in B_j$ we have

$$\left\| h(t) - \frac{1}{n}\sum_{i=1}^{n} g_i(t) \right\| = \frac{1}{n}\|h(t) + f(t)\| \leqslant \frac{2}{n},$$

and for $t \notin \bigcup_{j=1}^{n} B_j$ we have

$$h(t) = \frac{1}{n}\sum_{i=1}^{n} g_i(t).$$

Since $\|f - g_j\| > 2 - \varepsilon$, the above lemma shows that $L_\infty(\mu, X)$ has the Daugavet property.

To finish the proof, we write $L_\infty(\mu, X)$ in the form

$$L_\infty(\nu, X) \oplus_\infty \left[\bigoplus_{i \in I} X\right]_{l_\infty}$$

for a suitable set $I \subseteq \mathbb{N}$ and an atomless measure ν. Now, it should be noted that an l_∞-sum of Banach spaces has the Daugavet property if and only if every summand has [**13**]. □

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