J. Austral. Math. Soc. Ser. B 35(1994), 445–468

PERTURBATIONS OF NONLINEAR AUTONOMOUS OSCILLATORS

P. B. CHAPMAN[1]

(Received 23 January 1992; revised 5 November 1992)

Abstract

A general theory is given for autonomous perturbations of non-linear autonomous second order oscillators. It is found using a multiple scales method. A central part of it requires computation of Fourier coefficients for representation of the underlying oscillations, and these coefficients are found as convergent expansions in a suitable parameter.

1. Introduction

Autonomous perturbations of the equation

$$\frac{d^2u}{dt^2} = -f'(u) \tag{1.1}$$

are considered when f' (the $'$ denotes derivative with respect to argument) is such that (1.1) has a continuum of oscillatory solutions - or equivalently, a continuum of closed trajectories in the phase $(u, \frac{du}{dt})$ plane. Normally, f will also be analytic and it is reasonable to assume this property.

Thus it is required that the second derivative f'' is positive in a neighbourhood of an simple zero u_o of f' and then, perhaps also provided suitable bounds are placed on initial conditions $u(0)$ and $\frac{du}{dt}(0)$, there will be some continuum of oscillatory solutions of (1.1). Since perturbations of the (linear) harmonic oscillator

$$f'(u) = k^2u, \qquad k \text{ constant},$$

[1]Mathematics Department, University of Western Australia, Nedlands, 6009, Western Australia.

are well studied (exhaustive references are given by Nayfeh [8]) , it is envisaged that f' is other than linear, although the theory of the perturbed linear oscillator will follow as a special case.

The perturbed equation is taken to be

$$\frac{d^2u}{dt^2} + f'(u) = \epsilon \frac{du}{dt} g\left(u, \frac{du}{dt}\right), \tag{1.2}$$

where g is Lipschitz-continuous in its arguments in a region of the phase plane containing the continuum of closed trajectories of (1.1), and ϵ is a small parameter which may be taken to be positive. It is thus tacitly assumed that perturbations depending solely on u are analytic in u and absorbed in f' on the left hand side of (1.2). The perturbation term in (1.2) is more general than the corresponding term $\epsilon(du/dt)G(u)$ in [4] .

For many purposes the existence theory of (1.2), obtained by phase plane analysis, is sufficient. But sometimes approximations to solutions are required, and these problems are discussed below using a multi-scale analysis based on one employed by Kuzmak [4] in describing the behaviour of the Duffing equation when its coefficients are slowly varying. Thus, in Section 3, the approximate theory of (1.2) will be shown to reduce its solution to sequential integration of two first order equations. The result of this calculation will allow description of the evolution of solutions of the perturbed (1.2) in terms of solutions of the unperturbed (1.1).

There is no reason in principle that the perturbed (1.2) must be autonomous, but the complication possible with non-autonomous, fast perturbation of (1.2) is illustrated in Section 5. The essential difficulty is that the first order non-linear equations obtained on reduction may be awkwardly coupled, so that the resulting problem is as complicated as the original. Even so, this reorganization of the problem may lead to useful insights.

General statements of results are obviously unlikely for such an open problem as specified by (1.2), but when $g \equiv \pm 1$ a concise result is available. The oscillatory solutions of (1.1) are expressible as integrals of the first order equation

$$\frac{du}{dt} = \pm(c - 2f(u))^{1/2}, \tag{1.3}$$

where $c > 2f(u_o)$ is a constant. In the phase plane these oscillations map onto closed curves, symmetric about the u axis, with area given by the integral

$$A(c) = \oint (c - 2f(u))^{1/2} du \tag{1.4}$$

whose value depends only on the value of c. The solutions of the perturbed (1.2) with $g \equiv \pm 1$ evolve with c, and consequently A, now slowly varying in such a way that

$$A(t) = A(0)\exp(\pm\epsilon t) \tag{1.5}$$

approximately.

Even for the linearly damped non-linear oscillator the evolutionary principle suggested by (1.4) is not generally computable in closed form so an alternative, possibly more tractable formulation, is also given. This is based on Fourier representations of the solutions of (1.1), and these are discussed at length in Section 4. It will enable the evolution of solutions of (1.2) to be described in terms of the evolution of a parameter (Δ or D, below), now slowly varying, which in turn can be used to calculate the Fourier coefficients, and a slowly modulated, fast phase (ψ below).

The rationale for the method is given in Section 6, where it is also placed in context with other studies.

2. The unperturbed oscillator

Solutions of (1.2) are described in this section in terms of a representation of those of (1.1) with the parameter c, introduced in (1.3), allowed to vary slowly and with a new t-like, fast variable. Thus, suppose the angular frequency function $\nu(c)$ is known for the oscillatory solution (1.3). Then all oscillatory solutions of (1.1) can be expressed in the form

$$u(t) = U(c; (t+k)\nu), \tag{2.1}$$

where c and k are constants which are fixed by the initial conditions. By use of a Fourier synthesis, the underlying function U can be expressed as

$$U(c; \nu t) = \sum_{0}^{\infty} a_n(c)\cos(n\nu t), \tag{2.2}$$

where the coefficients a_n are analytic in c on some region including an appropriate *open* interval of the real c axis,

$$\mathcal{R}I(c) > 2f(u_o)$$

https://doi.org/10.1017/S0334270000009541 Published online by Cambridge University Press

since f is analytic in u. (Of course, c is restricted to real values in the following.) Later a reparametrisation will establish analyticity of the coefficients in one or other new parameter in an interval including the map of $c = f(u_o)$, but for the present it is more natural to use c as the describing parameter.

From its analyticity f is infinitely differentiable in u, so then U is an infinitely differentiable function of t, and its Fourier coefficients consequently decay rapidly: $| a_n(c) |$ will be $o(n^{-j})$ for any integer j. The solution of the perturbed equation (1.2) is specified below using the Fourier series representation (2.2) of the solution of (1.1)

$$U(c, \tau) = \sum_0^\infty a_n(c) \cos(n\tau), \tag{2.3}$$

so that

$$\mathcal{U}(\tau) = U(c, \tau) \tag{2.4}$$

satisfies the equation

$$\nu^2 \frac{d^2\mathcal{U}}{d\tau^2} + f'(\mathcal{U}) = 0 \tag{2.5}$$

when ν is evaluated at the chosen value of c.

Suppose a value of c has been fixed, and $U(c, \tau)$ calculated in accordance with (2.3). Then the odd, co-periodic function (period unity)

$$w(\tau) = \frac{\partial U}{\partial \tau}(c, \tau) = -\sum_1^\infty n a_n \sin(n\tau) \tag{2.6}$$

satisfies the first variation equation associated with (2.5), that is the linear equation

$$\nu^2 \frac{d^2 x}{d\tau^2} + f''(U)x = 0. \tag{2.7}$$

A second, linearly independent solution s of (2.7) is even but non-periodic (except for the linear problem $f''(U) = k^2$) since w (the first derivative of an even periodic function) is odd and periodic, and $f''(U)$ is co-periodic. So the exceptional, resonant, case of Floquet theory [7] applies, and s can be expressed as

$$s(\tau) = (K\tau w(\tau) + W(\tau)), \tag{2.8}$$

where

$$K(c) = \nu^{-1} \frac{d\nu}{dc} \tag{2.9}$$

and the even, co-periodic W is a Fourier cosine series

$$W(\tau) = \sum_{0}^{\infty} \frac{da_n}{dc} \cos(n\tau). \tag{2.10}$$

These results (2.6, 2.8-10) follow on differentiating (1.1) with respect to t, and partially with respect to c at the solution of the one parameter family of initial value problems

$$u(0) = f^{-1}(c/2), \qquad \frac{du}{dt}(0) = 0, \tag{2.11}$$

where f^{-1} denotes the branch of the inverse function with the greater values.

The subsequent calculations will require explicit evaluation of the necessarily constant Wronskian $V(c)$ of w and s. Since it is a constant, it is the same as its mean on $[-\pi, \pi]$ and it follows that the Wronskian

$$V(c) = \frac{dw}{d\psi}s - w\frac{ds}{d\psi} = -\left(\sum_{1}^{\infty} n^2 \frac{da_n^2}{dc}\right)/4. \tag{2.12}$$

Finally it is noted that all the above results need no formal change if the problem is non-singularly reparametrised; if a new parameter Δ (or D) were to replace c then Δ- (or D-) derivatives would replace c-derivatives, as the common factor $d\Delta/dc$ (or dD/dc) merely scales the solution (2.8).

3. The perturbation problem

To construct systematic approximations to solutions of (1.2), suppose that u can be expanded as

$$u = \sum_{0}^{\infty} \epsilon^n U_n, \tag{3.1}$$

and that each of the U_n have dependence on two variables, which are

$$c(\sigma) \stackrel{\text{def}}{=} c(\epsilon t) \tag{3.2}$$

(σ is a slow variable) and, as a first guess,

$$\psi(\sigma) = \epsilon^{-1}\phi(\sigma) \tag{3.3}$$

(a fast variable). Essentially, the aim is to prescribe the evolution of c and to identify ϕ so that solutions of (1.2) are approximated by U_0 on a useful interval of ψ. Kuzmak [4] suggests that this will be so if the U_n are constructed so that they all have period 2π in ψ.

Upon substituting (3.1) into (1.2) and arranging the resulting expression as a power series in ϵ, one obtains equations

$$\dot{\phi}^2 \frac{\partial^2 U_0}{\partial \psi^2} + f'(U_0) = 0 \tag{3.4}$$

and

$$\dot{\phi}^2 \frac{\partial^2 U_1}{\partial \psi^2} + f''(U_0) U_1 = \dot{\phi} \frac{\partial U_0}{\partial \psi} g\left(U_0, \dot{\phi} \frac{\partial U_0}{\partial \psi}\right) - \left(2\dot{c}\dot{\phi} \frac{\partial^2 U_0}{\partial c \partial \psi} + \dot{\phi} \cdot \frac{\partial U_0}{\partial \psi}\right) \tag{3.5}$$

by equating to zero the coefficients of the leading powers. In these equations (3.4-5) and below, the " $\dot{}$ " notation indicates derivative with respect to σ (although this new notation is technically superfluous).

By reference to (2.7), (3.4) would be satisfied if ϕ were chosen such that

$$\dot{\phi} = \nu(c(\sigma)), \qquad \phi = \int_0^\sigma \nu(c(\mu)) d\mu \tag{3.6}$$

(an arbitrary constant could be added, if desired) and if U_0 were identified through (2.4-5) as

$$U_0(c, \psi) = U(c, \psi). \tag{3.7}$$

But it turns out that, because there may be two conditions to be satisfied for 2π, ψ-periodicity of U_1, a perturbation of the choice (3.6) is demanded so choose instead

$$\dot{\phi}(\sigma) = \nu(c(\sigma)) - \epsilon \dot{m}(\sigma) + \sum_2^\infty \epsilon^j \dot{m}_j(\sigma), \tag{3.6a}$$

where m is, and eventually the m_j are, to be fixed; this means that (3.5) must now be

$$\begin{aligned} &\nu^2 \frac{\partial^2 U_1}{\partial \psi^2} + f''(U_0) U_1 \\ &\quad = \nu \frac{\partial U_0}{\partial \psi} g\left(U_0, \nu \frac{\partial U_0}{\partial \psi}\right) - \left(2\dot{c}\nu \frac{\partial^2 U_0}{\partial c \partial \psi} + \dot{\nu} \frac{\partial U_0}{\partial \psi}\right) + 2\nu \dot{m} \frac{\partial^2 U_0}{\partial \psi^2} \\ &\quad \equiv \mathbf{R}. \end{aligned} \tag{3.5a}$$

Likewise, linear equations

$$\nu^2 \frac{\partial^2 U_j}{\partial \psi^2} + f''(U_0) U_j = \mathbf{R}_j \tag{3.5b}$$

govern the U_j for $j > 1$, where the $\mathbf{R}_j$ contain U_k only for $k < j$, and m_k only for $k \leq j$.

In order that the evolution of solutions of (1.2) be usefully approximated by U_0 (3.7), it is necessary that the 2π, ψ-periodic, right hand side $\mathbf{R}$ of (3.5a) satisfy conditions involving the solutions (2.6, 2.8) of the homogeneous (2.7)

$$\int \nu^{-2} \mathbf{R}\, w\, d\psi = 0, \qquad \int \nu^{-2} \mathbf{R}\, s\, d\psi = 0, \tag{3.8}$$

the integrals each being taken over the period of $U[-\pi, \pi]$. These two conditions (3.8) will allow the solution U_1 of (3.5a) to be also periodic in ψ, and this in turn admits a corollary of Kuzmak's [4] implied result to justify using the approximation $u \approx U_0$ on an interval at least $O(\epsilon^{-1})$ of ψ and hence t. Thus conditions (3.8) determine the slowly varying functions c and m. Since ν is assumed to be slowly varying, a factor can be taken through the integrals and the two conditions (3.8) reduced to

$$\int \mathbf{R}\, w\, d\psi = 0, \qquad \int \mathbf{R}\, s\, d\psi = 0. \tag{3.8a}$$

Kuzmak [4] used the first part of this observation (3.8a) to discuss the evolution of solutions of equations not very dissimilar to (1.2), but not the second, for it was unnecessary in the restricted class of perturbations he considered. As will be demonstrated by an example in Section 5 below, the second condition may be necessary for a complete result even when the underlying oscillator (1.1) is linear (so that the first variation operator corresponding to (2.5) is the same as its generator). The argument leading to (3.8-8a) is given in Section 6. When they are satisfied, a ψ-periodic particular integral U_{1P} can be calculated, and the most general ψ-periodic solution of (3.8a) with period 2π is

$$U_1(\psi, \sigma) = A_1(\sigma) w(\psi) + U_{1P}(\psi, \sigma),$$

where A_1 and m_2 can be found by applying conditions to the coperiodic $\mathbf{R}_2$

$$\int \mathbf{R}_2\, w\, d\psi = 0, \qquad \int \mathbf{R}_2\, s\, d\psi = 0, \tag{3.8b}$$

the integrals again being over a ψ period. This pattern of calculation can then be iterated to arbitrary order.

The first of the conditions (3.8a) reduces to

$$\begin{aligned}\int \mathbf{R}\, w\, d\psi &= \int \left(\nu \frac{\partial U}{\partial \psi} g\left(U, \nu \frac{\partial U}{\partial \psi}\right) - \left(2\dot{c}\nu \frac{\partial^2 U}{\partial c \partial \psi} + \dot{\nu} \frac{\partial U}{\partial \psi}\right) + 2\nu\dot{m} \frac{\partial^2 U}{\partial \psi^2}\right) \frac{\partial U}{\partial \psi}\, d\psi \\ &= \int \left(\nu \frac{\partial U}{\partial \psi} g\left(U, \nu \frac{\partial U}{\partial \psi}\right) - \left(2\dot{c}\nu \frac{\partial^2 U}{\partial c \partial \psi} + \dot{\nu} \frac{\partial U}{\partial \psi}\right)\right) \frac{\partial U}{\partial \psi}\, d\psi = 0 \end{aligned} \tag{3.9}$$

on using the results (2.6, 3.7) and appealing to the slow variation property of m and ν, which allows the factor $\nu\dot{m}$ to be treated as a constant, to show that the last term in the period integral makes a zero contribution.

The result (3.6a) enables the second part of (3.9) to be expressed as

$$\begin{aligned}\int \left(2\dot{c}\nu \frac{\partial^2 U}{\partial c \partial \psi} + \dot{\nu} \frac{\partial U}{\partial \psi}\right) \frac{\partial U}{\partial \psi}\, d\psi &= \int 2\dot{c}\nu^{1/2} \left(\nu^{1/2} \frac{\partial^2 U}{\partial c \partial \psi} + \frac{1}{2}\nu^{-1/2} \frac{d\nu}{dc} \frac{\partial U}{\partial \psi}\right) \frac{\partial U}{\partial \psi} d\psi \\ &= 2\dot{c} \int \nu^{1/2} \frac{\partial}{\partial c} \left(\nu^{1/2} \frac{\partial U}{\partial \psi}\right) \frac{\partial U}{\partial \psi} d\psi \\ &= \dot{c} \frac{d}{dc} \left(\nu \int \left(\frac{\partial U}{\partial \psi}\right)^2 d\psi\right) \\ &= \frac{d}{d\sigma} \left(\nu \int \left(\frac{\partial U}{\partial \psi}\right)^2 d\psi\right), \end{aligned} \tag{3.10}$$

as in Kuzmak's [4] analysis. The last expression can be interpreted in at least two ways. Since the integral is over a period, the Fourier representation (2.3) of U gives

$$\int \left(\frac{\partial U}{\partial \psi}\right)^2 d\psi = \pi \sum_1^\infty (n a_n)^2 \tag{3.11}$$

and likewise, in terms of the first integral (1.3) of (1.1), it is

$$\nu \int \left(\frac{\partial U}{\partial \psi}\right)^2 d\psi = \oint (c - 2f(u))^{1/2} du \stackrel{\text{def}}{=} A(c), \tag{3.12}$$

https://doi.org/10.1017/S0334270000009541 Published online by Cambridge University Press

the area of the loop in the phase plane representing the cyclic solution of (1.1) parametrised by the value of c. Thus, when $g \equiv 1$, the condition (3.9) is

$$\frac{dA}{d\sigma} - A = 0$$

leading to the result foreshadowed in (1.5). Alternatively, this condition could be expressed as

$$\frac{dc}{d\sigma} = A\left(\frac{dA}{dc}\right)^{-1},$$

where $A(c)$ is assumed to be known. Any more complicated form of g will usually require some manipulation to obtain a result in closed form, if this is possible.

The scheme proposed here is to here is to utilise the identification (3.11), and calculate the first period integral in (3.9)

$$\int \left(\nu\frac{\partial U}{\partial \psi} g\left(U, \nu\frac{\partial U}{\partial \psi}\right)\right)\frac{\partial U}{\partial \psi} d\psi = \nu \int \left(\frac{\partial U}{\partial \psi} g\left(U, \nu\frac{\partial U}{\partial \psi}\right)\right)\frac{\partial U}{\partial \psi} d\psi \equiv B(c)$$

approximately using truncated Fourier series representations of U, and finite approximations to $\nu(c)$. To justify this procedure qualitatively, appeal is made to the rapidity of convergence of the Fourier series representation of U, which follows from the analyticity of the non-linear term f' in the governing equation. Then the evolutionary equation

$$\frac{dA}{d\sigma} = B, \tag{3.10a}$$

where A and B are in principle known functions of c, may be used to summarise (3.10). Similar results are common in the literature.

Now the consequences of the second of the conditions (3.8a) are examined on the assumption that $c(\sigma)$ has been calculated as described above. This condition is that the integral over the period $[-\pi, \pi]$

$$\int \mathbf{R}\, s\, d\psi = 2\nu\dot{m} \int \frac{\partial^2 U}{\partial \psi^2} s\, d\psi + \int (\ldots) s\, d\psi \tag{3.13}$$

vanishes, where the slow variation of ν and m allows their factoring through the integral. The second integral above reduces because s in (2.8) is even on

$[-\pi, \pi]$, while $w = (\partial U \partial \psi)(2.6)$ is a sine series, so the condition requiring the vanishing of the expression (3.13) becomes just

$$\nu \int \left(\frac{\partial U}{\partial \psi} g \left(U, \nu \frac{\partial U}{\partial \psi} \right) \right) s \, d\psi + 2\nu \dot{m} \int \frac{\partial^2 U}{\partial \psi^2} s \, d\psi = 0. \tag{3.14}$$

The first integral in (3.14) is in principle a known function of c; it is now argued that the integral coefficient of $\dot{m}$ in it is also known, so (3.13) provides an equation for m. The period-integral $[-\pi, \pi]$ coefficient of m is

$$\int \frac{\partial^2 U}{\partial \psi^2} s \, d\psi = \int \frac{\partial w}{\partial \psi} s \, d\psi. \tag{3.15}$$

But also, integration of the Wronskian V of $w = \partial U / \partial \psi$ and s

$$V = \frac{dw}{d\psi} s - \frac{ds}{d\psi} w \tag{3.16}$$

over $[-\pi, \pi]$ shows that the period integral (3.15) satisfies

$$2 \int \frac{\partial^2 U}{\partial \psi^2} s \, d\psi = \int (V) d\psi = 2\pi V, \tag{3.17}$$

where $V(c)$ is to be determined from (2.12). Equation (3.14) can thus be summarised as

$$\dot{m} \equiv \frac{dm}{d\sigma} = C(c), \tag{3.14a}$$

where $C(c)$ is a known function. Equations (3.10a) and (3.14a) control the approximation U_0 (3.7) through the parameter c, and the argument ψ. They can be integrated sequentially for an explicit result.

To carry out the evaluation implied by (3.14) requires an extra computational effort and it may not be justified. This assessment is made on the qualitative grounds that the useful information resides in a knowledge of $c(\sigma) = c(\epsilon t)$, and this does not require an evaluation of m. It suffices for many purposes, in particular proof of approximation, to show only the existence of an m, which contributes a slow modulation to the phase integral

$$\phi(\sigma) = \int_0^{c(\sigma)} \big(\nu(\mu) - \epsilon \dot{m}(\mu)\big) \big(\dot{c}(\mu)\big)^{-1} d\mu.$$

Again, any non-singular reparametrisation of the solutions of the unperturbed (1.1) would only require replacing c with the new parameter wherever it occurred.

4. Calculation of Fourier coefficients for underlying oscillation

The first aim is to describe a procedure for expanding the Fourier coefficients a_n of U (and hence of $w = \partial U/\partial\tau$) as analytic functions of a parameter which depends on c and is not necessarily small. The usual descriptions of these calculations either do not carry the calculation beyond the third harmonic in the general case (the fifth harmonic in special ones) or, when they do, are restricted to specific cases such as, for example, the standard Fourier representations [10] of Jacobian Elliptic functions. In the following a method is described which gives sequential calculations of finite truncations of convergent power series representations of the successive Fourier coefficients; the latter are analytic functions of a new parameter Δ which replaces c. (Δ is roughly proportional to $(c - 2f(u_o))^{1/2}$, and some of the coefficients are consequently singular in c at the zero of the surd.) Of course, for utility of the power series coefficient representation, Δ should not be too large but the common radius of convergence of all the power series is not necessarily small. In principle these calculations can be extended to arbitrary powers of the parameter.

Because it turns out that Δ is not the most convenient parameter for calculations (as opposed to proof construction or data reduction), a further reparametrisation introduces the amplitude of the fundamental D $(= a_1)$ as parameter. This is shown to be an analytic function of Δ, and vanishes linearly with Δ as $\Delta \to 0$, so the two parametrisations are equivalent. The use is made of D as parameter in some cases in the literature, but without formal justification.

The following results will be obtained below: a one parameter family of periodic solutions of (1.1) can be expressed in terms of a parameter

$$\Delta = (\max u - \min u)/2$$

as

$$u(t) = u_0 + \Delta \sum_{0}^{\infty} \Delta^{|n-1|}\alpha_n \cos(n\nu t),$$

where ν and all the α_n are analytic functions of Δ^2. As a corollary it will be shown that an alternative representation is

$$u(t) = u_0 + D\cos(\nu t) + D\sum_{\substack{0 \\ n\neq 1}}^{\infty} D^{|n-1|}\alpha_n \cos(n\nu t),$$

where ν and α_n are analytic functions of D^2, and D is the amplitude of the fundamental. (A reader for whom these results are unexceptional is advised to skip the demonstration.)

A solution of (1.1) in the form (2.3) can be uniquely specified by choice of the parameter c, the initial condition

$$\frac{du}{dt}(0) = 0, \tag{4.1}$$

and the sign of the second derivative, say

$$\frac{d^2u}{dt^2}(0) < 0. \tag{4.2}$$

The basic criterion in the choice of a new parameter is that the Fourier coefficients are analytic in it wherever they exist *including* $c = 2f(u_o)$. Assume that u_o, the zero of the derivative f', is known. Then for a range of values of $c \geq 2f(u_o)$ there will be two solutions $u = H$, $u = L$ of the equation

$$f(u) - f(u_o) = c/2 - f(u_o) \stackrel{\text{def}}{=} \delta^2 \tag{4.3}$$

which fix the extreme (local and global) values of u on a cycle by specifying the values of u at which its derivative vanishes. The property that there are only two zeros of the derivative per cycle is generic for periodic solutions of (1.1). As f is analytic, there exists an analytic function H in the variable δ such that

$$H(\delta) - u_o = \delta(h_0 + h_1\delta + h_2\delta^2 + \ldots + h_n\delta^n + \ldots), \tag{4.4}$$

$$L(\delta) - u_o = H(-\delta) - u_o, \tag{4.5}$$

the series having at least a finite radius of convergence. The coefficients h_j in the series depend on the evaluations of the derivatives of f at u_o, for example

$$\begin{aligned} h_0 &= \big(2/f''(u_o)\big)^{1/2} > 0, \\ h_1 &= -f'''(u_o)/\big(3(f''(u_o)^2\big), \\ h_2 &= \big(f'''(u_o)\big)^2 5/\big(18\sqrt{2}(f''(u_o))^{7/2}\big) - \big(f''''(u_o)\big)/\big(6\sqrt{2}(f''(u_o))^{5/2}\big). \end{aligned} \tag{4.6}$$

In terms of the Fourier expansion (2.3) one has

$$H(\delta) = \sum_0^\infty a_n, \qquad L(\delta) = \sum_0^\infty (-)^n a_n, \tag{4.7}$$

https://doi.org/10.1017/S0334270000009541 Published online by Cambridge University Press

so the relations

$$\frac{H(\delta)-L(\delta)}{2}=\sum_0^\infty a_{2k+1}=\delta\sum_0^\infty h_{2k}\delta^{2k}, \tag{4.8}$$

$$\frac{H(\delta)+L(\delta)}{2}=\sum_0^\infty a_{2k}=\delta\sum_0^\infty h_{2k+1}\delta^{2k+1}+u_o \tag{4.9}$$

follow. To set the problem in the desired form change the variable to

$$u-u_o=\Delta v, \tag{4.10}$$

where the new parameter, depending on c, is

$$\Delta=\frac{H(\delta)-H(-\delta)}{2}=\delta(h_0+\delta^2h_2+\ldots) \tag{4.11}$$

and it is a natural one. The inverse of this relation is

$$\delta=\Delta(h_0^{-1}-h_2h_0^{-4}\Delta^2+\ldots), \tag{4.12}$$

this series being also in even powers of Δ so that δ/Δ is analytic in c but more importantly, the ratio $(H-u_o)/\Delta$ is analytic in δ and thus also in Δ, in an interval including $\Delta=0$. The Fourier coefficients a_n of u and A_n of v are simply related:

$$a_n=\Delta A_n,\quad n=1\ldots\infty,\quad \text{and}\quad a_0-u_0=\Delta A_0. \tag{4.13}$$

If the substitution (4.10) is made in (1.1) and the non-linear term expanded about u_o the result is

$$\frac{d^2v}{dt^2}+f''(u_o)\sum_2^\infty \Delta^{(n-2)}f^{(n)}(u_o)\frac{v^{n-1}}{f''(u_o)(n-1)!}=0. \tag{4.14}$$

This equation is more general than the usual example chosen for discussion which has only a cubic non-linearity (the Duffing equation, or its variants), and its solution is consequently more elaborate. The essential observation to be made is that its solutions satisfying initial conditions

$$v(0)=(H-u_o)/\Delta,\qquad \frac{dv}{dt}(0)=0,\qquad \frac{d^2v}{dt^2}<0$$

are analytic in the parameter Δ at $\Delta = 0$. This follows from the analyticity of $(H - u_0)/\Delta$, and standard theory for initial value problems in ordinary differential equations. As a consequence, the Fourier coefficients of periodic solutions of (4.14) and the frequency are also analytic in Δ in an interval including $\Delta = 0$, so their formal power series representations have finite convergence radius.

Following the usual Poincaré-Lindstedt [8] approach, introduce a new variable

$$\tau = \nu t, \tag{4.15}$$

with the frequency ν expressed through an expansion

$$\nu^2/f''(u_o) = \left(1 + \sum_1^\infty \nu_k \Delta^{2k}\right), \tag{4.16}$$

where the ν_k are constants to be determined.

A description is given of the processes by which the corresponding expansion of the Fourier coeffients is inferred. The assumption

$$A_n = \Delta^{|n-1|}\alpha_n \tag{4.17}$$

on the Fourier coefficients (4.15) leads to a consistent solution proceedure. This is so because, on this (4.17) basis, expansions of terms in (4.14) are

$$\begin{aligned}\Delta^{n-1}v^n = \Delta^{n-1}\Big\{&q_{n,n}\cos(n\tau) + \Delta q_{n,n-1}\cos((n-1)\tau) + q_{n,n-2}\cos((n-2)\tau)\\ &+ \Delta q_{n,n-3}\cos((n-3)\tau) + q_{n,j}\cos(j\tau) \times \text{ terms factored}\\ &\text{alternately by 1 and } \Delta \text{ down to either } \Delta q_{n,0} \text{ or } q_{n,0}\Big\}\\ &+ \sum_n^\infty \Delta^k q_{n,k+1}\cos((k+1)\tau),\end{aligned} \tag{4.18}$$

as can be verified by induction. In this last expression the coefficients $q_{n,j}$ are each formal power series in even powers of Δ only, with coefficients depending on the α_j. Next, use the Fourier representation of v suggested by (4.17) to satisfy (4.14) to some fixed order in Δ, in terms of the Fourier basis set. Thus at O(1) the result is

$$\alpha_1 \text{ arbitrary}, \tag{4.19}$$

on consideration of the fundamental. (Naturally, this arbitrariness cannot persist, for the solution (4.17) would then have a free parameter extra to Δ. α_1 must be chosen eventually so that (4.8) and (4.11) are satisfied.) At $O(\Delta)$, in addition to the previous, on considering the zeroth and second modes, determine both of

$$\alpha_0 = -f'''(u_o)\alpha_1^2/(4f''(u_o)) \quad \text{and} \quad \alpha_2 = f'''(u_o)\alpha_1^2/(12f''(u_o)) \tag{4.20}$$

are determined; at $O(\Delta^2)$ the coefficients of the zeroth and second modes are unchanged, but a requirement is put on each of the fundamental and third modes resulting in the condition

$$\nu_1 = \left(\alpha_1^3 f^{(4)}(u_o) + 4f'''(u_o)[2\alpha_0 + \alpha_2]\alpha_1\right)/\left(8f''(u_o)\right) \tag{4.21}$$

and

$$\alpha_3 = \alpha_1\left(\alpha_1^2 f^{(4)}(u_o) + 12\alpha_2 f'''(u_o)\right)/\left(192f''(u_o)\right). \tag{4.22}$$

If the calculation is taken to a higher order, the situation is rather more complicated. Thus, if it is desired to satisfy (4.14) to $O(\Delta^3)$, the equations used to determine ν_1 and α_3 are formally unaltered, but the equations used above to make vanish the coefficients of $\cos(0\tau)$ and $\cos(2\tau)$ are modified by the addition of terms factored by Δ^2. These equations are in the form (omitting the argument u_o of the derivatives)

$$\alpha_0 + \frac{f'''\alpha_1^2}{4f''} = \Delta^2 \times \text{ a polynomial expression in variables } \alpha_0, \alpha_1, \alpha_2, \nu_1,$$

$$3\alpha_2 - \frac{f'''\alpha_1^2}{4f''} = \Delta^2 \times \text{ a polynomial expression in variables } \alpha_0, \alpha_1, \alpha_2, \alpha_3, \nu_1$$

and they can be solved for new expressions for α_0 and α_2 by using the already obtained results (4.19-21) on their right hand sides. The result will be the first two terms in the series expansion in even powers of Δ of α_0 and α_2, in terms of α_1, with the coefficients of Δ^0 given by the appropriate right hand sides of (4.20). There is also obtained a linear equation for a further coefficient (that of $\cos(4\tau)$)

$$15\alpha_4 = \frac{f'''}{2f''}(\alpha_1\alpha_3 + \alpha_2^2/2) + \frac{f^{(4)}}{8f''}\alpha_1^2\alpha_2 + \frac{f^{(5)}}{192f''}\alpha_1^4 \tag{4.23}$$

which can be solved (for what is now seen to be the coefficient of Δ^0 in the power series expansion in even powers of Δ of α_4, as a function of α_1) using (4.19-21) for the α_j on the right hand side.

Symbolic manipulation software might be used to continue the procedure suggested by the foregoing - that is, successively satisfying equation to progressively increased powers of Δ, and so obtaining at each stage an extra term in the expansions of previously calculated partial expansions of the odd, or of the even coefficients α_j (taking the expansion of $\nu^2/f''(u_o)$ in place of one for α_1), together with the coefficient of Δ^0 in the expansion of the next unknown coefficient. In effect then, appeal has been made to the analyticity of the coefficients α_n in Δ – in fact they are analytic in Δ^2 – so each of the α_n, $n \neq 1$, should have been expanded as power series

$$\alpha_n(\Delta;\alpha_1) = \sum_0^\infty \Delta^{2k}\alpha_{n,k}(\alpha_1). \tag{4.24}$$

(An illustrative example of the Fourier expansion

$$U = \Delta \sum_0^\infty \Delta^{|n-1|}\alpha_n \cos(n\nu t)$$

is supplied by the Jacobian Elliptic function

$$cn(t) = bq^{1/2}\sum_0^\infty \frac{(q^{1/2})^{2n}}{1+(q^{1/2})^{4n-2}}\cos((2n+1)\nu t)$$

given in [10]. Here $b(q)$ is a constant, and the parameter of the expansion q has been chosen so that its coefficients have a very simple form. Then $q^{1/2}$ corresponds to the present parameter Δ, and there must be a locally linear, analytic connection between the two.)

Thus closed, consistent equations determining each of the $\alpha_{k,n}$ and the ν_k can be obtained by substituting the expansions (2.2, 4.13, 4.16, 4.17, 4.24) in (4.14) and then equating to zero the coefficients of $\cos(j\tau)$ occurring to order Δ^1, Δ^2, $\Delta^3, \ldots, \Delta^n, \ldots$ sequentially, with α_1 arbitrary.

The essence of the calculation is in this sequencing. Suppose, for example $n > 3$, and the computation has been completed to $O(\Delta^{n-2})$ so that the following coefficients are known (using $\lfloor k \rfloor$ to denote the integer part of $k \geq 0$)

$$\alpha_{n-2,0};\ \alpha_{n-3,0};\ (\alpha_{n-4,0},\alpha_{n-4,1});\ \ldots;\ (\alpha_{n-r,0},\ldots,\alpha_{n-r,\lfloor\frac{r-2}{2}\rfloor});\ \ldots$$
$$\ldots;\ (\alpha_{2,0},\ldots,\alpha_{2,\lfloor\frac{n-3}{2}\rfloor});\ (\alpha_{0,0},\ldots,\alpha_{0,\lfloor\frac{n-3}{2}\rfloor});\ \text{and } (\nu_1,\ldots,\nu_{\lfloor\frac{n-2}{2}\rfloor})$$

in terms of α_1. The coefficients $\alpha_{n,0}$; $\alpha_{n-r,\lfloor r-12\rfloor}$, $r = 1,\ldots,(n-1)$; $\alpha_{0,\lfloor\frac{n-2}{2}\rfloor}$, and $\nu_{\lfloor\frac{n-1}{2}\rfloor}$ are then determined from new equations, or reconfirmed from existing equations, as follows. First recompute all the terms $\Delta^{(k-1)}v^k$, $k = 1,\ldots,n$ to $O(\Delta^{n-1})$. This calculation will require the above specified set of unknown coefficients only in the recomputation of the linear term v, and will generate no harmonics $\cos(m\tau)$, $m > n$, at $O(\Delta^{n-1})$ or larger. There will be no alteration to existing calculations of $\Delta^{(k-1)}v^k$, $k = 1, \ldots, (n-1)$ at $O(\Delta^{n-1})$ or larger. Similarly the unknown set of coefficients only occur linearly in the recomputation of

$$\left(1 + \sum_{1}^{\lfloor \frac{n-1}{2} \rfloor} \nu_k \Delta^{2k}\right) \frac{d^2 v}{d\tau^2}$$

at $O(\Delta^{n-1})$ and harmonics higher than the n^{th} are not generated at $O(\Delta^{n-1})$, or larger, and there are no other changes at $O(\Delta^{n-2})$ or larger. Thus those of the specified set of coefficients which are new are calculated from a set of independent equations obtained by substituting the finite series described above into the truncation of (4.14) after the Δ^{n-1} term. A similar calculation scheme is found in Milne-Thomson's solution [6] of an integral equation (Nekrasov [9]) describing a steady water wave profile.

But if, for simplicity, attention is retricted to the coefficients α_0 to α_3 inclusive (that is, the truncation of the calculation at $O(\Delta^2)$) then using the truncation of the relation resulting from (4.8) and the definitions (4.10, 4.11, 4.13, 4.17)

$$\alpha_1 + \Delta^2 \alpha_3 \approx \sum_{0}^{\infty} \Delta^{2k} \alpha_{2k+1} = 1, \tag{4.24}$$

the result follows

$$\alpha_1 = 1 - \Delta^2\big((f''')^2 (f'')^{-1} + f^{(4)}\big)(192 f'')^{-1} + O(\Delta^4). \tag{4.25}$$

In the foregoing description the results have been made possible from the analyticity property of the solution of initial value problems for ordinary differential equations. This leads to Fourier coefficients which are analytic functions of a parameter Δ containing what is in effect an initial condition c. But it is possible, and very convenient, to take the reparametrisation a stage further. Noticing that the amplitude $D = \Delta\alpha_1$ of the fundamental $\cos(\tau)$ is an analytic function of Δ with a particular form (4.25), then it follows from the inverse function theorem that Δ is an analytic function of D with the same form, differing only in the coefficients. This implies that descriptions of the solution can equally well be parametised by the amplitude $D = \Delta\alpha_1 = a_1$ of the fundamental, and so one has now the solution description

$$u - u_o = Dv, \qquad v(\tau) = \sum_0^\infty D^{|n-1|} \alpha_n \cos(n\tau), \tag{4.26}$$

where

$$A_1 = \alpha_1 = 1; \qquad \alpha_n = \sum_0^\infty D^{2k} \alpha_{n,k}(1), \quad n \neq 1, \tag{4.27}$$

and

$$\nu^2 / f''(u_o) = \left(1 + \sum_1^\infty D^{2k} \nu_k(1)\right). \tag{4.28}$$

Thus, by a somewhat circuitous route, a convenient parametrisation of the solutions of (1.1) is found. It would have been pleasing to be able to go directly to the final result, but the analyticity properties were not apparent. It is remarked that, for analysis of data from numerical simulations of non-linear oscillators, the parameter (4.4-5, 4.11) $\Delta = (\max(u) - \min(u))/2$ on a cycle, is the more easily inferred and is equally useful. But D is the better one for theoretical studies.

5. Examples

Examples of the application of the method described in Section 3 above to perturbations of non-linear oscillatory problems are of necessity elaborate, and it may suffice to demonstrate the method's use to look instead at a perturbed linear oscillator. Essentially the same but slightly extended principles as used in Section 2 can be used to study solution behaviour of the equation

$$\frac{d^2u}{dt^2} + u = \epsilon \frac{du}{dt}(1 - u^2) + \epsilon F \cos(\lambda t + \theta) \tag{5.1}$$

– the weakly non-linear, forced van der Pol oscillator. The parameters F, θ are O(1), and if λ is near resonant,

$$\lambda = (1 + \epsilon L), \tag{5.2}$$

the problem is that of "soft resonant" excitation discussed by Nayfeh [8] . The excitation or forcing term is included to indicate the escalation of difficulty

https://doi.org/10.1017/S0334270000009541 Published online by Cambridge University Press

caused by the presence of such terms, which act to couple the amplitude (3.9) and phase (3.14) equations. It also shows a case where the second of the two conditions (3.8a) must be used in order to arrive at the correct description of the interaction.

In the notation of Sections 2 and 3 above, the Fourier representation of the solution for the unperturbed oscillator can be taken to be

$$U_0 = a_1 \cos(\psi). \tag{5.2}$$

and since, for this linear oscillator ,

$$\nu = 1, \tag{5.3}$$

it follows (3.3, 3.6a) that

$$\psi = t - \int_0^\sigma \dot{m}(\sigma) d\sigma = t - m(\sigma) \tag{5.4}$$

with the slow variable (3.2)

$$\sigma = \epsilon t. \tag{5.5}$$

(Use of a_1 rather than D to denote the amplitude of the fundamental is only made so as to reflect the special property of the underlying oscillator.)

In this case the first variation (2.7) is the same as that governing the underlying oscillator, and a linearly independent pair of its solutions can be chosen with unit Wronskian V. They are

$$w(\psi) = a_1 \sin(\psi) \quad \text{and} \quad s(\psi) \equiv W(\psi) = (a_1)^{-1} \cos(\psi), \tag{5.6}$$

since the K in (2.8) is identically zero in this case. The first condition given in (3.8), modified by the addition of the non-homogeneous forcing term in (5.1), and using (3.11), is

$$\pi \frac{da_1^2}{d\sigma} = a_1^2 \int_{-\pi}^{\pi} (1 - a_1^2 \cos^2(\psi)) \sin^2(\psi)\, d\psi$$
$$+ a_1 F \int_{-\pi}^{\pi} \sin^2(\psi) \sin(L\sigma + m(\sigma) + \theta) d\psi. \tag{5.6}$$

Evaluation of the integrals in the preceding line yields the evolutionary equation

$$\frac{da_1^2}{d\sigma} = a_1^2(1 - \frac{a_1^2}{4}) + a_1 F \sin(L\sigma + m(\sigma) + \theta) \tag{5.7}$$

or

$$\frac{da_1}{d\sigma} = \frac{a_1}{2}(1 - \frac{a_1^2}{4}) + \frac{F}{2}\sin(L\sigma + m + \theta). \tag{5.8}$$

The equation following from the condition (3.14) is

$$\frac{dm}{d\sigma} = -(2a_1)^{-1}F\cos(L\sigma + m + \theta). \tag{5.9}$$

These equations (5.8-9) become (6.2.68-69) in [8] on introducing a new variable $(L\sigma + m + \theta)$, with $\omega_0 = 1$.

An example of calculating the Fourier representation of non-linear oscillators is now given. The exercise is masochistic, but the Fourier coefficients for the solution of the Duffing equation

$$\frac{d^2u}{dt^2} + u + u^3 = 0 \tag{5.10}$$

correct to $O(\Delta^7)$ are found. Following the procedure given in Section 4 it is seen straight away that all the even Fourier coefficients are zero, and that the odd coefficients in the expansion of

$$u(t) = \Delta \sum_0^{\infty} \Delta^{|n-1|}\alpha_n \cos(n\nu t)$$

are, explicitly to $O(\Delta^7)$ in the coefficient $\Delta^{1+|n-1|}\alpha_n$,

$$\begin{aligned}\alpha_3 &= (\alpha_1^3/2^5) - (21\alpha_1^5\Delta^2/2^{10}) + (417\alpha_1^7\Delta^4/2^{15}) + O(\Delta^6),\\ \alpha_5 &= \alpha_1^5/2^{10}) - (27\alpha_1^7\Delta^2/2^{15}) + O(\Delta^4),\\ \alpha_7 &= (\alpha_1^7/2^{15}) + O(\Delta^2)\end{aligned} \tag{5.12}$$

and all others are of smaller order. Also the expansion of ν is

$$\nu = 1 + (3\alpha_1^2\Delta^2/2^3) - (15\alpha_1^4\Delta^4/2^8) + (303\alpha_1^6\Delta^6/2^{14}) + O(\Delta^8). \tag{5.13}$$

(If the amplitude of the fundamental D had been used as a parameter in place of Δ, (5.12-13) would have the same numerical coefficients but α_1 replaced by unity and Δ by D.) These results are an extension of those given by Nayfeh [8](pp. 171-173) and agree with them in the terms in common, with $\omega_0 = 1$. The $\alpha_1(\Delta)$ relation required by (4.23) is supplied by solving

$$\alpha_1\left\{1 + (\alpha_1^2\Delta^2/32) - (5\alpha_1^4\Delta^4/2^8) + (391\alpha_1^6\Delta^6/2^{15}) + O(\Delta^8)\right\} = 1$$

and the result is

$$\alpha_1 = 1 - (\Delta^2/2^5) + (23\Delta^4/2^{10}) - (563\Delta^6/2^{15}) + O(\Delta^8). \tag{5.14}$$

It follows that the required frequency relation is

$$\nu(\Delta) = 1 + (3\Delta^2/2^3) - (21\Delta^4/2^8) + (705\Delta^6/2^{14}) + O(\Delta^8), \tag{5.15}$$

where $\Delta(c)$ as defined in (4.3, 4.5, 4.11) is

$$\Delta(c) = \left((1+2c)^{1/2} - 1\right)^{1/2} \tag{5.16}$$

and Δ^2 is analytic in c at $c = 0$. The quantity A defined in (3.12) can be evaluated to $O(\Delta^6)$ as

$$A(\Delta) = 2\pi^2\nu(\Delta)\left(\Delta^2\alpha_1^2(\Delta) + 9\Delta^6\alpha_3^2(\Delta) + \ldots\right) \tag{5.17}$$

using (5.12-14). The basic quantity is thus more sensitive to ν at small Δ as (5.17) expands to

$$\begin{aligned} A(\Delta) = 2\pi^2 &\left(1 + (3\Delta^2/2^3) - (21\Delta^4/2^8) + (705\Delta^6/2^{14}) + \ldots\right) \\ &\times \left(D^2 + (9D^6/2^{10}) - (189D^8/2^{14}) + (466D^{10}/2^{20}) + \ldots\right), \end{aligned}$$

with $D = (\alpha_1\Delta)$.

The advantage of working with D, the amplitude of the fundamental, as parameter is further emphasised when calculating the second solution s as given by (2.8-10).

6. Discussion of earlier work

Kuzmak's theorem [4] does not supply an error bound for the approximation (here U_0) on the specified interval, but one could probably be constructed along the lines of the proof given by Guckenheimer and Holmes [3] for the averaging method. (Rough but conservative calculations suggest the approximation is valid on an $O(\epsilon^{-1})$ interval of t.) The condition (3.8) that controls the evolution of the system, of which Kuzmak's [4] (1.16-17) is a special case, can be obtained as follows.

Because (1.2) is autonomous and its solutions have an essentially oscillatory character under the assumptions of Section 1, the evolution of all its trajectories

https://doi.org/10.1017/S0334270000009541 Published online by Cambridge University Press

can be studied by examination of the set of initial value problems for a range of u values, with $du/dt = 0$ and $d^2u/dt^2 > 0$; and it is assumed that the initial conditions on U_0 are chosen from this set. Then the particular integral U_{1P} of (3.5a) must satisfy the null initial conditions $U_{1P}(I) = U'_{1P}(I) = 0$ appropriate to the perturbation problem, where in the light of the prescription above $I = -\pi$. This solution is

$$V\nu^2 U_{1P}(\psi) = w\int_I^{\psi} \mathbf{R}\, s\, d\psi - s\int_I^{\psi} \mathbf{R}\, w\, d\psi,$$
$$V\nu^2 U'_{1P}(\psi) = w'\int_I^{\psi} \mathbf{R}\, s\, d\psi - s'\int_I^{\psi} \mathbf{R}\, w\, d\psi, \tag{6.1}$$

where V is the Wronskian (2.12), $\mathbf{R}$ and w are period 2π, ψ-periodic functions, but from (2.8)

$$s(\psi) = K\psi w(\psi) + W(\psi)$$

is not (although the even function W is). If this solution U_{1P} returns to its initial values after a period of the coefficient $f''(U_0)$ in (3.5a) – that is, when $\psi = I + 2\pi = \pi$ – then it too will be 2π periodic in ψ, since the differential equation (3.5a) which is solved by (6.1) has only coefficients with a 2π period in that variable. So necessary and sufficient conditions for periodicity of U_{1P} in ψ are that the integrals over the period vanish:

$$\int_I^{I+2\pi} \mathbf{R}\, w\, d\psi = 0 \tag{6.2}$$

(Kuzmak's condition), and the new condition

$$\int_I^{I+2\pi} \mathbf{R}\, s\, d\psi = 0. \tag{6.3}$$

The most general U_1 that is 2π periodic in ψ is $U_1 = A_1 w + U_{1P}$ where $A_1(\sigma)$ is at present arbitrary. (It will be found, along with $m_2(\sigma)$, as a consequence of applying (3.8b).)

This (U_1 2π-periodic in ψ) is the requirement for application of the Kuzmak Theorem [4]. So necessary and sufficient conditions for periodicity of U_1 in ψ are given by (6.2-3). It is noted that while (6.2) is a Fredholm alternative condition arising from the Sturm-Liouville problem for the homogeneous equation (2.7) with periodic boundary conditions, (6.3) is not, since s is not periodic and hence not an eigenfunction.

Clearly, the same argument will require conditions

$$\int_I^{I+2\pi} \mathbf{R}_j\, w\, d\psi = \int_I^{I+2\pi} \mathbf{R}_j\, s\, d\psi = 0, \qquad j = 2, 3, \ldots$$

to be satisfied by the co-periodic right hand sides $\mathbf{R}_j$ of (3.5b). These conditions will determine sequentially $A_j(\sigma)$ and $m_{j+1}(\sigma)$, where the former are coefficients of ϵ^j in an expansion of the coefficient of w in a representation of solutions of (1.2).

The last condition (6.3) can be reformulated. Using (2.8), integrate by parts the expression

$$Q(\psi) = \int_I^{\psi} \mathbf{R}(K\psi w + W)d\psi \equiv \int_I^{\psi} \mathbf{R}\, s\, d\psi$$

and obtain

$$Q(\psi) = K\psi \int_I^{\psi} \mathbf{R}\, w\, d\psi - \int_I^{\psi} \left(\left(K \int_I^{\psi} \mathbf{R}\, w\, d\psi\right) - \mathbf{R} W\right) d\psi, \qquad (6.4)$$

so that (6.3) is

$$Q(I + 2\pi) = -\int_I^{I+2\pi} \left(\left(K \int_I^{\psi} \mathbf{R}\, w\, d\psi\right) - \mathbf{R} W\right) d\psi = 0. \qquad (6.5)$$

From (6.2), the function

$$P(\psi) = \int_I^{\psi} \mathbf{R}\, w\, d\psi$$

is periodic in ψ, and hence its integral over a period is proportional to its mean $\langle P \rangle$. So (6.5) can be restated as

$$(2\pi)^{-1} \int_I^{I+2\pi} \mathbf{R}\, W\, d\psi = K \langle P \rangle \qquad (6.6)$$

and this form may be more convenient for evaluation.

There are a number of recent studies [1, 2] which use the Kuzmak technique to calculate the evolution of oscillators whose underlying oscillation is non-linear, and older work is summarised in [8]. These studies are restricted to problems for which (3.8, 3.8a) is not required. For example Bourland and Haberman [1] specify the perturbing function to be odd in the first derivative, and while they calculate a phase perturbation it is found from considerations other than the use of (6.3) at the first order. They lift the parity restriction in [2] , but the method used here (employing 6.3) is apparently more concise and systematic.

Although it seems a natural way of tackling the problem, use of the Fourier representation of the underlying oscillation seems to be novel in the present context. It makes routine the evaluation of the integrals which are consequences of the Kuzmak theory.

References

[1] F. J. Bourland and R. Haberman, "Separatrix crossing: Time invariant potentials with dissipation", *SIAM J. Appl Math.* **50** (1990) 1716–1744.

[2] F. J. Bourland, R. Haberman and W. Kath, "Averaging methods for the phase shift of arbitrarily perturbed nonlinear oscillators with application to capture", *SIAM J. Appl. Math.* **52** (1991) 1150–1167.

[3] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* (Springer - Verlag, New York, 1983).

[4] G. E. Kuzmak, "Asymptotic solutions of second order differential equations with variable coefficients", *P. M. M. J. Appl. Math. Mech.* **23** (1959) 730–744.

[5] L. Meirovitch, *Methods of analytical dynamics* (McGraw - Hill, New York, 1970).

[6] L. M. Milne-Thomson, *Theoretical hydrodynamics* (Macmillan, London, 1960).

[7] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, Volume 1 (McGraw - Hill, New York, 1953).

[8] A. Nayfeh, *Perturbation methods* (Wiley, New York, 1973).

[9] A. I. Nekrasov, "On waves of permanent type", *Izv. Ivanov - Vosnosonk. Politehn. Inst.* **6** (1922) 155– 171.

[10] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed. (The University Press, Cambridge, 1958).