ON COMMON MULTIPLES OF TRANSFINITE NUMBERS

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In his well known monograph [1] (p. 81) H. Bachmann indicates that two ordinal numbers > 1 always have a common left multiple > 1, but not always have a right multiple (RM). The monograph does not, however, contain any further analysis of right multiples. The purpose of the present note is to supplement this by formulating the following propositions which, despite their simplicity, seem not yet to be known. [1)]

THEOREM 1. If two ordinals, α and β ($\alpha \geq \beta > 0$), have a CRM, then their least CRM μ has the form $\mu = c\alpha = \alpha + c_1$ with c and c_1 finite. (Thus the least CRM differs only by a finite quantity from the larger number, α.)

THEOREM 2. Two ordinals, α and β ($\alpha \geq \beta > 0$), have a CRM if and only if either (i) α is itself a RM of β, or (ii) α is of the form $\alpha = \beta + c$ with $c < \omega$, and (in case (ii)) β is no limit ordinal. [2)]

A more explicit version of theorem 2 is this.

THEOREM 3. Two ordinals, α and β ($\alpha \geq \beta > 0$), have a CRM if and only if either

(i) β is a limit ordinal, and $\alpha = \omega^{\nu}\beta$ for some ν;

or (ii) β is no limit ordinal, and $\alpha = \beta + c$ with $c < \omega$;

or (iii) β is no limit ordinal, and $\alpha = \omega^{\nu}(\beta + qb) + \lambda$ ($q < \omega$; $\lambda < \omega^{\nu}$),

where b is the last (finite) term in the expansion $\beta = \omega\gamma + b$. [3)]

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1) We write "CRM" for "common right multiple", and use Bachmann's terminology, i.e., a RM of $\alpha \neq 0$ is any ordinal of the form $\sigma\alpha$ with $\sigma \neq 0$.

2) Conditions (i) and (ii) in theorem 2 do not exclude each other. Example: $\alpha = 2(\omega + 1) = \omega + 2$; $\beta = \omega + 1$.

3) Theorem 3, though less concise than theorem 2, is more convenient in applications, for it reduces the verification of the existence of a CRM to a simple inspection of the normal forms of α and β. Cf. also theorem 3' below.

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The proof of the theorems is based on three very simple lemmas.

LEMMA 1. If $\beta + \alpha = \alpha$, then $(\alpha + \beta)\gamma = \alpha\gamma + \beta N_\gamma$ where $N_\gamma = 1$ or $N_\gamma = 0$ according as γ is, or is not, a number of the form $\gamma = \lambda + 1$.

LEMMA 2. Every equation of the form $\tau\alpha = \sigma\beta$ $(\alpha \geq \beta > 0, \tau, \sigma \neq 0)$ can be reduced to $\tau'\alpha = \sigma'\beta$ where $0 < \tau' \leq \omega$ and $0 < \sigma' \leq \sigma$.

LEMMA 3. Every product $c\alpha$, with $0 < c < \omega$, can be represented in the form $c\alpha = \alpha + c_1$, with $c_1 < \omega$.

For the proof of lemma 1, see [1], p. 52. Lemma 2 was obtained in [2] as a corollary of another proposition; it can also easily be deduced from lemma 1. To prove lemma 3, put $\alpha = \omega\lambda + n$ $(n < \omega)$. We then have $c\alpha = c\omega\lambda + cn = (\omega\lambda + n) + (c-1)n = \alpha + c_1$ where $c_1 = (c-1)n < \omega$, as required.

We now proceed to prove our theorems. For the proof of theorem 1, take any CRM of α and β, say $\mu = \tau\alpha = \sigma\beta$, and use lemma 2 to reduce it to the least CRM, $\mu' = c\alpha = \sigma'\beta$ with $0 < c < \omega$. Then use lemma 3 to obtain the result.

To prove theorem 2 (necessity of conditions), let the least CRM of α and β be $\mu = \alpha + c = \sigma\beta$ $(c < \omega)$, and put $\sigma = \omega^\nu p + \varepsilon$ $(0 < p < \omega, \varepsilon < \omega^\nu)$. By lemma 1,

$$(1) \qquad \alpha + c = \omega^\nu p\beta + \varepsilon N_\beta \quad (\varepsilon < \omega^\nu).$$

Now if $N_\beta = 0$, equation (1) implies $c = 0$, so that theorem 2 (i) holds. If $N_\beta = 1$ and $\nu > 0$, it follows from (1) that ε has the form $\varepsilon = \lambda + c$ $(\lambda < \omega^\nu)$, so that (1) reduces to

$$(2) \qquad \alpha = \omega^\nu p\beta + \lambda N_\beta = (\omega^\nu p + \lambda)\beta$$

(see lemma 1), and again theorem 2 (i) holds. Finally, if $N_\beta = 1$, $\nu = 0$, then $\varepsilon < \omega^\nu = 1$, i.e., $\varepsilon = 0$, so that (1) yields $\alpha + c = p\beta = \beta + p_1$ (see lemma 3), where $c \leq p_1 < \omega$ (for $\alpha \geq \beta$), and theorem 2 (ii) follows. As regards the sufficiency of the conditions, case (i) of theorem 2 is obvious; so suppose that (ii) holds. Then we may set $\alpha = \lambda + m$, $\beta = \lambda + n$ where λ is a limit ordinal or 0, and $m, n < \omega$. Clearly, $n\alpha = n(\lambda + m) = \lambda + mn = m\beta$, so that $\mu = n\alpha = m\beta$ is a CRM of α and β. Thus theorem 2 is proved. The same proof applies to theorem 3 as well; in particular, theorem 2 (iii) is easily obtained from the first part of (2) by setting $p = 1+q$ (recall that $0 < p < \omega$).[4)]

From this proof it is obvious that theorem 2 (i) holds if and only if either theorem 3 (i) or theorem 3 (iii) is fulfilled. Hence, α is a RM of $\beta \neq 0$ if and only if $\alpha = \omega^{\nu}(\beta + qb) + \lambda N_{\beta}$, with q, b, λ and N_{β} defined as in theorem 3 and lemma 1 (note that b vanishes if $N_{\beta} = 0$). This can be used to obtain a more concise formulation of theorem 3, to wit:

THEOREM 3'. Two ordinals, α and β ($\alpha \geq \beta > 0$) have a CRM if and only if $\alpha = \omega^{\nu}(\beta + qb) + N_{\beta}(\lambda + c)$, ($\lambda < \omega^{\nu}$, $q, c < \omega$), with b and N_{β} defined as above. (This formulation, obviously, covers all three cases of theorem 3.)

REFERENCES

1. H. Bachmann, Transfinite Zahlen, (Berlin, 1955).

2. E. Zakon, On the relation of "similarity" between transfinite numbers, Mathematical Quarterly, Jerusalem, 7 (1953). English summary quoted in Math. Reviews, 15 (1954), 409.

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4) We prefer this proof to other possible proofs because, besides its simplicity, it yields theorem 2 and theorem 3 simultaneously, without requiring additional arguments or propositions to pass from one of them to the other one. It also yields necessary and sufficient conditions for α to be a RM of β in a simplified form (see below).

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