4
Decay of a Meta-stable State

In this chapter we will consider the decays of meta-stable states and calculate the lifetime for such a state using instanton methods. A meta-stable state arises due to the existence of a local minimum of the potential, which is not the global minimum. This corresponds to a potential having the form given in Figure 4.1. The potential rises steeply to infinity to the left and to the right; after the potential barrier, it goes down well below the energy of the meta-stable state, either eventually going to constant or it may even rise to plus infinity again in order to give an overall stable quantum mechanical problem. However, exactly what the potential does to the right is considered not to be important; the behaviour of the potential to the right is assumed to have a negligible effect on the tunnelling amplitude for a particle initially in the local minimum at $z = 0$ escaping to the right. We have drawn the potential, in Figure 4.1, so that it simply drops off to the right and we have normalized the potential by adding a constant such that the local minimum has $V(0) = 0$. Physically we are considering a potential of the type where a particle is trapped in a local potential well, but once the particle tunnels out of the well, it is free. The probability that the initial state is regenerated from the decay products is assumed to be negligible. This is in contra-distinction to the problem considered in Chapter 3 with two symmetric wells. Here the tunnelling-back amplitude was sizeable, corresponding to the anti-instanton, and had to be taken into account.

4.1 Decay Amplitude and Bounce Instantons

In this chapter, we will attempt to calculate the amplitude

$$< z = 0|e^{-\frac{\beta}{\hbar}\hat{h}(\hat{X},\hat{P})}|z = 0 >= \mathcal{N}\int \mathcal{D}z(\tau)e^{-\frac{S_E[z(\tau)]}{\hbar}} = e^{-\frac{\beta E_0}{\hbar}}\left|\langle E_0\,|\,z = 0\rangle\right|^2 + \cdots . \tag{4.1}$$

From this amplitude we expect to be able to identify and calculate the energy

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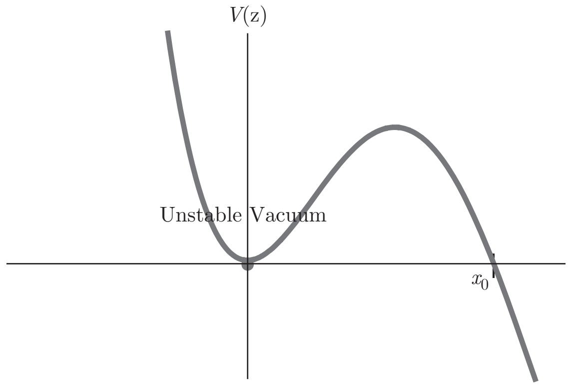

Figure 4.1. A potential with a meta-stable state at $z = 0$ that will decay via tunnelling

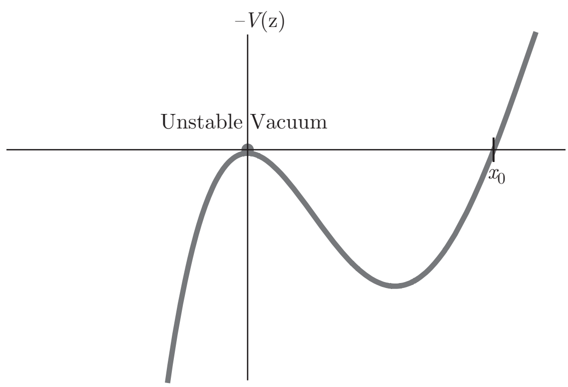

Figure 4.2. The flipped potential for instanton Euclidean classical solution

E_0 for the ground state. For a stable state, localized at $z = 0$, we expect E_0, in first approximation, to correspond to the ground-state energy of the harmonic oscillator appropriate to the well at $z = 0$, and $|\langle E_0 \,|\, z = 0\rangle|$ to be the magnitude of the ground-state wave function at $z = 0$. Now because of tunnelling we imagine that E_0 gains an imaginary part, $E_0 \to E_0 + i\Gamma/2$. We will directly attempt to use the path integral, and calculate it in a Gaussian approximation about an appropriate set of critical points, as in Chapter 3.

The equation of motion corresponds to particle motion in the inverted potential $-V(z)$, as depicted in Figure 4.2, with boundary condition that $z(\pm\frac{\beta}{2}) = 0$. There are two solutions, the trivial one $z(\tau) = 0$ for all τ, and a non-trivial true instanton solution $\bar{z}(\tau)$. Here the particle begins at $\tau = -\frac{\beta}{2}$ with a small positive velocity at $z = 0$, falls through the potential well and rises again to height zero at $z = x_0$, at around $\tau = 0$, and then bounces back, reversing its

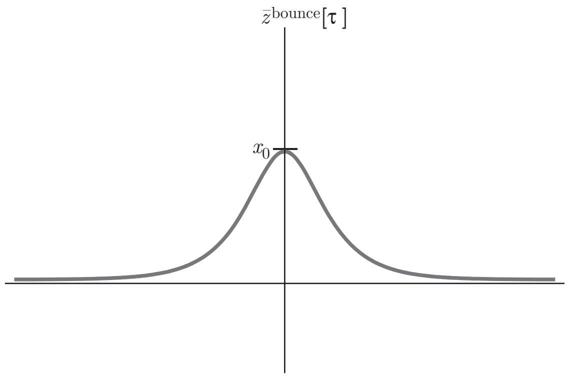

Figure 4.3. The bounce instanton which mediates tunnelling of a meta-stable state

steps and arriving at $z=0$ again at $\tau=\frac{\beta}{2}$. Clearly from symmetry such a solution exists if β is sufficiently large. We call this instanton, after Coleman, the bounce, $\bar{z}^{\text{bounce}}(\tau)$. The action for the bounce essentially comes from the short time interval during which the particle is significantly away from $z=0$. One can easily show that the bounce is exponentially close to zero except for a region around $\tau=0$ of size $\frac{1}{\omega}$, where again $\omega^2=V''(0)$. We call the action for the bounce $S_0=S_E[\bar{z}^{\text{bounce}}(\tau)]$ for the case $\beta=\infty$. Due to the time translation invariance in the $\beta=\infty$ case, again, there exists a one-parameter family of configurations, approximate bounces, which correspond to the bounce occurring at any time $\tau_0\in[-\frac{\beta}{2},\frac{\beta}{2}]$. The action for these configurations is exponentially close to S_0 and hence the degeneracy is β. Furthermore, approximate critical configurations also exist corresponding to n bounces occurring at widely separated times with action exponentially close to nS_0. The degeneracy of these configurations is $\frac{\beta^n}{n!}$ as they are exactly analogous to identical particles. Thus we expect the matrix element to be expressable as

$$\begin{aligned}
\langle_{z=0}|e^{-\frac{\beta}{\hbar}\hat{h}(\hat{X},\hat{P})}|_{z=0}\rangle &= \mathcal{N}\left(\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\left(\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}}\beta\right)^n\frac{e^{-\frac{nS_0}{\hbar}}}{n!}\times \\
&\quad\times\left(\frac{\det'\left[-\frac{d^2}{d\tau^2}+V''(\bar{z}_n^{\text{bounce}}(\tau))\right]}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]}\right)^{-\frac{1}{2}} \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}e^{-\omega\beta/2}e^{\beta\sqrt{\frac{S_0}{2\pi\hbar}}Ke^{-S_0}},
\end{aligned} \tag{4.2}$$

where

$$K=\frac{\left(\det'\left[-\frac{d^2}{d\tau^2}+V''(\bar{z}^{\text{bounce}}(\tau))\right]\right)^{-\frac{1}{2}}}{\left(\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]\right)^{-\frac{1}{2}}}. \tag{4.3}$$

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Here the prime signifies omitting only the zero mode. We will find that the situation is not that simple, and we must also deal with a negative mode. Then we would find

$$E_0 = \hbar\left(\frac{\omega}{2} + K\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{S_0}{\hbar}}\right) \tag{4.4}$$

and we look for an imaginary contribution to K.

4.2 Calculating the Determinant

The situation is actually more complicated than is apparent. K comes from the determinant corresponding to integration over the fluctuations around the critical bounce

$$\left(\det{}'\left[-\frac{d^2}{d\tau^2} + V''(\bar{z}^{\text{bounce}}(\tau))\right]\right)^{-\frac{1}{2}} = \int \prod_{\substack{n \\ \lambda_n \neq 0}} \frac{dc_n}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar}\frac{1}{2}\sum_n \lambda_n c_n^2} = \prod_{\substack{n \\ \lambda_n \neq 0}} \frac{1}{\sqrt{\lambda_n}}. \tag{4.5}$$

The n's corresponding to vanishing λ_n's are excluded, which is the meaning of the primed determinant. This time, however, the problem is much more serious. One of the λ_n's is actually negative. For this λ_n the integration over the c_n simply does not exist, and hence the determinant, as we wish to calculate it, does not exist. It seems our original idea is doomed. But there is a possible solution: perhaps we can define the integration by analytic continuation. Indeed, analytic continuations of real-valued functions often gain imaginary parts, exactly what we desire. This analytic continuation is in fact possible and we will see how we can perform it appropriately.

4.3 Negative Mode

First we will establish the existence of the negative mode. For $\beta = \infty$ we have an exact zero mode due to time translation invariance

$$\begin{aligned} &\left(-\frac{d^2}{d\tau^2} + V''(\bar{z}^{\text{bounce}}(\tau))\right)\frac{d}{d\tau}\bar{z}^{\text{bounce}}(\tau) \\ &\quad = \frac{d}{d\tau}\left(-\frac{d^2}{d\tau^2}\bar{z}^{\text{bounce}}(\tau) + V'(\bar{z}^{\text{bounce}}(\tau))\right) = 0, \end{aligned} \tag{4.6}$$

where the second term vanishes as it is the equation of motion. Since $\bar{z}^{\text{bounce}}(\tau)$ has the increasing and then decreasing form given in Figure 4.3, this implies $\dot{\bar{z}}^{\text{bounce}}(\tau)$ has the form given by Figure 4.4. In contra-distinction to the zero mode of Chapter 3, this zero mode has a node, *i.e.* it has a zero. This is intuitively obvious, the velocity of the particle executing the bounce will vanish exactly when it reverses direction. The analogous quantum mechanical Hamiltonian

$$-\frac{d^2}{d\tau^2} + V''(\bar{z}^{\text{bounce}}(\tau)) \tag{4.7}$$

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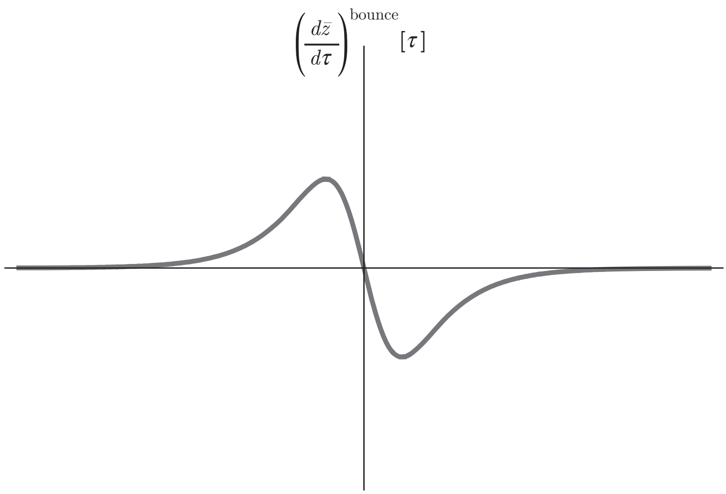

Figure 4.4. The derivative of the bounce $\frac{d\bar{z}^{bounce}(\tau)}{d\tau}$

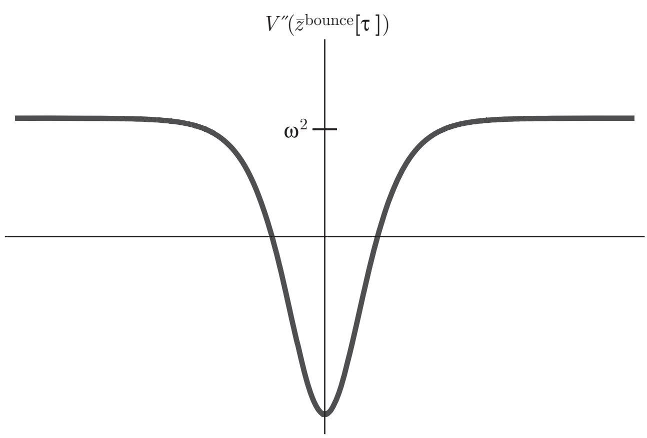

Figure 4.5. The form of the potential $V''(\bar{z}^{bounce}(\tau))$

has the potential given by Figure 4.5. One expects the spectrum to consist of a finite number of bound states and then a continuum beginning at ω^2. The ground-state wave function must have no nodes. The next bound energy level, if it exists, will have one node. We have already found a bound-state wave function with energy exactly zero, but it has one node. Thus there exists exactly one bound-state level, the nodeless ground state, with negative energy. The Gaussian integral in this direction in function space does not exist, and we must only define it through analytic continuation.

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4.4 Defining the Analytic Continuation

The original idea that the matrix element has an expansion of the form

$$\langle z=0|\, e^{-\frac{\beta \hat{h}(\hat{X},\hat{P})}{\hbar}}\, |z=0\rangle = e^{-\frac{(E+i\Gamma)\beta}{\hbar}}\, \langle 0|\, E+i\Gamma\, \rangle\, \langle E+i\Gamma|\, 0\rangle + \cdots \tag{4.8}$$

was ill-conceived. There is no eigenstate of the Hamiltonian corresponding to the meta-stable state. The Hamiltonian is a hermitean operator with all eigenvalues real, an eigenstate with a complex eigenvalue simply does not exist. We can only obtain the imaginary energy of the meta-stable state through analytic continuation. We imagine the analytic continuation in a parameter α which starts at $\alpha=0$ with a potential with a stable bound state localized at $z=0$, but yields our original potential at $\alpha=1$. The energy of the bound state will also be an analytic function of the parameter α. As long as a true bound state exists around $z=0$, this energy will be a real function of the parameter α. When the parameter is continued to yield our original potential where the bound state becomes meta-stable, we expect that this energy as an analytic function of the parameter α will not remain real and will gain an imaginary part. This imaginary part should correspond to the width of the meta-stable state. These general considerations correspond to a sequence of potentials, as shown in Figures 4.6, 4.7 and 4.8.

4.4.1 An Explicit Example

We will confirm these ideas with an explicit demonstration in a specific solvable potential. The example we consider is

$$V(\alpha,z) = -\left(\alpha - \frac{1}{2}\right) z^4 + \omega^2 z^2 \tag{4.9}$$

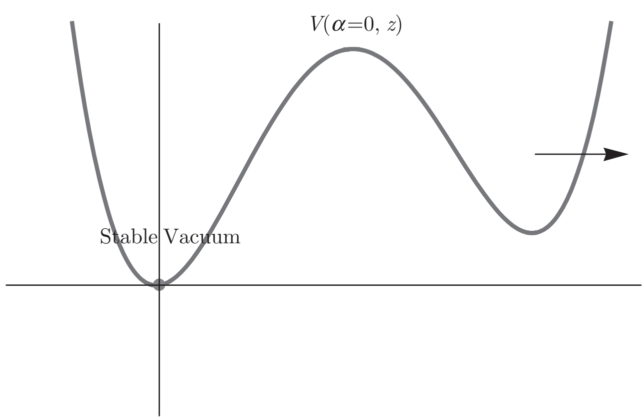

Figure 4.6. The potential with a stable state at $z=0$ for $\alpha=0$

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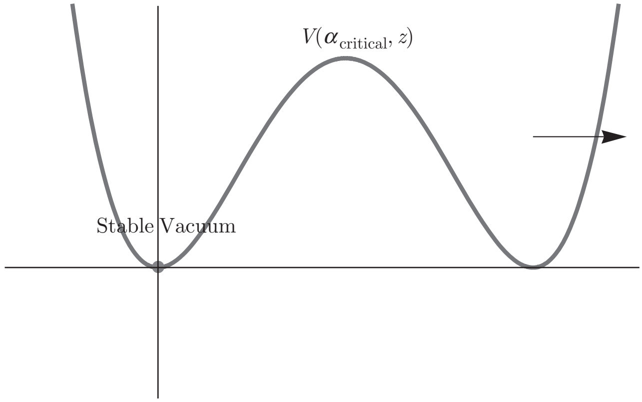

Figure 4.7. The critical potential with a stable state at $z = 0$ for $\alpha_{critical}$

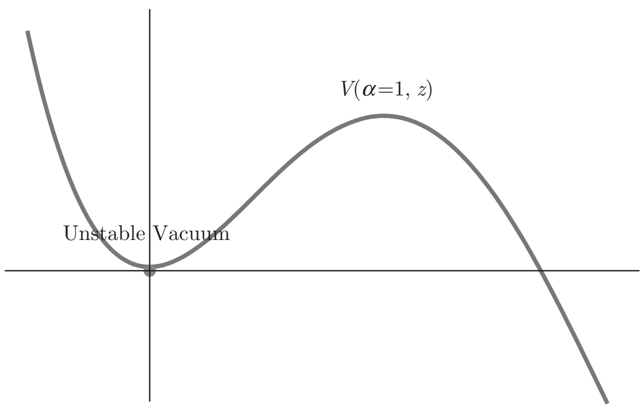

Figure 4.8. The potential with a meta-stable state at $z = 0$ for $\alpha = 1$

and the integral

$$\mathcal{I}(\alpha,\omega) = \int_{-\infty}^{\infty} dz e^{-\frac{1}{\hbar}\left(-\left(\alpha-\frac{1}{2}\right)z^4+\omega^2 z^2\right)}, \tag{4.10}$$

which is analogous to the integral over the direction corresponding to the negative mode in the definition of the determinant, when $\alpha = 1$, as depicted in Figures 4.9 and 4.10. The integral is defined for $\alpha \leq \frac{1}{2}$ and, specifically, it is not defined for $\alpha = 1$. The integral is actually well-defined for complex α, with the condition $\Re e\{\alpha\} \leq \frac{1}{2}$. We can define the analytic function $\mathcal{I}(\alpha,\omega)$ for $\Re e\{\alpha\} > \frac{1}{2}$ by analytic continuation. In this simple case we have no difficulty whatsoever, for $\Re e\{\alpha\} \leq \frac{1}{2}$, the integral is known in terms of special functions,

$$\mathcal{I}(\alpha,\omega) = \frac{1}{2}\sqrt{\frac{\omega^2}{\left(\frac{1}{2}-\alpha\right)}}e^{\left(\frac{\omega^4}{8\hbar\left(\frac{1}{2}-\alpha\right)}\right)} K_{\frac{1}{4}}\left(\frac{\omega^4}{8\hbar\left(\frac{1}{2}-\alpha\right)}\right), \tag{4.11}$$

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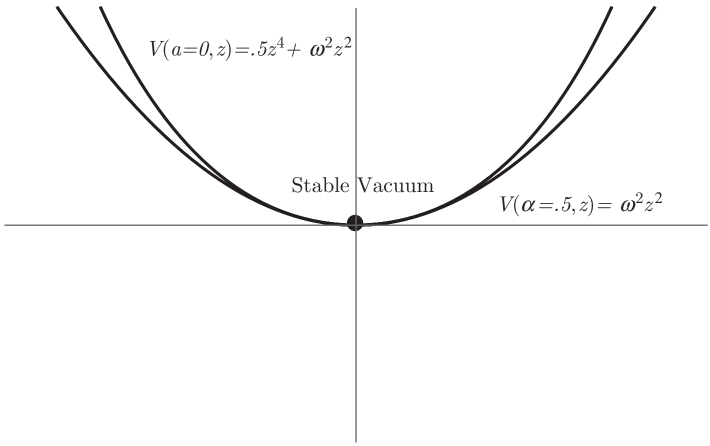

Figure 4.9. The potential with a stable state at $z=0$ for $\alpha=0$ and $\alpha=.5$

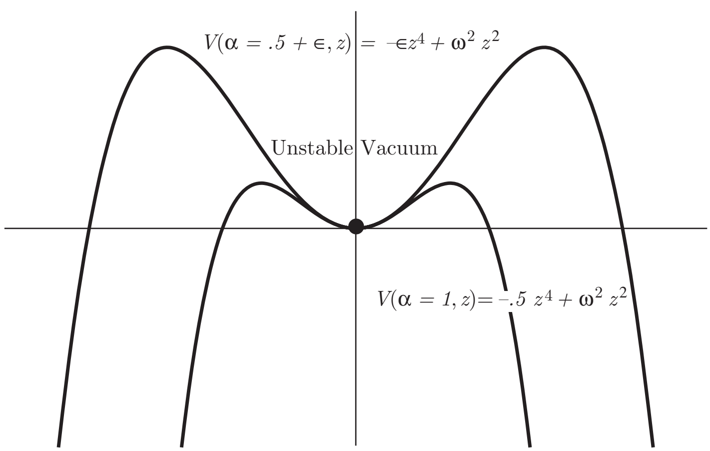

Figure 4.10. The potential with a meta-stable state at $z=0$ for $\alpha=.5+\epsilon$ and for $\alpha=1$

where

$$K_\nu(z)=\frac{\pi i}{2}e^{\frac{\pi}{2}\nu i}\left(J_\nu(iz)+N_\nu(iz)\right) \tag{4.12}$$

is the modified Bessel function of imaginary argument. The expression in Equation (4.11) has a well-defined analytic continuation throughout the complex α-plane, except on the real α-axis, where starting at $\alpha=\frac{1}{2}$, there is a branch cut.

But in general, we do not have the luxury of knowing the integral exactly. There is, however, a method for performing the analytic continuation more implicitly. Happily, this method allows us to extract the information that we actually seek, the imaginary part of the energy. We apply the method to the specific, exactly

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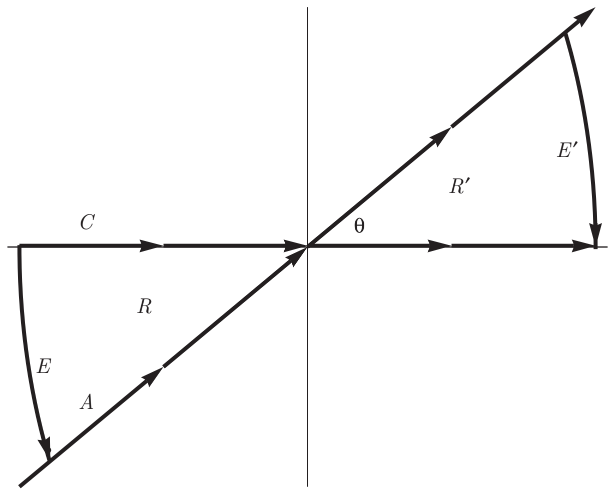

Figure 4.11. The original integration contour C, along the real line and the deformed contour A, a straight line at an angle θ, for the analytic continuation

solvable integral of Equation (4.11) to see in detail how the implicit method works. Indeed, we can obtain the analytic continuation of a function defined by a contour integral, by deforming the integration contour. In our example

$$\mathcal{I}(\alpha,\omega) = \int_{-\infty}^{\infty} dz e^{-\frac{1}{\hbar}\left(-\left(\alpha-\frac{1}{2}\right)z^4+\omega^2 z^2\right)} \quad \text{for} \quad \text{Real}(\alpha) \leq \frac{1}{2} \tag{4.13}$$

corresponds to the integration contour C along the real axis, in Figure 4.11. The integration is defined for $\left|\arg\left(-\left(\alpha-\frac{1}{2}\right)\right)\right| < \frac{\pi}{2}$. We deform the contour to $E + A + E'$ as in Figure 4.11, the integral is invariant since there are no poles in regions R and R' and if the contributions from the circular arcs E, E', vanish for infinite radius, which is assumed to be true, we get

$$\mathcal{I}(\alpha,\omega) = \int_{z=re^{i\theta}} dz e^{-\frac{1}{\hbar}\left(-\left(\alpha-\frac{1}{2}\right)z^4+\omega^2 z^2\right)}. \tag{4.14}$$

But now the integration converges for $\left|\arg\left(-\left(\alpha-\frac{1}{2}\right)\right)+4\theta\right| < \frac{\pi}{2}$ since after replacing $z = re^{i\theta}$ we must have that $-(\alpha-\frac{1}{2})e^{i4\theta}r^4$ has a positive real part. Thus a deformation of the contour defines an analytic continuation of the integral in the parameter α. If we take $\theta = \frac{\pi}{4}$ then $\left|\arg\left(-\left(\alpha-\frac{1}{2}\right)\right)+\pi\right| < \frac{\pi}{2}$. This implies

$$\arg\left(-\left(\alpha-\frac{1}{2}\right)\right) \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right) \tag{4.15}$$

hence the integral is now defined for $\mathfrak{Re}\{-\left(\alpha-\frac{1}{2}\right)\} < 0$, which is negative. This means $\mathfrak{Re}\{\alpha\} > \frac{1}{2}$. Thus we define, with A corresponding to the contour with

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$\theta = \pi/4$,

$$\mathcal{I}(\alpha > \frac{1}{2}, \omega) = \int_A dz e^{-\frac{1}{\hbar}\left(-\left(\alpha - \frac{1}{2}\right) z^4 + \omega^2 z^2\right)}. \tag{4.16}$$

This is an exact expression for the analytic continuation of the original integral $\mathcal{I}(\alpha < \frac{1}{2}, \omega) \to \mathcal{I}(\alpha > \frac{1}{2}, \omega)$, and there is no question as to its existence. However, we wish to actually evaluate the integral in the approximation as $\hbar \to 0$ and extract only the imaginary part.

4.5 Extracting the Imaginary Part

Consider the part of the contour A from 0 to ∞ in the first quadrant. The other half of the contour clearly gives the same contribution. We will calculate this integral approximately using the method of steepest descent, which is the indicated approximation method in the limit $\hbar \to \infty$. To do this, we further deform the contour from its present path between 0 and $\infty \times e^{i\frac{\pi}{4}}$ to the path of the steepest descent between these points. As there are no poles in the integrand, the integral clearly is invariant under this additional deformation.

4.5.1 A Little Complex Analysis

A contour of the steepest descent for the real part of a complex analytic function keeps the imaginary part constant (and vice versa). We can easily demonstrate this fact. If we have $f(x,y) = R(x,y) + iI(x,y)$ and a curve parametrized by a variable t, $(x(t), y(t))$ with tangent vector $\overrightarrow{(\dot{x}(t), \dot{y}(t))}$, the curve will correspond to the steepest descent of the real part $R(x,y)$ if the tangent vector is anti-parallel to its gradient, as the gradient points in the direction of maximum change. Therefore,

$$\overrightarrow{(\partial_x R(x,y), \partial_y R(x,y))} \times \overrightarrow{(\dot{x}(t), \dot{y}(t))} = \partial_x R(x,y)\dot{y}(t) - \partial_y R(x,y)\dot{x}(t) = 0. \tag{4.17}$$

Due to analyticity, the Cauchy–Riemann equations give

$$\partial_x R(x,y) = \partial_y I(x,y) \quad \text{and} \quad \partial_x I(x,y) = -\partial_y R(x,y) \tag{4.18}$$

thus Equation (4.17) gives

$$\partial_y I(x,y)\dot{y}(t) - (-\partial_x I(x,y))\dot{x}(t) \equiv \frac{d}{dt} I(x,y) = 0. \tag{4.19}$$

But this means $I(t) = \text{constant}$, demonstrating that the imaginary part of the complex analytic function remains constant on the paths of steepest descent of the real part.

In general for an integral of the form

$$\mathcal{I} = \int_a^b dz e^{\lambda f(z)} \tag{4.20}$$

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we can describe the process of the method of steepest descent as follows. For the application of the method of steepest descent, a should be a critical point of $f(z)$. We assume $\Re e\{f(a)\} > \Re e\{f(b)\}$ and $\Im m\{f(a)\} > \Im m\{f(b)\}$ and we start from a following the path of steepest descent of the $\Re e\{f(z)\}$ to a' where $\Re e\{f(a')\} = \Re e\{f(b)\}$ and then we append the path of steepest descent of $\Im m\{f(z)\}$ from a' to b along a path where now only the imaginary part of $f(z)$ changes. If $\Im m\{f(a)\} < \Im m\{f(b)\}$, we obviously ascend the appropriate portion. In the limit $\lambda \to \infty$ it is only the first contour which is important, since the second is multiplied by $e^{\lambda \Re e\{f(a')\}} \ll e^{\lambda \Re e\{f(a)\}}$. Finally we perform the integration over the first contour in the Gaussian approximation about $z = a$.

There are two further points to be made. First, we are actually only interested in the imaginary part of the integral, as it is only this part that we believe will have a leading contribution that is non-perturbative in $\hbar$. Second, and this is very important to the first, if the path of steepest descent of the real part of $f(z)$ passes through an ordinary critical point of $f(z)$, it abruptly changes direction by 90°. We can demonstrate this easily. An ordinary critical point of $f(z)$, which requires $f'(z_0) = 0$ and assumes $f''(z_0) \neq 0$, implies the behaviour

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \cdots. \tag{4.21}$$

Replacing $z - z_0 = x + iy$ we get

$$f(z_0 + x + iy) = f(z_0) + \frac{1}{2} f''(z_0)(x^2 - y^2 + 2ixy) + \cdots. \tag{4.22}$$

Then paths of steepest descent passing through the critical point are paths of the constant imaginary part of $f(z_0 + x + iy)$ passing through $x = y = 0$, *i.e.* $\Im m\{f(z_0 + x + iy)\} = \Im m\{f(z_0)\}$. Therefore, to lowest non-trivial order, we need paths with $\Im m\{f''(z_0)(x^2 - y^2 + 2ixy)\} = 0$. If $f''(z_0) = r + is$ this gives

$$s(x^2 - y^2) + 2rxy = 0. \tag{4.23}$$

If $s = 0$, the solutions are $x = 0$ or $y = 0$, which are perpendicular horizontal or vertical lines, respectively, hence crossing at 90°. Assuming $s \neq 0$,

$$x^2 + 2\frac{r}{s}xy + \left(\frac{ry}{s}\right)^2 = y^2\left(1 + \left(\frac{r}{s}\right)^2\right). \tag{4.24}$$

This gives

$$\left(x + \frac{ry}{s}\right) = \pm y\left(1 + \left(\frac{r}{s}\right)^2\right)^{\frac{1}{2}} \tag{4.25}$$

that is the curves, which are just the straight lines

$$x = \pm y\left(\left(1 + \left(\frac{r}{s}\right)^2\right)^{\frac{1}{2}} \mp \frac{r}{s}\right) \tag{4.26}$$

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with the $\pm$ signs correlated. The tangents at $x = y = 0$ are given by the directions $\left(\pm\left(\left(1+\left(\frac{r}{s}\right)^2\right)^{\frac{1}{2}} \mp \frac{r}{s}\right), 1\right)$. These are clearly orthogonal, as their scalar product vanishes,

$$-\left(\left(1+\left(\frac{r}{s}\right)^2\right)^{\frac{1}{2}} - \frac{r}{s}\right)\left(\left(1+\left(\frac{r}{s}\right)^2\right)^{\frac{1}{2}} + \frac{r}{s}\right) + 1 = 0. \tag{4.27}$$

Thus in complete generality, the paths of steepest descent turn abruptly by $90°$ as they pass through an ordinary critical point.

The real or imaginary parts of complex analytic functions are called harmonic functions, which means that they satisfy $\nabla^2 R(x,y) = \nabla^2 I(x,y) = 0$. As should be well known, all critical points of the real or imaginary parts of complex analytic functions are saddle points. Then a path of steepest descent, which descends through an ordinary critical point, must change direction by $90°$, since continuing in the same direction through the critical point would correspond to ascending the other side of the saddle. Turning through $90°$ continues the descent through the saddle point. The above analysis shows that for an ordinary critical point, $f'(z_0) = 0$ but $f''(z_0) \neq 0$, the minimum and maximum directions are at $90°$ to each other.

For our integral Equation (4.16), the real part of the exponent changes from $0 \to -\infty$ as z varies from $0 \to \infty \times e^{i\frac{\pi}{4}}$, while the imaginary part of the exponent is equal to 0 at $z = 0$ but becomes arbitrarily large at $z = \infty \times e^{i\frac{\pi}{4}}$. Thus in this case, along the path of steepest descent of the real part, the imaginary part of the exponent will always be equal to 0, since it must remain constant and it vanishes at the initial point. Such a path will reach a point where $\mathfrak{Re}\{f(z_0)\} = -\infty$. Then further following a contour with fixed real part, equal to $-\infty$, but changing imaginary part will be irrelevant since the factor corresponding to the exponential of the real part will already be zero.

Our function actually has three critical points. Indeed,

$$\frac{d}{dz}\left(\left(\alpha - \frac{1}{2}\right) z^4 - \omega^2 z^2\right) = 4\left(\alpha - \frac{1}{2}\right) z^3 - 2\omega^2 z = 0 \tag{4.28}$$

has the solutions $z = 0$ and $z = \frac{\pm\omega}{\sqrt{2\left(\alpha - \frac{1}{2}\right)}}$ for the case at hand, $\alpha > \frac{1}{2}$. Thus the point $z = 0$ happens also to be a critical point, and it is easy to check that the path of steepest descent of the real part from $z = 0$ proceeds along the positive real axis, instead of the contour A, until it reaches the critical point at $z = \frac{\omega}{\sqrt{2\left(\alpha - \frac{1}{2}\right)}}$, and then turns by $90°$ into the complex plane.

The path of steepest descent can be explicitly computed in our special case. The condition that the imaginary part be constant and equal to zero gives, with $z = x + iy$,

$$\mathfrak{Im}\left\{\left(\alpha - \frac{1}{2}\right)(x^2 - y^2 + 2ixy)^2 - \omega^2(x^2 - y^2 + 2ixy)\right\} = 0. \tag{4.29}$$

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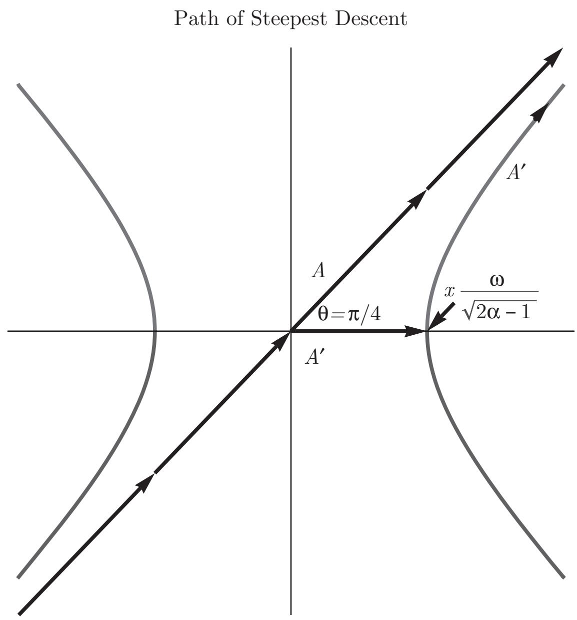

Figure 4.12. The integration contour along the path of steepest descent

Thus

$$\left(4\left(\alpha-\frac{1}{2}\right)(x^2-y^2)-2\omega^2\right)xy=0 \tag{4.30}$$

$$\Rightarrow x=0, \quad \text{or} \quad y=0, \quad \text{or} \quad (2\alpha-1)(x^2-y^2)=\omega^2 \tag{4.31}$$

The first two solutions simply describe the x and y axes, the third solution corresponds to a hyperbola. Note that all of these curves intersect at 90° as we expect. The path of steepest descent, starting at the origin and going out to infinity at $\infty\times e^{i\frac{\pi}{4}}$, corresponds to the curve A', as depicted in Figure 4.12. Asymptotically the arcs of the hyperbola converge to the lines $y=\pm x$ which is the original contour A. The turn by 90° occurs at the critical point at $z=x=\frac{\omega}{\sqrt{(2\alpha-1)}}$.

But now, where does the imaginary part to the integral come from? The integrand is always real, and the imaginary part of the function is always zero along the contour of steepest descent of the real part. It can only come from the integration measure dz when the contour follows the hyperbola in the complex plane. The contribution from $z=0$ to $z=\frac{\omega}{\sqrt{(2\alpha-1)}}$ along the real axis has no imaginary part, thus we are not interested in it. The integration along the hyperbola we perform in the Gaussian approximation about the critical point at $z=\frac{\omega}{\sqrt{(2\alpha-1)}}$. We have $x=\sqrt{y^2+\frac{\omega^2}{(2\alpha-1)}}$, $dx=\frac{ydy}{\sqrt{y^2+\frac{\omega^2}{(2\alpha-1)}}}$,

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so $dz = dx + idy = \left(\frac{y}{\sqrt{y^2+\frac{\omega^2}{(2\alpha-1)}}} + i\right) dy$ and the integral is

$$\int_0^\infty dy \left(\frac{y}{\sqrt{y^2+\frac{\omega^2}{(2\alpha-1)}}} + i\right) e^{\frac{1}{\hbar}\left(-\frac{\omega^4}{(4\alpha-2)} - 2\omega^2 y^2 + o(y^4)\right)}. \tag{4.32}$$

Therefore, the imaginary part comes only from the second term, and is given by

$$\frac{i}{2}\frac{\sqrt{2\pi\hbar}}{2\omega} e^{-\frac{\omega^4}{\hbar(4\alpha-2)}}, \tag{4.33}$$

where the factor of $1/2$ in front comes because we are only integrating over half the Gaussian peak, while the full Gaussian integral gives $\frac{\sqrt{2\pi\hbar}}{2\omega}$. Then for our original integral we get

$$\Im m \left\{ \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar}\left(-\left(\alpha-\frac{1}{2}\right)z^4 + \omega^2 z^2\right)} \right\}_{\alpha\to 1} = \frac{1}{2}\frac{1}{2\omega} e^{-\frac{\omega^4}{2\hbar}} \times 2, \tag{4.34}$$

where the factor of 2 arrives because we have the integral over the full contour of Figure 4.11, whereas the analysis above was only for half of the contour, the part in the first quadrant. We point out that the imaginary part of the integral simply corresponds to the formal expression of Equation (4.5) with $\lambda_{-1} \to |\lambda_{-1}|$.

4.6 Analysis for the General Case

Now, in the general case, we know what we must do. In order to do the path integral, we parametrize the space of all paths which satisfy the required boundary conditions for $z(\alpha, \tau = -\beta/2)$ and $z(\alpha, \tau = \beta/2)$ (β can be effectively taken to be ∞). We do this parametrization with one special, specific contour $z(\alpha, \tau)$ in the space of all paths, and augmented to this contour, we add the subspace of all paths orthogonal to this contour (which we will label as $z_\perp(\tau)$). To be very clear, a contour is not a path, it is a curve, itself parametrized by α, in the space of paths, where each point along the contour corresponds to a path $z(\alpha, \tau)$. The specific contour will contain two critical points

$$z(\alpha = 0, \tau) = \bar{z}(\tau) = 0, \tag{4.35}$$

which is the “instanton” corresponding to the particle just sitting on top of the unstable initial point in Figure 4.2 and never moving, and the point

$$z(\alpha = 1, \tau) = \bar{z}^{\text{bounce}}(\tau), \tag{4.36}$$

which corresponds to the instanton that we have called the “bounce”. This contour is represented pictorially in Figure 4.13 while the corresponding action is represented in Figure 4.14. We will see that the actual paths that the contour passes through are unimportant except for the two critical points. We also insist

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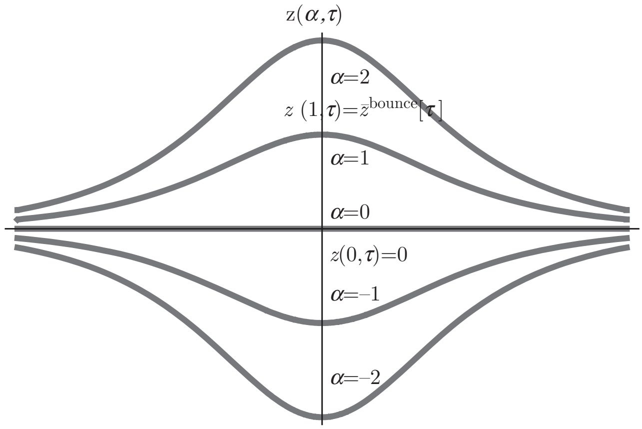

Figure 4.13. The path in function space as a function of α and τ

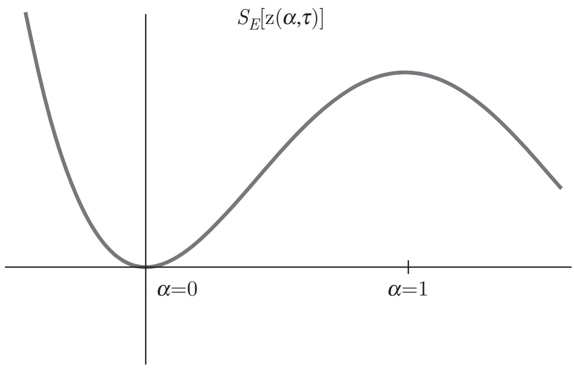

Figure 4.14. The Euclidean action as a function of α

that the "tangent" to the contour at $\alpha = 1$ corresponds to the negative energy mode

$$\frac{d}{d\alpha}\, z(\alpha,\tau)|_{\alpha\to 1} = z_{-1}(\tau). \tag{4.37}$$

In this way, the orthogonal directions never contain a negative mode and the determinant (path integral over $\mathcal{D}z_\perp$) can be done in principle. We then write the path integral as a nested product of two integrals

$$\mathcal{N}\int \mathcal{D}z(\tau)e^{-\frac{1}{\hbar}S_E[z(\tau)]} = \mathcal{N}\int \frac{d\alpha}{\sqrt{2\pi\hbar}}\mathcal{D}z_\perp(\tau)e^{-\frac{1}{\hbar}S_E[z(\tau)]}. \tag{4.38}$$

It is important to note that the path integral over the transverse directions is α-dependent. However, we will find that, since we are actually only interested in finding the imaginary part of the full integral, we will need to evaluate this transverse integral only at $\alpha = 1$.

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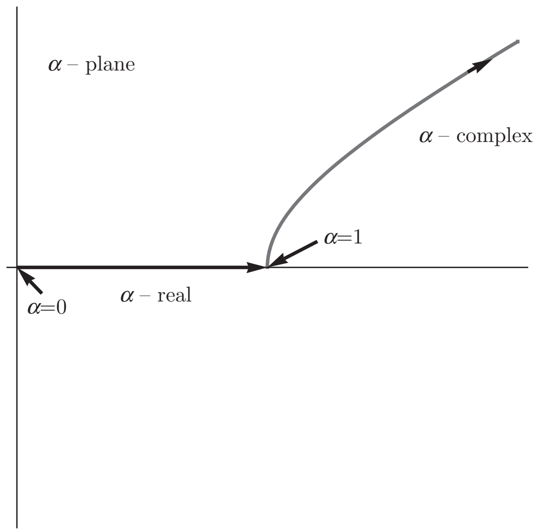

Figure 4.15. The contour for α from the origin, along the real axis and then jutting out into the complex plane at 90° at $\alpha = 1$

Now the integral over α is, however, ill-defined due to the existence of the negative mode at $\alpha = 1$. As $\alpha = 0$ is a critical point which is a local minimum, the action increases as we pass from $\alpha = 0$ to $\alpha = 1$ through real values of α. Hence the path in function space is defined as the path of steepest descent of $-S_E[z(\alpha,\tau]$, the exponent (up to the trivial factor of $1/\hbar$) in the integral Equation (4.38). But then we encounter the second critical point at $\alpha = 1$, which is a local maximum of the action, again for real α. The action behaves as depicted in Figure 4.14. Hence continuing the integral past $\alpha = 1$ to $\alpha = \infty$, it fails to converge and give a sensible answer. However, we are actually only trying to find an imaginary component of the original expression. If in fact we could integrate from $\alpha = 1$ on to $\alpha = \infty$, the expression would remain completely real. Thus we can only be content that we must define the integral via analytic continuation, since that is the only possible way that the integral could obtain an imaginary component.

This analytic continuation is expressed as a deformation of the contour of integration into the complex α-plane as we saw in the previous section. From $\alpha = 1$ we must follow along the contour of steepest descent of $-S_E[z(\alpha,\tau)]$. The important point, as we have seen, is that for an ordinary critical point, which is generic and that we assume, this corresponds to a 90° turn into the complex plane, as depicted in Figure 4.15. We start at $\alpha = 0$ and go till $\alpha = 1$ on the real α line, then we continue out at 90° into the complex α-plane following the line of steepest descent of $-S_E[z(\alpha,\tau)]$.

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As before, the imaginary part only comes from the integration measure; the imaginary part of $-S_E[z(\alpha,\tau)]$ on the path of steepest descent is constant and hence always zero. This gives for the imaginary part of the path integral for the fluctuations about one bounce (using the notation $\mathcal{A.C.}$ to mean "analytic continuation"),

$$
\begin{aligned}
&\Im m\left\{\mathcal{A.C.}\left(\mathcal{N}\int\frac{d\alpha}{\sqrt{2\pi\hbar}}\mathcal{D}z_\perp(\tau)e^{-\frac{1}{\hbar}S_E[z(\alpha,\tau)]}\right)\right\}=\\
&=\Im m\left\{\mathcal{A.C.}\int\frac{d\alpha}{\sqrt{2\pi\hbar}}e^{-\frac{1}{\hbar}\left(S_E[\bar{z}^{bounce}(\tau)]+\frac{1}{2}\frac{d^2}{d\alpha^2}S_E[z(\alpha,\tau)]|_{\alpha=1}(\alpha-1)^2+\cdots\right)}\right\}\times\\
&\quad\times\mathcal{N}\int\mathcal{D}z_\perp(\tau)e^{-\frac{1}{\hbar}\int d\tau' d\tau''\left(\frac{1}{2}\left.\frac{\delta^2 S_E[z(\tau)]}{\delta z_\perp(\tau')\delta z_\perp(\tau'')}\right|_{z(\tau)=\bar{z}^{bounce}(\tau)}\delta z_\perp(\tau')\delta z_\perp(\tau'')+\cdots\right)}\\
&=\Im m\left\{\mathcal{A.C.}\int\frac{d\alpha}{\sqrt{2\pi\hbar}}e^{-\frac{1}{\hbar}\left(S_0+\frac{1}{2}\lambda_{-1}(\alpha-1)^2+\cdots\right)}\right\}\times\\
&\quad\times\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}}\beta\mathcal{N}\left(\det{}'\left[-\frac{d^2}{d\tau^2}+V''(\bar{z}^{\text{bounce}}(\tau))\right]\right)^{-\frac{1}{2}}\\
&=\frac{1}{2}\times e^{-\frac{S_0}{\hbar}}\times\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}}\beta\times\frac{1}{\sqrt{|\lambda_{-1}|}}\times\mathcal{N}\left(\det{}'\left[-\frac{d^2}{d\tau^2}+V''(\bar{z}^{\text{bounce}}(\tau))\right]\right)^{-\frac{1}{2}},
\end{aligned}
\tag{4.39}
$$

where $\frac{d^2}{d\alpha^2}S_E[z(\alpha,\tau)]|_{\alpha=1}=\lambda_{-1}$ is the negative eigenvalue and $\det'$ now means the determinant is calculated excluding both the zero eigenvalue and the negative eigenvalue. In the last line of Equation (4.39), each factor separated by the $\times$ signs correspond, respectively, to: the factor of one-half since we are integrating over only half of the Gaussian peak, the exponential of minus the action of the bounce divided by $\hbar$, the factor corresponding to the Jacobian of the change of variables and the factor of β when we integrate over the position of the bounce rather than its translational zero mode, the factor of one over the square root of the magnitude of the negative eigenvalue which is the upshot of our analysis of the analytic continuation, and finally the primed determinant over the orthogonal directions in the space of paths where the negative mode and the zero mode are removed. Taking into account the contribution from the multi-bounce sector, the one-bounce contribution, including its imaginary part, just exponentiates as before.

Thus K, as defined in Equation (4.3), changes as $K\to\Re e\{K\}+i\Im m\{K\}$ and we find

$$
\Im m\{K\}=\frac{1}{2}\frac{1}{\sqrt{|\lambda_{-1}|}}\left(\frac{\det'\left[-\frac{d^2}{d\tau^2}+V''(\bar{z}^{\text{bounce}}(\tau))\right]}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]}\right)^{-\frac{1}{2}},
\tag{4.40}
$$

where we now understand the factor of $\frac{1}{2}$ as coming from integrating, in the Gaussian approximation, over just half of the saddle point descending into the

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complex α-plane and the primed determinant is now understood to exclude both the zero mode and the negative mode. Thus the original matrix element that we wish to calculate, Equation (4.1), is obtained from an analytic continuation

$$\begin{aligned}\mathcal{A.C.}\left\{\langle z{=}0|\, e^{-\frac{\beta}{\hbar}\hat{h}(\hat{X},\hat{P})}\, |z{=}0\rangle\right\} &= \mathcal{A.C.}\left\{\mathcal{N}\int_{z\left(\pm\frac{\beta}{2}\right)=0}\mathcal{D}z(\tau)e^{-\frac{1}{\hbar}S_E[z(\tau)]}\right\}\\ &= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\beta\omega}{2}} e^{\beta K\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{S_0}{\hbar}}} + \cdots\\ &= e^{-\beta(E_0+i\Gamma/2)/\hbar}\mathcal{A.C.}\left\{|\langle E_0|\,0\rangle|^2 + \cdots\right\}, \end{aligned} \tag{4.41}$$

where

$$K = \Re e(K) + i\frac{1}{2}\frac{1}{\sqrt{|\lambda_{-1}|}}\left(\frac{\det'\left(-\frac{d^2}{d\tau^2}+V''\left(\bar{z}^{bounce}(\tau)\right)\right)}{\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)}\right)^{-\frac{1}{2}}. \tag{4.42}$$

This yields the imaginary part to the energy, $i\Gamma/2$, with the width of the state

$$\Gamma = \hbar\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}}\frac{e^{-\frac{S_0}{\hbar}}}{\sqrt{|\lambda_{-1}|}}\left(\frac{\det'\left(-\frac{d^2}{d\tau^2}+V''\left(\bar{z}^{bounce}(\tau)\right)\right)}{\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)}\right)^{-\frac{1}{2}}. \tag{4.43}$$