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FORMATION OF SINGULARITIES IN A STRATIFIED FLUID IN THE PRESENCE OF A CRITICAL LEVEL

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Abstract

The paper is concerned with formation of singularities in a density stratified fluid subject to a monochromatic point source of frequency σ. The frequency of the source is assumed to be such that the steady-oscillation equation is hyperbolic in the neighbourhood of the source and degenerates at a critical level. We obtain asymptotic formulae demonstrating how the solution diverges as $t \to \infty$ on the characteristic surface emanating from the source. It is shown that, at points of the surface that belong to the critical level, the solution behaves as $t^{2/3} \exp\{i(\sigma t + \pi/2)\}$ as $t \to \infty$, whereas its large time behaviour at the other points of the surface is given by $t^{1/2} \exp\{i(\sigma t + \pi/2 \pm \pi/4)\}$.

1. Introduction

Processes of propagation and stabilization of waves in a stratified fluid possess a number of peculiarities connected with the anisotropic character of its dispersion relations. For instance, in the case of the Boussinesq approximation, it follows from the dispersion relations that the energy of oscillations of frequency σ can propagate from a point $x = (x_1, x_2, x_3)$ solely along the directions forming an angle $\arccos(\sigma/N(x_3))$ with the vertical line. Throughout the paper $N(x_3)$ denotes the buoyancy frequency (*cf.* [10]). In a stratified fluid subject to a monochromatic source the property mentioned results in the appearance of structures which consist of points connected with the source by rays of energy propagation and which are characterized by greater amplitude of oscillation. Such structures were observed experimentally (see [12]); their formation in time was studied theoretically in [8, 2, 11, 14, 15].

In the present paper the formation of such structures is studied on the assumption that the forcing is a monochromatic point source. As in [14], the study is based upon the properties of the steady-oscillation equation. The cone of directions along which energy of short waves corresponding to the frequency of the monochromatic

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source propagates is nondegenerate if and only if the steady-oscillation equation is hyperbolic. In [14] the problem was considered under the restriction that the steady-oscillation equation was hyperbolic in the whole space. In this paper that restriction is lifted, which necessitates taking into account the presence of a critical level where the degeneration of the steady-oscillation equation and the turn of its characteristic rays take place.

2. Statement of the problem and auxiliary results

We study properties of the solution to the equation

$$\frac{\partial^2}{\partial t^2}(\nabla^2 - \beta^2)u + N^2(x_3)\hat{\nabla}^2 u = \delta(x - x^0)e^{i\sigma t}, \tag{1}$$
$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad \hat{\nabla}^2 \equiv \nabla^2 - \frac{\partial^2}{\partial x_3^2}$$
$$x \equiv (\hat{x}, x_3), \qquad \hat{x} \equiv (x_1, x_2), \qquad x^0 \equiv (0, 0, h),$$

where $\delta(x)$ is Dirac's delta-function, subject to the zero initial conditions $u|_{t=0} = u_t|_{t=0} = 0$ and the condition at infinity $|u| \to 0$ as $|x| \to \infty$. Equation (1) simulates waves in a density-stratified fluid and turns into the equation of internal waves in the Boussinesq approximation when $\beta = 0$. The buoyancy frequency is assumed throughout to be a smooth function satisfying the conditions

$$N_{\inf} \equiv \inf N(x_3) > 0, \qquad N_{\sup} \equiv \sup N(x_3) < \infty, \qquad \frac{dN(x_3)}{dx_3} < 0. \tag{2}$$

Regarding the frequency σ, we assume

$$N_{\inf} < \sigma < N(h). \tag{3}$$

Condition (3) corresponds to the situation when the rays bearing the energy of short-wave oscillations of frequency σ emanate from x^0 and turn at the critical level $x_3 = h_\sigma$, where $N(h_\sigma) = \sigma$.

The solution possessing the necessary properties is of the form

$$u(x, t) = \int_0^t e^{i\sigma(t-\tau)}\varepsilon(x, x^0, \tau)\, d\tau, \tag{4}$$

where $\varepsilon(x, x^0, \tau)$ is the fundamental solution of (1) that equals zero when $t < 0$ and vanishes as $|x| \to \infty$. The fundamental solution of (1) with $N(x_3) \equiv \text{const}$ was constructed in [4] (see also [16]). A derivation of $\varepsilon(x, x^0, \tau)$ and study of some of its

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properties for $N(x_3) \neq$ const were carried out in [5]. Its Laplace transform $\bar{\varepsilon}(x, x^0, p)$ is an analytic function of p at $p \notin [-iN_{\sup}, iN_{\sup}]$. Also, the formula

$$\bar{\varepsilon}(x, x^0, p) = -\frac{\exp(-\beta|x - x^0|)}{4\pi p^2 |x - x^0|}(1 + O(|p|^{-2})), \tag{5}$$

is valid for fixed x, x^0 and $|p| \to \infty$. Taking advantage of the analyticity of $\bar{\varepsilon}(x, x^0, p)$, (5) and the convolution theorem for the Laplace transform we get from (4) that

$$u(x,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{(e^{pt} - e^{i\sigma t})}{(p - i\sigma)}\bar{\varepsilon}(x, x^0, p)\, dp, \qquad a > 0.$$

After that, putting $a = 0$, we rewrite $u(x, t)$ as

$$u(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(e^{i\omega t} - e^{i\sigma t})}{(\omega - \sigma)}\bar{\varepsilon}(x, x^0, i\omega)\, d\omega. \tag{6}$$

It is assumed in (6) that $\bar{\varepsilon}(x, x^0, i\omega) = \lim_{\delta \to 0+0} \bar{\varepsilon}(x, x^0, i\omega + \delta)$ where the limit exists. The function $\bar{\varepsilon}(x, x^0, i\omega)$ satisfies the steady-oscillation equation

$$L_x^\omega \bar{\varepsilon}(x, x^0, i\omega) = \delta(x - x^0), \tag{7}$$

where $L_x^\omega \equiv -\omega^2(\nabla^2 - \beta^2) + N^2(x_3)\hat{\nabla}^2$.

The large-time behaviour of (6) can be studied on the basis of information about singularities of $\bar{\varepsilon}(x, x^0, i\omega)$. That information is obtained from analysis of (7).

3. Singularities of $\bar{\varepsilon}(x, x^0, i\omega)$

Let $\omega \in (N_{\inf}, N(h))$ and introduce

$$q(\omega, x_3) = \frac{N^2(x_3)}{\omega^2} - 1, \qquad S_1(x_3, h, \omega) = \int_h^{x_3} q^{1/2}(\omega, \eta)\, d\eta,$$

$$S_2(x_3, h, \omega) = \int_h^{h_\omega} q^{1/2}(\omega, \eta)\, d\eta + \int_{x_3}^{h_\omega} q^{1/2}(\omega, \eta)\, d\eta,$$

where h_ω satisfies the relation $N(h_\omega) = \omega$. The functions $S_{1,2}(x_3, h, \omega)$ will be used only for $x_3 \le h_\omega$. Let us fix $H \in (h, h_\omega)$. The main order singularities of $\bar{\varepsilon}(x, x^0, i\omega)$ are described by the function

$$W^h(x,\omega) = -\frac{1}{4\pi\omega^2} q^{-1/4}(\omega, h) q^{-1/4}(\omega, x_3) \times \left\{ \left(S_1^2(x_3, h, \omega) - |\hat{x}|^2\right)^{-1/2} + e^{-i\pi/2}\left(S_2^2(x_3, h, \omega) - |\hat{x}|^2\right)^{-1/2} \right\} \tag{8}$$

when $x_3 \le H$, and

$$W^h(x,\omega) = \frac{e^{i\pi/4}}{4\pi^{3/2}\omega^2}(q(\omega,h))^{-1/4}\left\{-\frac{\zeta(x_3,\omega)}{q(\omega,x_3)}\right\}^{1/4}$$
$$\times\int_{R_2} d^2\hat{k}\ \mathrm{Ai}\left(|\hat{k}|^{2/3}\zeta(x_3,\omega)\right)|\hat{k}|^{-5/6}\exp\{i\hat{k}\hat{x}+i|\hat{k}|S_1(h_\omega,h,\omega)\} \qquad (9)$$

when $x_3 > H$. In (9) Ai(z) denotes Airy's function (see, for example, [13]) and function $\zeta(x_3,\omega)$ given by

$$\zeta(x_3,\omega) = \begin{cases} -\left(\frac{3}{2}\int_{x_3}^{h_\omega} q^{1/2}(\omega,\eta)\,d\eta\right)^{2/3}, & x_3 \le h_\omega \\ \left(\frac{3}{2}\int_{x_3}^{h_\omega}(-q(\omega,\eta))^{1/2}\,d\eta\right)^{2/3}, & x_3 > h_\omega. \end{cases}$$

We use also the notation $\hat{x} = (x_1,x_2)$, $\hat{k} = (k_1,k_2)$, $\hat{k}\hat{x} = k_1x_1 + k_2x_2$. When $|\hat{x}| > S_n$ the argument of the complex number $\left(S_n^2 - |\hat{x}|^2\right)^{-1/2}$ in (8) is assumed to equal $(-\pi/2)$. If $x_3 = h_\omega$ the expression $\{-\zeta/q\}$ is taken to equal its limiting value as $x_3 \to h_\omega$:

$$\lim_{x_3\to h_\omega}\left\{-\frac{\zeta(x_3,\omega)}{q(\omega,x_3)}\right\} = \left(-\frac{\omega}{2N'(h_\omega)}\right)^{2/3}. \qquad (10)$$

At any fixed x, $W^h(x,\omega)$ differs from $\bar{\varepsilon}(x,x_0,i\omega)$ by a function bounded within a neighbourhood of $\omega \in (N_{\inf}, N(h))$. Expressions (8)-(9) are derived by considering the problem ($w^h = w^h(x_3,|\hat{k}|,\omega)$)

$$\frac{d^2w^h}{dx_3^2} + \hat{k}^2q^2(x_3,\omega)w^h - \beta^2w^h = 0, \qquad x_3 \ne h;$$
$$w^h\Big|_{x_3=h-0} = w^h\Big|_{x_3=h+0}; \qquad \frac{dw^h}{dx_3}\Big|_{x_3=h+0} - \frac{dw^h}{dx_3}\Big|_{x_3=h-0} = -\omega^2;$$
$$|w^h| \text{ is bounded as } |x_3| \to \infty. \qquad (11)$$

It is clear that the solution to this problem satisfies the equation

$$\frac{d^2w^h}{dx_3^2} + \hat{k}^2q^2(x_3,\omega)w^h - \beta^2w^h = -\omega^2\delta(x_3-h)$$

and that the inverse Fourier transform of w^h with respect to $\hat{k}$,

$$F^{-1}_{\hat{k}\to\hat{x}}[w^h] \equiv \frac{1}{(2\pi)^2}\int_{R^2} d^2\hat{k}\, w^h(x_3,|\hat{k}|,\omega)\exp(i\hat{k}\hat{x}), \qquad (12)$$

satisfies (7). The character of the singularities of (12) is determined by the behaviour of w^h at large $|\hat{k}|$. This information can be obtained with the aid of the Liouville-Green (or WKB) method (see, for example, [13]). In the case under study three intervals of variation of x_3 must be considered: $(-\infty, h)$, (h, H) and $(H, +\infty)$. The asymptotic solution of (11) corresponding to the fundamental solution of (1) vanishing at negative t and at $|\hat{x}| \to \infty$ has the form

$$\widetilde{W}^h(x_3, \hat{k}, \omega) = \frac{\exp\{i|\hat{k}||S_1(x_3, h, \omega)| + i\pi/2\}}{2\omega^2 q^{1/4}(\omega, h) q^{1/4}(\omega, x_3)|\hat{k}|} + A\frac{\exp\{i|\hat{k}|S_2(x_3, h, \omega)\}}{q^{1/4}(\omega, x_3)} \tag{13}$$

when $x_3 \leq H$ and

$$\widetilde{W}^h(x_3, \hat{k}, \omega) = B\left\{-\frac{\zeta(x_3, \omega)}{q(\omega, x_3)}\right\}^{1/4} \mathrm{Ai}\left(|\hat{k}|^{2/3}\zeta(x_3, \omega)\right) \tag{14}$$

when $x_3 > H$. The values A and B, which depend only on ω and $|\hat{k}|$, are determined from the condition that, at $x_3 = H$, (13) and (14) are to match as $|\hat{k}| \to \infty$. The inverse Fourier transform with respect to $\hat{k}$ then leads us to (8)-(9).

If $\omega \in (0, N_{\mathrm{inf}})$, then instead of (8)-(9), we obtain for $W^h(x, \omega)$ just the first term in the braces on the right-hand side of (8). Such $W^h(x, \omega)$ can also be obtained by a method based on Hadamard's expansion of the fundamental solution of L_x^ω in powers of the geodesic distance generated by L_x^ω [7]. In deriving (8)-(9) we used the connection between singularities of solutions to hyperbolic equations and asymptotic solutions to oscillatory problems [9]. When deriving formulae (13)-(14) we also made use of [1] and [13].

Neither (13) nor (14) depend on β, which means that the final asymptotic formulae will not include β. However, we use the assumption $\beta \neq 0$, since it guarantees the existence of $\varepsilon(x, x^0, t)$ and provides its Laplace transform with the necessary analytical properties [5].

4. Large-time behaviour of the solution on the characteristic surface

Let us introduce surfaces $K_n^\sigma(x^0)$ whose points x $(x \neq x^0)$ satisfy the relations

$$|\hat{x}|^2 = S_n^2(x_3, h, \sigma) \qquad (n = 1, 2).$$

Each characteristic ray of L_x^σ emanating upwards from the source turns at the critical level $x_3 = h_\sigma$. The parts of such rays beyond the turning points form the surface $K_2^\sigma(x^0)$. The surface $K_1^\sigma(x^0)$ is made up of characteristic rays of L_x^σ emanating downwards from the source and segments of characteristic rays of L_x^σ connecting the source and the turning points. We denote the union of $K_1^\sigma(x^0)$ and $K_2^\sigma(x^0)$ by $K^\sigma(x^0)$.

Our aim is to describe the large-time behaviour of $u(x, t)$ at points of $K^\sigma(x^0)$. The principal term of the asymptotic expansion of (6) at large times is determined by the

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expression

$$u(x,t) \sim \frac{1}{2\pi i}\int_{\sigma-\alpha}^{\sigma+\alpha} W^h(x,\omega)\,(e^{i\omega t}-e^{i\sigma t})\,(\omega-\sigma)^{-1}\,d\omega, \tag{15}$$

for $x \in K^\sigma(x^0)$ where α is small and positive. The symbol $\sim$ indicates that the expressions on the two sides of the relation have the same principal terms in their large-time expansions. If x belongs to $K^\sigma(x^0)$ but does not lie on the critical level, then, in the neighbourhood of $\omega = \sigma$, the main-order singularities of $\bar{\varepsilon}(x, x^0, i\omega)$ are described by (8). In that case $W^h(x,\omega)$ behaves like $(\omega-\sigma)^{-1/2}$ near $\omega = \sigma$. On substituting (8) into (15) we find

$$u(x,t) \sim \mu_n^{-1}(x_3, h, \sigma)\,t^{1/2}\exp\{i(\sigma t + \varphi_n)\} \tag{16}$$

for $x \in K_n^\sigma(x^0)$, $x_3 \neq h_\sigma$, where $\varphi_1 = 3\pi/4$, $\varphi_2 = \pi/4$ and

$$\mu_n(x_3, h, \sigma) = 2\pi^{3/2}\sigma^2\left|\frac{\partial}{\partial\omega}S_n^2(x_3, h, \omega=\sigma)\right|^{1/2} q^{1/4}(\sigma, h)\; q^{1/4}(\sigma, x_3).$$

Now consider the situation when $x \in K^\sigma(x^0)$ and $x_3 = h_\sigma$. In this case we have to substitute the representation (9) for $W^h(x,\omega)$ in (15) and taking (10) into account, obtain

$$u(x,t) \sim m_\sigma \int_{\sigma-\alpha}^{\sigma+\alpha} d\omega \frac{(e^{i\omega t}-e^{i\sigma t})}{(\omega-\sigma)}\int_{R^2} d^2\hat{k}\,\mathrm{Ai}\left(|\hat{k}|^{2/3}\zeta(x_3,\omega)\right) \times |\hat{k}|^{-5/6}\exp\left\{i\left(|\hat{k}|S_1(h_\omega, h, \omega) + \hat{k}\hat{x}\right)\right\},$$

where

$$m_\sigma = \frac{e^{-i\pi/4}}{8\pi^{5/2}\sigma^2 q^{1/4}(\sigma,h)}\left(-\frac{\sigma}{2N'(h_\sigma)}\right)^{1/6}.$$

Passing to polar coordinates in the integral over $\hat{k}$ we evaluate the integral with respect to φ at large ρ by the stationary-phase method. This gives the formula

$$u(x,t) \sim m_\sigma\left(\frac{2\pi}{S(\sigma)}\right)^{1/2}\int_{\sigma-\alpha}^{\sigma+\alpha} d\omega\frac{(e^{i\omega t}-e^{i\sigma t})}{(\omega-\sigma)} \times \int_0^\infty d\rho\rho^{-1/3}\,\mathrm{Ai}\left(\rho^{2/3}\zeta(h_\sigma,\omega)\right)\exp\{i\rho(S(\omega)-S(\sigma))+i\pi/4\}, \tag{17}$$

where $S(\omega) \equiv S_1(h_\omega, h, \omega)$. Further transformations are made with the aid of the formulae

$$\zeta(h_\sigma,\omega) = \psi(\omega)(\omega-\sigma), \qquad S(\omega)-S(\sigma) = -\nu(\omega)(\omega-\sigma), \tag{18}$$

where $\psi(\omega)$ and $\nu(\omega)$ are differentiable functions of ω and

$$\psi(\sigma) = \left\{-4/N'(h_\sigma)\sigma^2\right\}^{1/3}, \qquad \nu(\sigma) = \sigma^{-3}\int_h^{h_\sigma} q^{-1/2}(\sigma,\eta)N^2(\eta)\,d\eta.$$

Using (18) and making a change of variables in the integral over ρ, transforms (17) into the form

$$u(x,t) \sim m_\sigma \left(\frac{2\pi}{S(\sigma)}\right)^{1/2} e^{i\pi/4}\int_{\sigma-\alpha}^{\sigma+\alpha} d\omega \frac{(e^{i\omega t}-e^{i\sigma t})}{(\omega-\sigma)^{5/3}}$$
$$\times \int_0^\infty dr\, r^{-1/3}\mathrm{Ai}\left(r^{2/3}(\omega-\sigma)^{1/3}\psi(\omega)\right)\exp\{-ir\nu(\omega)\mathrm{sgn}(\omega-\sigma)\}.$$

Expanding the integral over r in powers of $(\omega-\sigma)^{1/3}$

$$u(x,t) \sim |m_\sigma|\left(\frac{2\pi}{S(\sigma)}\right)^{1/2}\frac{2\pi}{\Gamma(5/3)(3\nu(\sigma))^{2/3}}\,t^{2/3}e^{i(\sigma t+\pi/2)}, \tag{19}$$

where Γ denotes Euler's gamma function.

5. Concluding remarks

A monochromatic point source of frequency $\sigma \in (N_{\mathrm{inf}}, N(h))$ operating in an inviscid stratified fluid causes infinite growth in the amplitude of the solution as $t \to \infty$ on the characteristic set $K^\sigma(x^0)$. In contrast to the case $\sigma \in (0, N_{\mathrm{inf}})$, the characteristic rays forming this surface experience reflection from the critical level $x_3 = h_\sigma$. If $x \in K_1^\sigma$, $x_3 \neq h, h_\sigma$, then according to (16), the solution behaves as $t^{1/2}e^{i(\sigma t+3\pi/4)}$ when $t \to \infty$. At points of the set $K_2^\sigma(x_0)$ $(x_3 \neq h_\sigma)$, composed of reflected characteristic rays, the solution behaves as $t^{1/2}e^{i(\sigma t+\pi/4)}$ when $t \to \infty$. At turning points we have by (19) that $u \sim |\mathrm{const}(\sigma)|t^{2/3}e^{i(\sigma t+\pi/2)}$ as $t \to \infty$.

We discussed a very special case of a monotone $N(x_3)$ admitting the presence of not more than one critical level. A situation exhibiting two critical levels can be observed if we take a buoyancy frequency having a unique maximum and $\sigma \in (\max_{+,-} N(\pm\infty), N(h))$. In that case a formal asymptotic solution of (11) leads to an amplitude singular on a surface composed of characteristic rays emanating from the source and experiencing successive turns at the critical levels. In a neighbourhood of each turning point the character of the singularity is the same as that of (9), whereas in the neighbourhood of the rest of the surface, the singularity is described by terms similar to those of (8). Therefore, the divergence rate results are expected to be like those we obtained above (that is, $t^{2/3}e^{i\sigma t}$ at turning points and $t^{1/2}e^{i\sigma t}$ on the rest of the characteristic surface). A rigorous consideration of the problem with two critical levels requires summation of an infinite number of remainder terms of the asymptotic solution to (11) and estimation of the sum at large $|\hat{k}|$. We do not dwell on it here.

The results presented in this paper can be arrived at if one takes advantage of the procedure proposed in [3] for determining the large-time behaviour of the Green's function of the equation of internal waves. On the other hand, the large-time behaviour of $\varepsilon(x, x^0, t)$ can be calculated with the aid of formulae (8)-(9), which contain information about singularities of $\bar{\varepsilon}(x, x^0, p)$ on the imaginary axis.

Our final remark concerning the model used here. The infinite growth of $|u(x, t)|$ considered is indicative of the limitations of the model's applicability. The results of this paper, as well as those of [2, 8, 11, 14 and 15], show where and when the model of an ideal stratified fluid fails to describe the wave process. A more realistic model of a viscous stratified fluid is governed by an equation whose steady-oscillation equation is not hyperbolic, which rules out the propagation of singularities of limiting amplitude. For instance, in the case of the Boussinesq approximation and a constant buoyancy frequency $N(x_3) = N =$ const, one should use $P_\gamma(i\partial/\partial x_j, \partial/\partial t)$ instead of the differential operator in the left-hand side of (1). Here γ denotes the kinematic viscosity and $P_\gamma(k_j, p) \equiv -k^2p^2 - \gamma k^4 p - N^2\hat{k}^2$ (see, for example, [6]). The solution of the problem analogous to (1) for such a fluid possesses limiting amplitude $w_\gamma(x) = F^{-1}_{k\to x}[1/P_\gamma(k_j, p)]$ continuous in the whole space, that is, the effects under study are smoothed out by the viscosity and, in viscous models, can no longer be associated with the formation of singularities. Rather, we can speak about greater values of $w_\gamma(x)$ in a neighbourhood of the characteristic surface. However, those values become infinite on that surface as γ vanishes.

References

[1] D. S. Ahluwalia and J. B. Keller, "Exact and asymptotic representations of the sound field in a stratified ocean", in *Wave Propagation and Underwater Acoustics* (eds. J. B. Keller and J. S. Papadakis), (Springer, 1977) 14–85.

[2] J. C. Appleby and D. G. Crighton, "Internal gravity waves generated by oscillations of a sphere", *J. Fluid Mech.* **183** (1987) 439–450.

[3] V. A. Borovikov, "The $t \to \infty$ asymptotic behaviour of the Green's function of the equation of internal waves", *Sov. Phys. Dokl.* **35** (1990) 631–633.

[4] L. A. Diky, *The theory of oscillations of the Earth's atmosphere* (Gidrometeoizdat, Leningrad, 1969).

[5] A. S. Dostovalova and S. T. Simakov, "The fundamental solution of the model equation of an incompressible stratified liquid", *Comp. Maths. Math. Phys.* **34** (1994) 1321–1326.

[6] S. A. Gabov and A. G. Sveshnikov, *Linear Problems of Theory of Transient Internal Waves* (Nauka, Moscow, 1990).

[7] J. Hadamard, *Le Problème de Cauchy et les Equations aux Dérivées Partielles Linéaires Hyperboliques* (Hermann et Cie, Paris, 1932).

[8] M. C. Hendershott, "Impulsively started oscillations in a rotating stratified fluid", *J. Fluid Mech.* **36** (1969) 513–527.
[9] P. D. Lax, "Asymptotic solutions of oscillatory initial value problems", *Duke Math. J.* **24** (1957) 627–646.
[10] M. J. Lighthill, *Waves in Fluids* (Cambridge University Press, 1978).
[11] S. A. Makhortykh and S. A. Rybak, "Effect of the near field of a point source on the generation of internal waves", *Izv. Atmos. Ocean. Phys.* **26** (1990) 194–198.
[12] D. E. Mowbray and B. S. H. Rarity, "A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid", *J. Fluid Mech.* **28** (1967) 1–16.
[13] F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).
[14] S. T. Simakov, "Initial and boundary value problems of internal gravity waves", *J. Fluid Mech.* **248** (1993) 55–65.
[15] S. T. Simakov, "Formation of singularities of limiting amplitude in a density stratified fluid disturbed by an extended monochromatic forcing", *Wave Motion* **19** (1994) 11–27.
[16] B. Voisin, "Internal wave generation in uniformly stratified fluids. Part 1. Green's function and point sources", *J. Fluid Mech.* **231** (1991) 439–480.