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THE FIRST-ORDER THEORY OF BRANCH GROUPS

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Abstract

It is shown that for many branch groups G the action on the ambient tree can be interpreted in G, in the sense of first-order model theory.

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1. Introduction

The class of branch groups was introduced by Grigorchuk in order to provide a framework for the simultaneous study of various important examples in group theory. Some of these examples were first drawn to the author's attention by Laci Kovács during a meeting in Oberwolfach in the 1980s. The structure theory of branch groups is now quite well developed; see, for example, [2, 4–6, 8]. In particular, it was shown in [6] that in many cases the structure of the group G determines the maximal tree T on which it acts as a branch group and also the action on T. Here we prove that, for such a branch group G, the tree and the action of G on it are interpretable in G in the sense of first-order model theory.

The trees on which branch groups act are infinite rooted trees such that all vertices at the same distance from the root vertex have the same finite valency. The distance of a vertex v from the root v_0 is called the *level* of v, and the set L_n of vertices of level n is called the nth *layer* of T. We can regard (the set of vertices of) T as a partially ordered set by writing $u \leq v$ if and only if the simple path from u to v_0 passes through v. Clearly the tree structure of T can be reconstructed from the partial order. Each vertex v is the root of the subtree $T_v = \{u \mid u \leq v\}$.

Let G be a group that acts faithfully on T, fixing v_0. For each vertex v write $\mathrm{rst}_G(v)$ for the subgroup of elements of G that fix all vertices outside T_v, and for each $n > 0$

write $\mathrm{rst}_G(n)$ for the direct product $\langle \mathrm{rst}_G(v) \mid v \in L_n \rangle$. The action of G on T is called a *branch action* if G acts transitively on L_n and $|G\colon \mathrm{rst}_G(n)|$ is finite for each $n > 0$, and G is called a *branch group* if it has a branch action on some tree.

Although most well-known branch groups G have a branch action on a unique maximal tree up to G-equivariant isomorphism, not all branch groups have this property. The possible actions are encoded in the action of G on its *structure graph*, discussed in Section 2. Our main result here can be described as follows. A stronger statement concerning structure graphs is given in Section 4.

THEOREM. *There are first-order formulae τ, $\beta(x)$ and $\delta(x, y)$ such that the following statements hold for each branch group G:*

(a) *G has a branch action on a unique maximal tree up to G-equivariant isomorphism if and only if G satisfies τ;*
(b) *the set $S = \{x \mid \beta(x)\}$ is a union of conjugacy classes, and so G acts on it by conjugation;*
(c) *the relation on S defined by $\delta(x, y)$ is a preorder preserved by G, and so the quotient $Q = S/{\sim}$, where $\sim$ is the equivalence relation defined by $\delta(x, y) \wedge \delta(y, x)$, is a partially ordered set on which G acts;*
(d) *if G satisfies τ then Q is G-equivariantly isomorphic as a partially ordered set to the maximal tree on which G acts as a branch group.*

In the important cases when G has a branch action on a unique maximal tree T, the theorem gives a representation of T as the quotient of a definable subset of G modulo a definable equivalence relation. This provides a parameter-free interpretation for the tree T, and also for the action on T. For a discussion of interpretability in the context of groups we refer the reader to [3, Ch. 3].

2. Basal subgroups and structure graph

Branch groups are subject to strong restrictions. Since they act faithfully with finite orbits of unbounded length on trees, they are infinite and residually finite. It was shown in [5] that their proper quotient groups are virtually abelian and their non-trivial normal subgroups are non-abelian. Further information about their subgroup structure was given in [4, 6, 8]. Write $L(G)$ for the family of subgroups of G having only finitely many conjugates. In [4] the following results were proved.

LEMMA 2.1. *Let G be a branch group.*

(a) [4, Proposition 2.2]. *If $H, K \in L(G)$ with $K \triangleleft H$ and H/K virtually nilpotent then* $\mathrm{C}_G(H) = \mathrm{C}_G(K)$. *In particular, $L(G)$ contains no non-trivial virtually nilpotent subgroups.*
(b) [4, Lemma 2.3]. *If $H_1, H_2 \in L(G)$ then $H_1 \cap H_2 = 1$ if and only if $[H_1, H_2] = 1$.*
(c) (See [4, Lemma 2.5]). *If $H, K \in L(G)$ with $K \triangleleft H$ and* $\mathrm{C}_G(H) = \mathrm{C}_G(K)$ *then H/K is virtually abelian.*

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The family $L(G)$ is a sublattice of the subgroup lattice of G, and the *structure lattice* $\mathcal{L}(G)$ of a branch group G may be defined to be the quotient of $L(G)$ modulo the relation $\sim$ defined by writing $H_1 \sim H_2$ if and only if $\mathrm{C}_G(H_1) = \mathrm{C}_G(H_2)$; this equivalence relation can be shown to be a congruence on the lattice $L(G)$. A subgroup B in $L(G)$ is called *basal* if its conjugates in G generate their direct product, or equivalently (for a branch group G), if $B^g = B$ or $B^g \cap B = 1$ for each $g \in G$. The *structure graph* of G is then the set of classes $[B] \in \mathcal{L}(G)$ containing some basal subgroup B; two classes $[B_1]$, $[B_2]$ are joined by an edge if one class is contained in the other and there are no intermediate classes in the ordering inherited from $\mathcal{L}(G)$. The conjugation action of G induces an action on both $\mathcal{L}(G)$ and the structure graph. The relevance of the structure graph stems from the fact that the tree on which G acts can be equivariantly embedded as a cofinal subset in the structure graph, and that in many important cases the embedding is an isomorphism of trees; see [6, 8].

We shall now give a different description of the structure graph. For a subset Y of G we write $\mathrm{C}^2_G(Y)$ for $\mathrm{C}_G(\mathrm{C}_G(Y))$. Thus $Y \subseteq \mathrm{C}^2_G(Y)$. We say that a subgroup H is C^2-*closed* if $H = \mathrm{C}^2_G(H)$. Since $\mathrm{C}_G(\mathrm{C}_G(\mathrm{C}_G(Y))) = \mathrm{C}_G(Y)$ for each subset Y, the C^2-closed subgroups can also be described as the subgroups that are centralizers.

Lemma 2.2. *Let G be a branch group.*

(a) *If $H_1, H_2 \in L(G)$ are subgroups with the same centralizer then $\mathrm{C}^2_G(H_1) = \mathrm{C}^2_G(H_2)$.*

(b) *If B is a basal subgroup then so is $\mathrm{C}^2_G(B)$.*

(c) *If B_1, B_2 are C^2-closed basal subgroups and $B_1 < B_2$ then $\mathrm{N}_G(B_1) < \mathrm{N}_G(B_2)$.*

Proof. Assertion (a) is clear.

To prove (b), suppose that $g \in G$ and $\mathrm{C}^2_G(B)^g \cap \mathrm{C}^2_G(B) \neq 1$. By Lemma 2.1(c), each of $\mathrm{C}^2_G(B)/B$ and $\mathrm{C}^2_G(B)^g/B^g$ is virtually abelian. Hence $(\mathrm{C}^2_G(B)^g \cap \mathrm{C}^2_G(B))/(B^g \cap B)$ is virtually abelian, and so $B^g \cap B \neq 1$ by Lemma 2.1(a). Therefore $B^g = B$ and so $\mathrm{C}^2_G(B)^g = \mathrm{C}^2_G(B)$.

Finally, suppose that B_1, B_2 are as in (c). If $g \in \mathrm{N}_G(B_1)$ then $B_2 \cap B_2^g \geq B_1$ and so $g \in \mathrm{N}_G(B_2)$. Suppose that $\mathrm{N}_G(B_1) = \mathrm{N}_G(B_2)$; thus $B_1^g \neq B_1$ if and only if $B_2^g \neq B_2$. We have

$$\begin{aligned} B_1 &\leq B_2 \cap \langle B_1^g \mid g \in G\rangle = B_1(B_2 \cap \langle B_1^g \mid B_1^g \neq B_1\rangle) \\ &\leq B_1(B_2 \cap \langle B_2^g \mid B_2^g \neq B_2\rangle) \leq B_1, \end{aligned}$$

and so B_2/B_1 embeds in the virtually abelian group $G/\langle B_1^g \mid g \in G\rangle$. Therefore by Lemmas 2.1(a) and 2.2(a) we have $B_1 = B_2$, a contradiction. □

Lemma 2.3. *Let G be a branch group acting on a tree T. Then $\mathrm{C}^2_G(\mathrm{rst}_G(v)) = \mathrm{rst}_G(v)$ for each vertex v.*

Proof. Clearly $\mathrm{rst}_G(v) \leq \mathrm{C}^2_G(\mathrm{rst}_G(v))$. Now let $h \in \mathrm{C}^2_G(\mathrm{rst}(v))$. It suffices to prove that h fixes every vertex not in T_v, and this will follow if h fixes every such vertex u of level at least the level of v. We have $\mathrm{rst}_G(u) \leq \mathrm{C}(\mathrm{rst}_G(v))$, and so h centralizes $\mathrm{rst}_G(u)$. Thus $\mathrm{rst}_G(u) = (\mathrm{rst}_G(u))^h = \mathrm{rst}_G(uh)$, and so $uh = u$. □

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Let $\mathcal{B}(G)$ be the graph with vertices the non-trivial C^2-closed basal subgroups of G, and with an edge between two vertices if one of them is contained as a maximal proper C^2-closed basal subgroup of the other. The group G acts on $\mathcal{B}(G)$ by conjugation.

PROPOSITION 2.4. *Let G be a branch group acting on a tree T.*

(a) *The map $B \mapsto [B]$ is a G-equivariant isomorphism from $\mathcal{B}(G)$ to the structure graph of G.*
(b) *The map $v \mapsto \mathrm{rst}_G(v)$ is a G-equivariant order-preserving injective map from T to $\mathcal{B}(G)$.*

PROOF.
(a) Since $C_G(C_G(C_G(D))) = C_G(D)$ for each subgroup D, it follows from Lemma 2.2 that each class in the structure graph of G contains precisely one C^2-closed basal subgroup; this is then the largest subgroup in its class. Therefore the map is bijective. It is clearly both G-equivariant and an isomorphism of partially ordered sets, and hence a graph isomorphism.
(b) Lemma 2.3 shows that $v \mapsto \mathrm{rst}_G(v)$ is an order-preserving map from T to $\mathcal{B}(G)$; equivariance is clear and injectivity follows from [6, Proposition 2]. □

Some basic facts concerning the structure graph are clear from Proposition 2.4 and Lemma 2.2(c):

PROPOSITION 2.5. *For a branch group G the graph $\mathcal{B}(G)$ has the following properties.*

(a) *G is the only vertex fixed in the action of G on $\mathcal{B}(G)$.*
(b) *The orbit $O(B)$ of each vertex B is finite.*
(c) *Each vertex B is connected to the vertex G by a finite path; all simple such paths have length at most $\log_2(|O(B)|)$.*

Our next task is to elucidate the relationship between rigid stabilizers and C^2-closed basal subgroups.

LEMMA 2.6. *Let G act as a branch group on T.*

(a) *Let $1 \neq B \in L(G)$. Then there exists some $v \in T$ such that $[\mathrm{rst}(v)] \leq [B]$.*
(b) *Let $B \in \mathcal{B}(G)$. Then G has a branch action for which B is the restricted stabilizer of a vertex.*
(c) *If $\mathcal{B}(G)$ is a tree then G acts on it as a branch group.*

PROOF. Assertion (a) is [4, Proposition 3.2].
(b) Find a vertex $v \in T$ with $[\mathrm{rst}_G(v)] \leq [B]$; suppose that v is in the nth layer L_n of T. Since B is the largest element of its class by Lemma 2.2 we have $\mathrm{rst}_G(v) \leq B$, and we can assume that $\mathrm{rst}_G(v) < B$. Let $\overline{T}$ be the set consisting of G, all conjugates of B, and the union U of all layers L_m of T with $m \geq n$. The maximal elements of U constitute the layer L_n; their restricted stabilizers are the subgroups $\mathrm{rst}_G(vg) = \mathrm{rst}_G(v)^g$, and each lies in a unique conjugate of B. We regard $\overline{T}$ as a tree, with the same edges between elements of U as in T, with edges linking G and all conjugates of B, and with each

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maximal element u of U joined by an edge to the conjugate of B containing $\mathrm{rst}_G(u)$. It is clear that $\overline{T}$ is a tree. To see that G acts on it as a branch group, we need to observe that the restricted stabilizer of the layer of $\overline{T}$ to which B belongs contains $\mathrm{rst}_G(n)$ and so has finite index in G. The elements of the restricted stabilizer of B in this action act trivially on all subtrees T_{vg} of T with $\mathrm{rst}_G(vg) \nleq B$, and so the only elements w of U that they can move satisfy $\mathrm{rst}_G(w) \leq B$. Thus B is the restricted stabilizer of B in the action on $\overline{T}$.

(c) We only need to prove that $\prod_{g\in G} \mathrm{rst}(B^g)$ has finite index in G for each $B \in \mathcal{B}(G)$. By (a) the subgroup B contains $\mathrm{rst}_G(v)$ for some vertex v of a tree on which G acts as a branch group, and the conclusion follows since $\prod \mathrm{rst}(vg) \leq \prod B^g$. □

We say that T is a *maximal tree* on which a group G acts as a branch group if T cannot be G-equivariantly embedded in a strictly larger tree on which G acts as a branch group. An easy application of Zorn's lemma shows that each tree on which G acts as a branch group is contained in at least one maximal such tree.

Corollary 2.7. *Let G be a branch group. The following are equivalent:*

(i) *there is a unique maximal tree up to G-equivariant isomorphism on which G acts as a branch group;*
(ii) *$\mathcal{B}(G)$ is a tree;*
(iii) *for all B, B_1, $B_2 \in \mathcal{B}(G)$ with $B \leq B_1$ and $B \leq B_2$, either $B_1 \leq B_2$ or $B_2 \leq B_1$.*

Proof. The implication (ii) ⇒ (i) follows from Proposition 2.4(b) and Lemma 2.6(c).

Suppose that (i) holds, so that G acts as a branch group on a unique maximal tree T. Then all rigid stabilizers in all branch actions of G arise as rigid stabilizers in the action on T, and hence by Lemma 2.6(b), all C^2-closed basal subgroups are rigid stabilizers in the action on T. Let $B, B_1, B_2 \in \mathcal{B}(G)$ be the restricted stabilizers of vertices v, v_1, v_2 and suppose that $B \leq B_1$, $B \leq B_2$. Then v_1, v_2 lie in the path from v to the root of T, so that $v_1 \in T_{v_2}$ or $v_2 \in T_{v_1}$, and consequently $B_1 \leq B_2$ or $B_2 \leq B_1$. Therefore (i) ⇒ (iii).

Finally, suppose that $\mathcal{B}(G)$ is not a tree. Let C be a simple cycle in $\mathcal{B}(G)$, and let B be a vertex of C with the greatest number of conjugates in G. The vertices B_1, B_2 adjacent to B in C are then incomparable elements of $\mathcal{B}(G)$ containing B and so (iii) does not hold. Thus (iii) ⇒ (ii). □

3. Identification of restricted stabilizers

We shall use the description of the structure graph in the previous section to address the issue of the definability of the tree on which a branch group acts. Throughout this section, G is a branch group that acts on a tree T.

It follows from a result of Hardy [7] or a more general result of Abért [1] that branch groups satisfy no group laws. We give an *ad hoc* proof relating to a special case.

Lemma 3.1. *Let u be a vertex of T and write $R = \mathrm{rst}_G(u)$. Then R has elements x, y with $(xy)^2 \neq y^2x^2$, and also elements z, t with $z^2t^2 \neq t^2z^2$.*

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PROOF. Suppose the first assertion false. Then $x^2y^3 = yxyxy = y^3x^2$ for all $x, y \in R$, and so the characteristic subgroups $U = \langle x^2 \mid x \in R\rangle$ and $V = \langle y^3 \mid y \in R\rangle$ of R commute with each other. Therefore $U \cap V$ is an abelian subgroup in $L(G)$, and so trivial by Lemma 2.1(a). Hence R embeds in $R/U \times R/V$. Since groups of exponents 2 and 3 are nilpotent of class at most 2, Lemma 2.1(a) gives $R = 1$, and this is a contradiction.

Similarly, if all squares in R commute, then the group that they generate is abelian, and so trivial; therefore R is abelian and $R = 1$. □

DEFINITION. For each $h \in G$ write

$$X_h = \{[h^{-1}, h^k] \mid k \in G\}, \quad Y_h = \{x_1x_2x_3 \mid x_i \in X_h\},$$
$$W_h = \bigcup\{Y_{h^g} \mid g \in G, [Y_h, Y_{h^g}] \neq 1\}.$$

LEMMA 3.2. *If $h \in \mathrm{rst}_G(v)$ for some vertex v of T then $\mathrm{C}^2_G(W_h) \le \mathrm{rst}_G(v)$.*

PROOF. We have $X_h \subseteq \mathrm{rst}_G(v)$, since $\mathrm{rst}_G(vk) = \mathrm{rst}_G(v)^k$, and if $vk \neq v$ then $[\mathrm{rst}_G(v), \mathrm{rst}_G(vk)] = 1$. Therefore $Y_h \subseteq \mathrm{rst}_G(v)$, and $Y_{h^g} \subseteq \mathrm{rst}_G(vg)$ for all $g \in G$. If $[Y_h, Y_{h^g}] \neq 1$ it follows that $g \in \mathrm{stab}_G(v)$ and $Y_{h^g} \subseteq \mathrm{rst}_G(v)$. Hence $W_h \subseteq \mathrm{rst}_G(v)$ and $\mathrm{C}^2_G(W_h) \le \mathrm{C}^2_G(\mathrm{rst}_G(v)) = \mathrm{rst}_G(v)$ by Lemma 2.3. □

Our object now is to prove the reverse inequality for certain choices of h.

PROPOSITION 3.3. *For each vertex $v \in T$ there exists $h \in \mathrm{rst}_G(v)$ with $\mathrm{rst}_G(v) = \mathrm{C}^2_G(W_h)$.*

We conduct part of the proof of Proposition 3.3 in the following lemma.

LEMMA 3.4. *Let $u \in T$, write $R = \mathrm{rst}_G(u)$ and let $h \in G$.*

(a) *If $uh^2 \neq u$ then Y_h contains $\{y^{-2}(xy)^2x^{-2} \mid x, y \in R\}$.*

(b) *If $u \in L_n$ and $uh \neq u$ then the projection of $X_h \cap \mathrm{rst}_G(n)$ in $\mathrm{rst}_G(u)$ contains all squares in R.*

(c) *If u is as in* (b) *and $g \in G$ satisfies $uh^g \neq u$ then $[X_h, X_{h^g}] \neq 1$.*

PROOF.

(a) We have

$$[h^{-1}, h^x] = (x^{-1})^{h^{-1}} x (x^{-1})^h x \quad \text{for all } x \in R. \tag{$*$}$$

The four terms in the product on the right lie respectively in $\mathrm{rst}_G(uh^{-1})$, $\mathrm{rst}_G(u)$, $\mathrm{rst}_G(uh)$ and $\mathrm{rst}_G(u)$. Since $uh^2 \neq u$, the three subgroups $\mathrm{rst}_G(uh^{-1})$, $\mathrm{rst}_G(u)$, $\mathrm{rst}_G(uh)$ are distinct and commute with each other. Therefore for $x, y \in R$ we have

$$[h^{-1}, h^{y^{-1}}][h^{-1}, h^{xy}][h^{-1}, h^{x^{-1}}] = (y(xy)^{-1}x)^{h^{-1}}(y(xy)^{-1}x)^h(y^{-2}(xy)^2x^{-2})$$
$$= y^{-2}(xy)^2x^{-2}.$$

(b) This follows immediately from ($*$).

(c) By (b), the projections P_1, P_2 of $X_h \cap \mathrm{rst}_G(n)$ in R and in $\mathrm{rst}_G(ug^{-1})$ contain respectively $\{x^2 \mid x \in R\}$ and $\{x^2 \mid x \in R^{g^{-1}}\}$. Since the projection of $X_{h^g} \cap \mathrm{rst}_G(n)$ in $\mathrm{rst}_G(u)$ is equal to P_2^g we conclude that the projection of $[X_h \cap \mathrm{rst}_G(n), X_{h^g} \cap \mathrm{rst}_G(n)]$ in $\mathrm{rst}_G(u)$ contains $\{[x^2, y^2] \mid x, y \in R\}$, and hence that $[X_h, X_{h^g}] \neq 1$ by Lemma 3.1. □

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Proof of Proposition 3.3. Write $Q(u) = \{y^{-2}(xy)^2x^{-2} \mid x, y \in \mathrm{rst}_G(u)\}$ for each vertex u. Thus $Q(u) \neq \{1\}$ by Lemma 3.1, and $\langle Q(ug) \mid g \in G\rangle = \prod_{g\in G} Q(u^g)$ is a nontrivial normal subgroup of G.

First we choose a suitable element h. We find an element $h_0 \in \mathrm{rst}_G(v)$ with $h_0^2 \neq 1$, and $u_0 \in T_v$ with $u_0h_0^2 \neq u_0$. Let $u_0 \in L_n$. For this choice of u_0, choose $h \in \mathrm{rst}_G(v)$ with $u_0h^2 \neq u_0$ and with h fixing as few elements of $T_v \cap L_n$ as possible.

Let $g \in \mathrm{stab}_G(v)$. Consideration of hh^g shows that the supports of h, h^g on $T_v \cap L_n$ cannot be disjoint. Therefore there is some $u \in T_v \cap L_n$ with $uh \neq u$ and $uh^g \neq u$. Thus from Lemma 3.4 we have $[Y_h, Y_{h^g}] \neq 1$. It follows that

$$W_h = \bigcup\{Y_{h^g} \mid g \in \mathrm{stab}_G(v)\}.$$

Since $u_0h^2 \neq u_0$, from Lemma 3.4 we have $Q(u_0) \subseteq Y_h$ and hence

$$Q(u_0g) = Q(u_0)^g \subseteq Y_h^g = Y_{h^g} \subseteq W_h$$

for each $g \in \mathrm{stab}_G(v)$. Being a branch group, G acts transitively on L_n, and so $\mathrm{stab}_G(v)$ acts transitively on $T_v \cap L_n$. Hence $Q(w) \subseteq W_h$ for each $w \in T_v \cap L_n$, and so $K \leq \mathrm{C}_G^2(W_h) \leq \mathrm{rst}_G(v)$, where $K = \prod_{w\in T_v\cap L_n}\langle Q(w)\rangle$. However,

$$K = \Big(\prod_{l\in L_n}\langle Q_l\rangle\Big) \cap \mathrm{rst}_G(v),$$

and it follows that $\mathrm{rst}_G(v)/K$ embeds in $G/(\prod_{l\in L_n}\langle Q_l\rangle)$, and so is virtually abelian. From Lemma 2.1(a) it follows that K, $\mathrm{C}_G^2(W_h)$ and $\mathrm{rst}_G(v)$ have the same centralizer, and hence, since $\mathrm{C}_G^2(U_h)$ and $\mathrm{rst}_G(v)$ are C^2-closed, that $\mathrm{C}_G^2(W_h) = \mathrm{rst}_G(v)$, as required. □

Corollary 3.5. *For each $B \in \mathcal{B}(G)$ there exists some $h \in G$ with $B = \mathrm{C}_G^2(W_h)$.*

Proof. The subgroup B is a rigid stabilizer in the branch action of G on some tree, by Proposition 2.6(b). □

4. Interpretability of structure graph and uniqueness of maximal tree

We prove the following result.

Theorem 4.1. *There are first-order formulae τ, $\beta(x)$ and $\delta(x, y)$ such that the following statements hold for each branch group G:*

(a) *G has a branch action on a unique maximal tree up to G-equivariant isomorphism if and only if G satisfies τ;*

(b) *the set $S = \{x \mid \beta(x)\}$ is a union of conjugacy classes, and so G acts on it by conjugation;*

(c) *the relation on S defined by $\delta(x, y)$ is a preorder preserved by G, and so the quotient $Q = S/\sim$, where $\sim$ is the equivalence relation defined by $\delta(x, y) \wedge \delta(y, x)$, is a partially ordered set on which G acts;*

(d) *Q is G-equivariantly isomorphic as a partially ordered set to the structure graph of G.*

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This result, together with facts in Section 2, implies the theorem stated in Section 1. Indeed, it differs from that result only in the fact that it applies for all branch groups, and not just for those that act on a unique maximal tree. Assertion (d) provides a parameter-free interpretation for the whole structure graph and for the action of G on this graph, since the action of G on Q is induced by conjugation in G.

Proof. It is convenient to define some intermediate formulae; for brevity we use the predicates $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$:

$$\varphi_h(x): \quad (\exists y_1 \exists y_2 \exists y_3)\ x = [h^{-1}, h^{y_1}][h^{-1}, h^{y_2}][h^{-1}, h^{y_3}];$$
$$\psi_h(x): \quad (\exists t \exists y_1 \exists y_2)\ \varphi_h(y_1) \wedge \varphi_{h^t}(y_2) \wedge ([y_1, y_2] \neq 1) \wedge \varphi_{h^t}(x);$$
$$\gamma^1_h(x): \quad (\forall y)(\psi_h(y) \to [x, y] = 1);$$
$$\gamma_h(x): \quad (\forall y)(\gamma^1_h(y) \to [x, y] = 1).$$

Thus, in the notation of Section 3, we have

$$\{x \mid \varphi_h(x)\} = Y_h, \quad \{x \mid \psi_h(x)\} = W_h \quad \text{and} \quad \{x \mid \gamma_h(x)\} = \mathrm{C}^2_G(W_h).$$

Now define $\beta(x)$ as follows:

$$x \neq 1 \wedge (\forall y)(((\exists z_1 \exists z_2)\ \gamma_x(z_1) \wedge \gamma_{x^y}(z_2) \wedge [z_1, z_2] \neq 1) \to ((\forall z_3)(\gamma_x(z_3) \to \gamma_{x^y}(z_3)))).$$

Thus the statement $\beta(h)$ holds if and only if $h \neq 1$ and the subgroup $B = \mathrm{C}^2_G(W_h)$ has the property that if $[B, B^g] \neq 1$ then $B \leq B^g$. This latter property is equivalent to the statement that B commutes with its distinct conjugates; then it follows from [6, Lemma 3] that B is in $L(G)$ and is a basal subgroup. Therefore Corollary 3.5 shows that the C^2-closed basal subgroups are precisely the sets $\{x \mid \gamma_h(x)\}$ with h in the definable set $S = \{x \mid \beta(x)\}$.

Now we define $\delta(x, y)$ as follows:

$$\delta(x, y): (\forall t)(\gamma_x(t) \to \gamma_y(t)).$$

For $h_1, h_2 \in S$, the statement $\delta(h_1, h_2)$ holds if and only if $\mathrm{C}^2_G(W_{h_1}) \leq \mathrm{C}^2_G(W_{h_2})$; in particular, δ defines a preorder on S. Moreover, $h_1, h_2 \in S$ satisfy $\delta(h_1, h_2) \wedge \delta(h_2, h_1)$ if and only if the basal subgroups $\mathrm{C}^2_G(W_{h_1}), \mathrm{C}^2_G(W_{h_2})$ are equal. This provides our interpretation of the graph $\mathcal{B}(G)$ as the definable quotient Q of S; the graph structure on Q can be reconstructed from the order induced by δ.

Finally, define τ as follows:

$$(\forall x)(\forall x_1)(\forall x_2)(\beta(x) \wedge \beta(x_1) \wedge \beta(x_2) \wedge \delta(x, x_1) \wedge \delta(x, x_2)) \to (\delta(x_1, x_2) \vee \delta(x_2, x_1)).$$

By Corollary 2.7, the sentence τ holds in G if and only there is (up to G-equivariant isomorphism) a unique maximal tree on which G acts as a branch group. □

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