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MORITA EQUIVALENCE FOR C^*-ALGEBRAS WITH THE WEAK BANACH–SAKS PROPERTY. II

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Abstract Let C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. Suppose that A or B is unital. It is shown that X has the weak Banach–Saks property if and only if it has the uniform weak Banach–Saks property. Thus, we conclude that A or B has the weak Banach–Saks property if and only if X does so. Furthermore, when C^*-algebras A and B are unital, it is shown that X has the Banach–Saks property if and only if it is finite dimensional.

Keywords: Banach–Saks property; Hilbert C^*-module; Morita equivalence

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1. Introduction

First we briefly review the definition of the Banach–Saks property in Banach spaces. Historically, Banach and Saks showed that every bounded sequence in $L^p([0,1])$ with $1 < p < \infty$ has a subsequence whose arithmetic means converge in the norm topology (see [**1**]). More generally, if every bounded sequence $\{x_n\}$ in a Banach space X has a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty} \left\| \frac{1}{k}(x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}) - y \right\| = 0$$

with some $y \in X$, we say that X has the *Banach–Saks property*. Note here that Banach spaces with the Banach–Saks property are reflexive.

It is well known that the weak Banach–Saks property is one of the most important properties in Banach spaces. We recall here the definition of the weak Banach–Saks property. Let X be a Banach space. If, given any weakly null sequence $\{x_n\}$ in X, one can extract a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty} \frac{1}{k} \|x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}\| = 0,$$

then we say that X has the *weak Banach–Saks property*. Furthermore, there is a slightly stronger version of the weak Banach–Saks property introduced by Nuñez [**10**]. We say that a Banach space X has the *uniform weak Banach–Saks property* if there is a null sequence $\{\delta_n\}$ of positive real numbers such that, for any weakly null sequence $\{x_n\}$ in X with $\|x\| \leqq 1$ and for any natural number k, there exist natural numbers $n(1) < n(2) < \cdots < n(k)$ such that

$$\frac{1}{k}\|x_{n(1)} + x_{n(2)} + \cdots + x_{n(k)}\| < \delta_k.$$

In the theory of C^*-algebras, Chu [**3**] has studied C^*-algebras with the weak Banach–Saks property in detail, as a noncommutative extension of characterizations of the Banach space, of complex continuous functions on a compact Hausdorff space, with the weak Banach–Saks property. Later, in [**5**] the author has shown that the weak Banach–Saks property in C^*-algebras is invariant under Morita equivalence. More precisely, let C^*-algebras A and B be Morita equivalent, that is, we suppose that there exists a Hilbert C^*-module X called an A–B-imprimitivity bimodule. We then consider the following conditions:

(1) A has the weak Banach–Saks property;

(2) B has the weak Banach–Saks property;

(3) X has the uniform weak Banach–Saks property.

In [**5**, Theorem 2.3], Kusuda has shown that (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3), and that if either A or B is unital, then conditions (1)–(3) are equivalent. We remark that the uniform weak Banach–Saks property always implies the weak Banach–Saks property. Hence, condition (3) implies that X has the weak Banach–Saks property. So, under the assumption that either A or B be unital, it still remains to ask whether, if the above imprimitivity bimodule X has the weak Banach–Saks property, then conditions (1) and (2) hold, or in other words, whether, if X has the weak Banach–Saks property, then it has the uniform weak Banach–Saks property. In Theorem 2.2, we shall show that this problem can be answered in the affirmative.

Note that the weak Banach–Saks property and the uniform weak Banach–Saks property are not equivalent in general, as is shown in [**10**, Theorem 7]. Hence, it would be interesting to investigate Banach spaces for which the weak Banach–Saks property and the uniform weak Banach–Saks property are equivalent. For example, C^*-algebras are such Banach spaces, which was shown by Chu [**3**, Theorem 2]. As a corollary to our main theorem, we obtain the result that a full Hilbert C^*-module X over a unital C^*-algebra has the weak Banach–Saks property if and only if X has the uniform weak Banach–Saks property.

In § 3, we discuss the Banach–Saks property in full Hilbert C^*-modules. In general, the Banach–Saks property in C^*-algebras cannot be preserved under Morita equivalence. But it will be shown that the Banach–Saks property is preserved under Morita equivalence for unital C^*-algebras. Furthermore, we then show that X has the Banach–Saks property if and only if it is finite dimensional.

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2. The weak Banach–Saks property in Hilbert C^*-modules

Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *left Hilbert A-module* (or a *left A-Hilbert module*), we mean a left A-module X equipped with an A-valued pairing $\langle \cdot\,,\cdot \rangle$, called an A-valued inner product, satisfying the following conditions:

(H1) $\langle \cdot\,,\cdot \rangle$ is sesquilinear. (We make the convention that $\langle \cdot\,,\cdot \rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(H2) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in X$.

(H3) $\langle ax, y\rangle = a\langle x, y\rangle$ for all $x, y \in X$ and $a \in A$.

(H4) $\langle x, x\rangle \geqq 0$ for all $x \in X$, and $\langle x, x\rangle = 0$ implies that $x = 0$.

(H5) X is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

Furthermore, X is said to be *full* if X satisfies the following additional condition:

(H6) the closed linear span of $\{\langle x, y\rangle \mid x, y \in X\}$ coincides with A.

Let B be a C^*-algebra. Right Hilbert B-modules are defined similarly except that we require that B should act on the right of X, that the B-valued inner product $\langle \cdot\,,\cdot \rangle$ should be conjugate-linear in the first variable, and that $\langle x, yb\rangle = \langle x, y\rangle b$ for all $x, y \in X$ and $b \in B$. We denote by ${}_A\langle \cdot\,,\cdot \rangle$ the A-valued inner product on the left Hilbert A-module and by $\langle \cdot\,,\cdot \rangle_B$ the B-valued inner product on the right Hilbert B-module. By an *A–B-imprimitivity bimodule*, we mean a full left Hilbert A-module and full right Hilbert B-module X satisfying the following condition:

(H7) ${}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B$ for all $x, y, z \in X$.

We recall here that two C^*-algebras A and B are said to be *Morita equivalent* if there exists an A–B-imprimitivity bimodule. We remark that, in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (see [**11**, Remark 3.15]). The reader is referred to [**9**, **11**] for Hilbert C^*-modules and Morita equivalence.

Let $\{X_i\}_{i=1}^m$ be a finite family of Hilbert A-modules X_i with A-valued inner product $\langle \cdot\,,\cdot \rangle_i$. Then we denote by $\bigoplus_{i=1}^m X_i$ the direct sum of $\{X_i\}$, which admits a Hilbert A-module structure in a natural way. Note that the A-valued inner product $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ on $\bigoplus_{i=1}^m X_i$ is defined by

$$\left\langle\!\!\left\langle \bigoplus_{i=1}^m y_i, \bigoplus_{i=1}^m z_i \right\rangle\!\!\right\rangle = \sum_{i=1}^m \langle y_i, z_i\rangle_i.$$

The following lemma plays a key role in the proof of Theorem 2.2. Throughout this paper, by a Hilbert A-module we mean either a left Hilbert A-module or a right Hilbert A-module unless we need to specify whether the action of A on the Hilbert A-module is left or right.

Lemma 2.1. *Let A be a C^*-algebra and let X and Y be Hilbert A-modules. Then both X and Y have the weak Banach–Saks property if and only if the direct sum $X \oplus Y$ does so as a Hilbert C^*-module.*

Proof. Regarding $X \oplus Y$ as the direct sum of Banach spaces X and Y, consider the norm $\|\cdot\|_1$ on $X \oplus Y$ given by $\|x \oplus y\|_1 = \|x\| + \|y\|$. We denote by the same symbol $\langle\cdot\,,\cdot\rangle$ each A-valued inner product on X and on Y unless there is a danger of confusion.

Suppose that both X and Y have the weak Banach–Saks property. For any $x \in X$ and $y \in Y$, we have

$$\langle x, x\rangle, \langle y, y\rangle \leqq \langle x, x\rangle + \langle y, y\rangle$$

in A. Hence, we see that

$$\|x\|^2, \|y\|^2 \leqq \|x \oplus y\|^2 = \|\langle x, x\rangle + \langle y, y\rangle\| \leqq \|\langle x, x\rangle\| + \|\langle y, y\rangle\|,$$

and furthermore that

$$\|\langle x, x\rangle\| + \|\langle y, y\rangle\| = \|x\|^2 + \|y\|^2 \leqq (\|x\| + \|y\|)^2 = \|x \oplus y\|_1^2.$$

Thus, we obtain

$$\tfrac{1}{2}\|x \oplus y\|_1 \leqq \|x \oplus y\| \leqq \|x \oplus y\|_1, \tag{$*$}$$

that is, the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

Let $\{x_n \oplus y_n\}$ be any weakly null sequence in the Hilbert C^*-module $X \oplus Y$. Then it follows from the equivalence of $\|\cdot\|_1$ and $\|\cdot\|$ that $\{x_n \oplus y_n\}$ is a weakly null sequence also in the Banach space $X \oplus Y$ with the norm $\|\cdot\|_1$. Since the Banach space $X \oplus Y$ with the norm $\|\cdot\|_1$ has the weak Banach–Saks property (see [**10**, Theorem 5]), there exists a subsequence $\{x_{n(k)} \oplus y_{n(k)}\}_k$ of $\{x_n \oplus y_n\}$ such that

$$\lim_{k\to\infty} \left\| \frac{1}{k} \sum_{i=1}^{k} (x_{n(i)} \oplus y_{n(i)}) \right\|_1 = 0.$$

Thus, it follows from $(*)$ that the subsequence $\{x_{n(i)} \oplus y_{n(i)}\}$ satisfies

$$\lim_{k\to\infty} \left\| \frac{1}{k} \sum_{i=1}^{k} (x_{n(i)} \oplus y_{n(i)}) \right\| = 0,$$

which shows that the Hilbert C^*-module $X \oplus Y$ has the weak Banach–Saks property.

The converse direction follows from the fact that the weak Banach–Saks property passes to closed subspaces. □

Since Morita equivalence is an equivalence relation between C^*-algebras, it is important to investigate which properties in C^*-algebras are invariant under Morita equivalence. Then it is natural to expect that imprimitivity bimodules must contain sufficient information on properties in C^*-algebras invariant under Morita equivalence (see [**5**–**8**] for some results with such a viewpoint). The following theorem means that the weak Banach–Saks property is one of such properties.

Theorem 2.2. *Let two C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. Consider the following conditions:*

(1) *A has the weak Banach–Saks property;*

(2) *B has the weak Banach–Saks property;*

(3) *X has the weak Banach–Saks property.*

Then we have the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *If either A or B is unital, then conditions* (1)–(3) *are equivalent.*

Proof. In [**5**, Theorem 2.3], it was shown that each of conditions (1) and (2) imply that X has the uniform weak Banach–Saks property. Hence, we have the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). Suppose that either A or B is unital. So, without loss of generality, we assume that A is unital, and now we show (3) $\Longrightarrow$ (1). Since X is a full left Hilbert A-module equipped with the A-valued inner product ${}_A\langle\cdot\,,\cdot\rangle$ and since A is unital, it follows from [**11**, Lemma 5.53] that there is a finite subset $\{y_k\}_{k=1}^m$ in X such that

$$\left\| \sum_{k=1}^{m} {}_A\langle y_k, y_k\rangle - 1 \right\| < 1.$$

Then $a = \sum_{k=1}^m {}_A\langle y_k, y_k\rangle$ is invertible in A. If we put $x_k = a^{-1/2}y_k$, then $\{x_k\}_{k=1}^m \subset X$ satisfies that $\sum_{k=1}^m {}_A\langle x_k, x_k\rangle = 1$. Now we fix the natural number m and $\{x_k\}_{k=1}^m$.

Take any weakly null sequence $\{a_n\}$ in A, that is, $a_n \to 0$ weakly in A. To show condition (1), we have to show that there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ such that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} a_{n(i)} \right\| \to 0, \quad k \to \infty.$$

We remark here that, for each $x \in X$ and each $\varphi \in X^*$, the linear functional $A \ni a \to \varphi(ax)$ belongs to A^*. Since $a_n \to 0$ weakly, we see that, for each $x \in X$, the sequence $\{a_n x\}_n$ in X weakly converges to 0.

Consider the finite direct sum

$$\bigoplus^{m} X = \overbrace{X \oplus X \oplus \cdots \oplus X}^{m \text{ times}}.$$

Recall now that $\bigoplus^m X$ has a left Hilbert A-module structure with the A-valued inner product $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ defined by

$$\left\langle\!\!\left\langle \bigoplus_{i=1}^{m} y_i, \bigoplus_{i=1}^{m} z_i \right\rangle\!\!\right\rangle = \sum_{i=1}^{m} {}_A\langle y_i, z_i\rangle.$$

It then follows from Lemma 2.1 that $\bigoplus^m X$ has the weak Banach–Saks property. Since $\{a_n(x_1 \oplus x_2 \oplus \cdots \oplus x_m)\}_n$ is weakly convergent to 0 in $\bigoplus^m X$, we can extract a subsequence

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$\{a_{n(k)}(x_1 \oplus x_2 \oplus \cdots \oplus x_m)\}_k$ from $\{a_n(x_1 \oplus x_2 \oplus \cdots \oplus x_m)\}_n$ such that

$$\lim_{k\to\infty} \frac{1}{k}\left\| \sum_{i=1}^{k} a_{n(i)}(x_1 \oplus x_2 \oplus \cdots \oplus x_m)\right\| = 0.$$

Then we see that

$$\left\|\left\langle\!\!\left\langle \frac{1}{k}\sum_{i=1}^{k} a_{n(i)}(x_1 \oplus x_2 \oplus \cdots \oplus x_m), x_1 \oplus x_2 \oplus \cdots \oplus x_m \right\rangle\!\!\right\rangle\right\| \to 0.$$

On the other hand, we have

$$\begin{aligned}
\left\langle\!\!\left\langle \frac{1}{k}\sum_{i=1}^{k} a_{n(i)}(x_1 \oplus x_2 \oplus \cdots \oplus x_m), x_1 \oplus x_2 \oplus \cdots \oplus x_m \right\rangle\!\!\right\rangle \\
&= \frac{1}{k}\sum_{i=1}^{k} \langle\!\langle a_{n(i)}(x_1 \oplus x_2 \oplus \cdots \oplus x_m), x_1 \oplus x_2 \oplus \cdots \oplus x_m\rangle\!\rangle \\
&= \frac{1}{k}\sum_{i=1}^{k} \left(a_{n(i)} \sum_{j=1}^{m} {}_A\langle x_j, x_j\rangle \right) \\
&= \frac{1}{k}\sum_{i=1}^{k} a_{n(i)}.
\end{aligned}$$

Hence, we obtain

$$\lim_{k\to\infty} \frac{1}{k}\left\| \sum_{i=1}^{k} a_{n(i)}\right\| = 0,$$

which completes the proof. □

Let A be a C^*-algebra and let X be a Hilbert A-module with A-valued inner product $\langle \cdot\,,\cdot\rangle$. For convenience, without loss of generality we suppose that X is a right Hilbert A-module. We define the linear operator $\theta_{x,y}$ by $\theta_{x,y}(z) = x \cdot \langle y, z\rangle$ for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\theta_{x,y} \mid xy \in X\}$ (see [**11**, Proposition 2.21 and Lemma 2.25]). Then X is a $\mathcal{K}(X)$–A-Hilbert bimodule. If X is a full right Hilbert A-module, then it is a $\mathcal{K}(X)$–A-imprimitivity bimodule [**11**, Proposition 3.8].

Chu [**3**, Theorem 2] showed that a C^*-algebra has the weak Banach–Saks property if and only if it has the uniform weak Banach–Saks property. Thus, we have reached a question of whether a Hilbert C^*-module with the weak Banach–Saks property has the uniform weak Banach–Saks property. An answer to the question is given by the following result.

Corollary 2.3. *Let A be a unital C^*-algebra and let X be a full Hilbert A-module. Then X has the weak Banach–Saks property if and only if it has the uniform weak Banach–Saks property.*

Proof. Suppose that X has the weak Banach–Saks property. Without loss of generality, we assume that X is a right Hilbert A-module. Since X is a full Hilbert A-module, it is a $\mathcal{K}(X)$–A-imprimitivity bimodule. It then follows from Theorem 2.2 that A has the weak Banach–Saks property. By [**5**, Theorem 2.3], this assertion implies that X has the uniform weak Banach–Saks property. □

3. The Banach–Saks property

First we give an easy characterization of C^*-algebras with the Banach–Saks property, which will be used below. If a Banach space has the Banach–Saks property, it is reflexive. Hence, we have the following.

Lemma 3.1. *Let A be a C^*-algebra. Then A has the Banach–Saks property if and only if A is finite dimensional.*

For a C^*-algebra A, we denote by $\hat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes $[\pi]$ of non-zero irreducible representations π of A equipped with the Jacobson topology. The reader is referred to [**11**] for the spectrum of a C^*-algebra.

Now we prove that the Banach–Saks property in unital C^*-algebras is preserved under Morita equivalence.

Lemma 3.2. *Let unital C^*-algebras A and B be Morita equivalent. Then the following conditions are equivalent:*

(1) *A is finite dimensional;*

(2) *B is finite dimensional.*

Proof. By symmetry, it suffices to show the implication (1) $\Longrightarrow$ (2). Since A is finite dimensional, A is isomorphic to $\bigoplus_{i=1}^{n} M_i$ with some matrix algebras M_i. Hence, it follows that the spectrum $\hat{A}$ of A is a finite set, which is clearly a discrete space. Since A and B are Morita equivalent, $\hat{A}$ and $\hat{B}$ are homeomorphic by the Rieffel homeomorphism. Hence, $\hat{B}$ is also a finite space equipped with discrete topology. Since type I-ness in C^*-algebras is preserved under Morita equivalence, B is also a C^*-algebra of type I. Thus, B is the finite direct sum of closed ideals $\{J_i\}$ each of which is isomorphic to the C^*-algebra of all compact operators on some Hilbert space. Since B is unital, each J_i has also an identity. Hence, each J_i must be isomorphic to some matrix algebra. Thus, B is finite dimensional. □

Remark 3.3. In the above lemma, the assumption that both C^*-algebras A and B be unital is essential. In fact, let H be an infinite-dimensional Hilbert space and $\mathcal{C}(H)$ be the C^*-algebra of all compact operators on H. Then H is a $\mathbb{C}$–$\mathcal{C}(H)$-imprimitivity bimodule. But $\mathcal{C}(H)$ is infinite dimensional.

Lemma 3.4. *Let unital C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. If either A or B is finite dimensional, then so is X.*

Proof. Let $\mathcal{L}(X)$ be the linking algebra for X (see [**11**] for linking algebras). Then the C^*-algebra $\mathcal{L}(X)$ is Morita equivalent to A and to B. In fact, we have $A = p\mathcal{L}(X)p$ and $B = q\mathcal{L}(X)q$ with $p + q = 1$ for projections p, q in the multiplier algebra of $\mathcal{L}(X)$. Since A and B are unital, so is $\mathcal{L}(X)$. It hence follows from Lemma 3.2 that $\mathcal{L}(X)$ is finite dimensional. Since X is isometrically isomorphic to $p\mathcal{L}(X)q$, X is finite dimensional. □

Let E be a Banach space and let F be a closed subspace of E. It is known that E has the Banach–Saks property if and only if F and the quotient space E/F have the same property (see [**4**]). This fact is used to prove Lemma 3.5.

Let A be a C^*-algebra and let X and Y be Hilbert A-modules. Denote by $X \oplus Y$ the direct sum of X and Y which becomes a Hilbert A-module in a canonical way. Recall that the norm on $X \oplus Y$ is defined by $\|x \oplus y\| = \|\langle x, x\rangle + \langle y, y\rangle\|^{1/2}$. Then $(X \oplus Y)/Y$ is isometrically isomorphic to X, from which Lemma 3.5 easily follows.

Lemma 3.5. *Let X and Y be Hilbert A-modules. If X and Y have the Banach–Saks property, then the direct sum $X \oplus Y$ also has the Banach–Saks property.*

Now we are in a position to establish one of the main results in this section.

Theorem 3.6. *Let unital C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. Then the following conditions are equivalent:*

(1) *A has the Banach–Saks property;*

(2) *B has the Banach–Saks property;*

(3) *X has the Banach–Saks property.*

Proof. (1) $\iff$ (2). This follows from Lemma 3.1 and Lemma 3.2.

(1) $\implies$ (3). Suppose that A has the Banach–Saks property. Then it follows from Lemmas 3.1 and 3.4 that X is finite dimensional. Hence, X has the Banach–Saks property.

(3) $\implies$ (1). Suppose that X has the Banach–Saks property. We regard X as a full left Hilbert A-module equipped with the A-valued inner product ${}_A\langle\cdot,\cdot\rangle$. Take a finite subset $\{x_k\}_{k=1}^m \subset X$ satisfying that $\sum_{k=1}^m {}_A\langle x_k, x_k\rangle = 1$ (cf. the proof of Theorem 2.2). Consider the finite direct sum

$$\bigoplus_{k=1}^m X = \overbrace{X \oplus X \oplus \cdots \oplus X}^{m \text{ times}}.$$

Then it follows from Lemma 3.5 that $\bigoplus_{k=1}^m X$ has the Banach–Saks property.

Define a linear map T from A into $\bigoplus_{k=1}^m X$ by

$$T(a) = a\biggl(\bigoplus_{k=1}^m x_k\biggr).$$

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Then we have

$$\langle\langle T(a), T(a)\rangle\rangle = \left\langle\!\!\left\langle a\left(\bigoplus_{k=1}^{m} x_k\right), a\left(\bigoplus_{k=1}^{m} x_k\right)\right\rangle\!\!\right\rangle = a\sum_{k=1}^{m} {}_A\langle x_k, x_k\rangle a^* = aa^*.$$

Thus, we see that T is an isometric isomorphism from A into $\bigoplus_{k=1}^{m} X$, that is, A is isometrically embedded into $\bigoplus_{k=1}^{m} X$. Hence, A has the Banach–Saks property. $\square$

Let E be a Banach space. We say that E has the *Schur property* if every weak convergent sequence in E converges in norm. Let F be a closed subspace of E. We remark that E has the Schur property if and only if the quotient space E/F and F have the same property (see [**2**, Theorem 6.1.a]). Lemma 3.7 easily follows from this fact.

Lemma 3.7. *Let A be a C^*-algebra and let X and Y be Hilbert A-modules. If X and Y have the Schur property, then the direct sum $X \oplus Y$ as a Hilbert A-module also has the Schur property.*

Lemma 3.8. *Let A be a C^*-algebra. Then A has the Schur property if and only if A is finite dimensional.*

Proof. Suppose that A has the Schur property. If A is infinite dimensional, by using functional calculus, we then obtain a sequence $\{a_n\}$ in A consisting of mutually orthogonal positive elements with norm 1. We now claim that $\{a_n\}$ weakly converges to 0. For this, assume that $\{a_n\}$ does not weakly converge to 0. Then we can choose a subsequence $\{a_{n_k}\}$ such that there exist $\delta > 0$ and a state φ of A which satisfy the condition that, for all k, $\varphi(a_{n_k}) > \delta$. Since $\{a_{n_k}\}$ is mutually orthogonal, we see that, for any N,

$$1 = \left\|\sum_{k=1}^{N} a_{n_k}\right\| \geqq \sum_{k=1}^{N} \varphi(a_{n_k}) > N\delta.$$

But this is impossible. Thus, $\{a_n\}$ weakly converges to 0, which implies that $\|a_n\| \to 0$. But this contradicts that $\|a_n\| = 1$. Hence, A is finite dimensional. $\square$

Theorem 3.9. *Let unital C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. Then the following conditions are equivalent:*

(1) *A has the Schur property;*

(2) *B has the Schur property;*

(3) *X has the Schur property.*

Proof. (1) $\iff$ (2). This follows from Lemmas 3.2 and 3.8.

(1) $\implies$ (3). Let $\mathcal{L}(X)$ be the linking algebra for X. Then the C^*-algebra $\mathcal{L}(X)$ is Morita equivalent to A and to B. Since A and B are unital, so is $\mathcal{L}(X)$. It hence follows from the equivalence of (1) and (2) that $\mathcal{L}(X)$ has the Schur property. Since X is identified with a closed subspace of $\mathcal{L}(X)$, X has the Schur property.

(3) $\implies$ (1). Suppose that X has the Schur property. We regard X as a full left Hilbert A-module equipped with the A-valued inner product ${}_A\langle\cdot\,,\cdot\rangle$. Take a finite subset $\{x_k\}_{k=1}^m \subset X$ such that

$$\sum_{k=1}^m {}_A\langle x_k, x_k\rangle = 1.$$

Take any sequence $\{a_n\}$ in A such that $a_n \to a$ weakly in A. We will show that $\|a_n - a\| \to 0$ $(n \to \infty)$.

Consider the finite direct sum

$$\bigoplus_{k=1}^m X = \overbrace{X \oplus X \oplus \cdots \oplus X}^{m \text{ times}}.$$

Then it follows from Lemma 3.7 that $\bigoplus_{k=1}^m X$ has the Schur property. It is easy to check that

$$a_n\biggl(\bigoplus_{k=1}^m x_k\biggr) \to a\biggl(\bigoplus_{k=1}^m x_k\biggr)$$

weakly in $\bigoplus_{k=1}^m X$ $(n \to \infty)$. Thus, we see that

$$\biggl\|a_n\biggl(\bigoplus_{k=1}^m x_k\biggr) - a\biggl(\bigoplus_{k=1}^m x_k\biggr)\biggr\| \to 0.$$

For any $b \in A$, we have

$$\begin{aligned}\biggl\|b\biggl(\bigoplus_{k=1}^m x_k\biggr)\biggr\|^2 &= \biggl\|\biggl\langle\biggl\langle b\biggl(\bigoplus_{k=1}^m x_k\biggr), b\biggl(\bigoplus_{k=1}^m x_k\biggr)\biggr\rangle\biggr\rangle\biggr\| \\ &= \biggl\|b\biggl(\sum_{k=1}^m {}_A\langle x_k, x_k\rangle\biggr)b^*\biggr\| \\ &= \|bb^*\| = \|b\|^2.\end{aligned}$$

Hence, we obtain

$$\|(a_n - a)\|^2 = \biggl\|(a_n - a)\biggl(\bigoplus_{k=1}^m x_k\biggr)\biggr\|^2 \to 0.$$

Thus we complete the proof. □

Corollary 3.10. *Let unital C^*-algebras A and B be Morita equivalent and let X be an A–B-imprimitivity bimodule. Then the following conditions are equivalent:*

(1) *X has the Banach–Saks property;*

(2) *X has the Schur property;*

(3) *X is finite dimensional.*

Proof. (1) $\iff$ (3). Suppose that X has the Banach–Saks property. Then it follows from Theorem 3.6 that A and B have the Banach–Saks property. Hence, Lemmas 3.1 and 3.4 show that X is finite dimensional. The converse is trivial.

(2) $\Longrightarrow$ (3). Theorem 3.9 implies that A and B have the Schur property; equivalently, the linking algebra $\mathcal{L}(X)$ for X has the Schur property. Hence, it follows from Lemma 3.8 that $\mathcal{L}(X)$ is finite dimensional, so that X is also finite dimensional.

(3) $\Longrightarrow$ (2). This is trivial. Thus, we complete the proof. □

In the above corollary, the assumption that both A and B be unital is essential. If either A or B is not unital, the assertion fails in general. For example, take $H = L^2([0,1])$ as an imprimitivity bimodule, $A = \mathbb{C} \cdot 1$, and $B = \mathcal{C}(L^2([0,1]))$ as C^*-algebras (cf. Remark 3.3). Then the A–B-imprimitivity bimodule H has the Banach–Saks property by the classical result of Banach–Saks, but H is not finite dimensional.

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