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ANZAI AND FURSTENBERG TRANSFORMATIONS ON THE 2-TORUS AND TOPOLOGICALLY QUASI-DISCRETE SPECTRUM

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ABSTRACT. Let ϕ_0 be an Anzai transformation on the 2-torus $\mathbf{T}^2$ defined by $\phi_0(x,y) = (e^{2\pi i\theta}x, xy)$ and ϕ_f a Furstenberg transformation on $\mathbf{T}^2$ defined by $\phi_f(x,y) = (e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)$ where θ is an irrational number and f is a real valued continuous function on the 1-torus $\mathbf{T}$. In the present note we will show that ϕ_f has topologically quasi-discrete spectrum if and only if ϕ_f is topologically conjugate to ϕ_0. Furthermore we will show that for any irrational number θ there is a real valued continuous function f on $\mathbf{T}$ such that ϕ_f does not have topologically quasi-discrete spectrum but is uniquely ergodic.

1. **Introduction.** Let ϕ be a homeomorphism on a compact topological space X. We say that ϕ is *minimal* if for any $x \in X$ the orbit $\{\phi^n(x)\}_{n\in Z}$ is dense in X. Hence it follows that if $f: X \longrightarrow \mathbf{C}$ is a continuous function and X is connected and if $f \circ \phi = f$, then f is constant. Two homeomorphisms ϕ_1 and ϕ_2 on X are said to be *topologically conjugate* if there is a homeomorphism ψ on X such that $\psi \circ \phi_1 = \phi_2 \circ \psi$.

Let $C(X)$ be the C^*-algebra of all complex valued continuous functions on X. For each homeomorphism ϕ on X we consider the following sets:

$$G_0(\phi) = \{\lambda \in \mathbf{C} : \lambda \text{ is an eigenvalue of } \phi \text{ and } |\lambda| = 1\},$$
$$G_1(\phi) = \{f \in C(X) : f \circ \phi = \lambda f \text{ for some } \lambda \in G_0(\phi) \text{ and } |f| = 1\},$$
$$\vdots$$
$$G_j(\phi) = \{g \in C(X) : g \circ \phi = fg \text{ for some } f \in G_{j-1}(\phi) \text{ and } |g| = 1\},$$

for $j \geq 1$.

Their union $\bigcup_{j\geq 0} G_j(\phi)$ is known as the set of quasi-eigenfunctions of ϕ. The homeomorphism ϕ is said to *have topologically quasi-discrete spectrum* if the C^*-algebra generated by its quasi-eigenfunctions is $C(X)$. It is easy to see that the property of having a topologically quasi-discrete spectrum is invariant under topological conjugation.

Let θ be an irrational number in $(0,1)$ and f a real valued continuous function on the 1-torus $\mathbf{T}$. Let ϕ_0 be an Anzai transformation on the 2-torus $\mathbf{T}^2$ defined by

$$\phi_0(x,y) = (e^{2\pi i\theta}x, xy)$$

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for any $x, y \in \mathbf{T}$. And let ϕ_f be a Furstenberg transformation on $\mathbf{T}^2$ defined by

$$\phi_f(x, y) = (e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)$$

for any $x, y \in \mathbf{T}$. By Rouhani [10] ϕ_0 and ϕ_f are minimal and ϕ_0 has a topologically quasi-discrete spectrum.

In [10] Rouhani proposed the following question: For any $j = 1, 2$ let ϕ_j be a Furstenberg transformation on $\mathbf{T}^2$ and $A(\phi_j)$ the associated crossed product C^*-algebra $C(\mathbf{T}^2) \times_{\phi_j} \mathbf{Z}$. If $A(\phi_1)$ is isomorphic to $A(\phi_2)$ and if ϕ_1 has topologically quasi-discrete spectrum, does it necessarily follow that ϕ_2 has topologically quasi-discrete spectrum?

In this note we attempt to shed some light on this question.

2. **Topological conjugation.** Let f and ϕ_0, ϕ_f be as in Section 1.

LEMMA 1. *We suppose that there is a real valued continuous function g on $\mathbf{T}$ such that*

$$g(x) - g(e^{2\pi i\theta}x) = f(x) - \int_{\mathbf{T}} f(z)\, dz$$

for any $x \in \mathbf{T}$. Then ϕ_f is topologically conjugate to ϕ_0.

PROOF. Let ψ be a homeomorphism on $\mathbf{T}^2$ defined by

$$\psi(x, y) = (e^{2\pi i\eta}x, e^{2\pi i g(x)}y)$$

for any $x, y \in \mathbf{T}$ where $\eta = \int_{\mathbf{T}} f(z)\, dz$. Then by an easy computation we see that $\phi_0 \circ \psi = \psi \circ \phi_f$. ■

LEMMA 2. *We suppose that ϕ_f is topologically conjugate to ϕ_0. Then there is a real valued continuous function g on $\mathbf{T}$ such that*

$$g(x) - g(e^{2\pi i\theta}x) = f(x) - \int_{\mathbf{T}} f(z)\, dz$$

for any $x \in \mathbf{T}$.

PROOF. Since ϕ_f is topologically conjugate to ϕ_0, there is a homeomorphism ψ on $\mathbf{T}^2$ such that $\phi_0 \circ \psi = \psi \circ \phi_f$. By the Homotopy Lifting Theorem we can write ψ as

$$\psi(x, y) = (x^{m_1}y^{n_1}e^{2\pi i F_1(x,y)}, x^{m_2}y^{n_2}e^{2\pi i F_2(x,y)})$$

for any $x, y \in \mathbf{T}$ where m_j, n_j ($j = 1, 2$) are integers and F_j ($j = 1, 2$) are real valued continuous functions on $\mathbf{T}^2$. By a routine computation

$$\begin{aligned}(\phi_0 \circ \psi)(x, y) &= \phi_0(x^{m_1}y^{n_1}e^{2\pi i F_1(x,y)}, x^{m_2}y^{n_2}e^{2\pi i F_2(x,y)}) \\ &= (e^{2\pi i\theta}x^{m_1}y^{n_1}e^{2\pi i F_1(x,y)}, x^{m_1+m_2}y^{n_1+n_2}e^{2\pi i\{F_1(x,y)+F_2(x,y)\}}), \\ (\psi \circ \phi_f)(x, y) &= \psi(e^{2\pi i\theta}x, e^{2\pi i f(x)}xy) \\ &= (x^{m_1+m_2}y^{n_1}e^{2\pi i\{m_1\theta+n_1 f(x)+F_1(\phi_f(x,y))\}}, x^{m_2+n_2}y^{n_2}e^{2\pi i\{m_2\theta+n_2 f(x)+F_2(\phi_f(x,y))\}}).\end{aligned}$$

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Since $\phi_0 \circ \psi = \psi \circ \phi_f$, we obtain

$$x^{m_1} y^{n_1} e^{2\pi i\{\theta + F_1(x,y)\}} = x^{m_1+n_1} y^{n_1} e^{2\pi i\{m_1\theta + n_1 f(x) + F_1(\phi_f(x,y))\}}, \tag{1}$$

$$x^{m_1+m_2} y^{n_1+n_2} e^{2\pi i\{F_1(x,y)+F_2(x,y)\}} = x^{m_2+n_2} y^{n_2} e^{2\pi i\{m_2\theta + n_2 f(x) + F_2(\phi_f(x,y))\}}. \tag{2}$$

By (1) we see that $n_1 = 0$ and that

$$\theta + F_1(x,y) = m_1\theta + F_1\big(\phi_f(x,y)\big) + k_1(x,y)$$

where k_1 is a $\mathbf{Z}$-valued continuous function on $\mathbf{T}^2$. But since $\mathbf{T}^2$ is connected, k_1 is a constant number. Hence we obtain that

$$\theta + F_1(x,y) = m_1\theta + F_1\big(\phi_f(x,y)\big) + k_1.$$

Furthermore since ϕ_f is measure-preserving,

$$\int_{\mathbf{T}^2} F_1(x,y)\,dy\,dx = \int_{\mathbf{T}^2} F_1\big(\phi_f(x,y)\big)\,dy\,dx.$$

Hence $\theta = m_1\theta + k_1$. Since θ is irrational, $k_1 = 0$ and $m_1 = 1$. Thus we obtain that

$$F_1(x,y) = F_1\big(\phi_f(x,y)\big)$$

for any $x, y \in \mathbf{T}$. Since ϕ_f is minimal and F_1 is continuous, $F_1 = c$, a real constant number. Since $m_1 = 1$, $n_1 = 0$ and $F_1 = c$, by (2) we see that $n_2 = m_1 = 1$ and that

$$c + F_2(x,y) = m_2\theta + f(x) + F_2\big(\phi_f(x,y)\big) + k_2 \tag{3}$$

where k_2 is a constant integer. Since ϕ_f is measure-preserving,

$$\int_{\mathbf{T}^2} F_2(x,y)\,dy\,dx = \int_{\mathbf{T}^2} F_2\big(\phi_f(x,y)\big)\,dx\,dy.$$

Thus

$$c = m_2\theta + \int_{\mathbf{T}} f(x)\,dx + k_2 = m_2\theta + \eta + k_2$$

where $\eta = \int_{\mathbf{T}} f(x)\,dx$. Let $g(x) = \int_{\mathbf{T}} F_2(x,y)\,dy$. Then g is a real valued continuous function on $\mathbf{T}$, and

$$\begin{aligned}\int_{\mathbf{T}} F_2\big(\phi_f(x,y)\big)\,dy &= \int_{\mathbf{T}} F_2(e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)\,dy \\ &= \int_{\mathbf{T}} F_2(e^{2\pi i\theta}x, y)\,dy = g(e^{2\pi i\theta}x)\end{aligned}$$

since $d(e^{2\pi i f(x)}xy) = dy$. Therefore by (3) we obtain that

$$c + g(x) = m_2\theta + f(x) + g(e^{2\pi i\theta}x) + k_2.$$

Furthermore since $c = m_2\theta + \eta + k_2$, we see that

$$\eta + g(x) = f(x) + g(e^{2\pi i\theta}x).$$

Thus

$$g(x) - g(e^{2\pi i\theta}x) = f(x) - \eta$$

for any $x \in \mathbf{T}$. ■

Combining Lemmas 1 and 2 we obtain the following theorem;

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THEOREM 3. *Let f and ϕ_0, ϕ_f be as above. Then ϕ_f is topologically conjugate to ϕ_0 if and only if there is a real valued continuous function g on $\mathbf{T}$ such that*

$$g(x) - g(e^{2\pi i\theta}x) = f(x) - \int_{\mathbf{T}} f(z)\,dz$$

for any $x \in \mathbf{T}$.

3. **Topologically quasi-discrete spectrum.** In this section we will show that ϕ_f has topologically quasi-discrete spectrum if and only if ϕ_f is topologically conjugate to ϕ_0.

LEMMA 4. *Let ϕ_f and ϕ_0 be homeomorphisms on $\mathbf{T}^2$ defined in Section 1. We suppose that ϕ_f is not topologically conjugate to ϕ_0. Then ϕ_f does not have topologically quasi-discrete spectrum.*

PROOF. By the proof of Rouhani [10, Theorem 2.1],

$$G_1(\phi_f) = \{au^k : k \in \mathbf{Z}\ |a| = 1\}$$

where $u(x, y) = x$ for any $x, y \in \mathbf{T}$.

Since the C^*-algebra generated by u is not all of $C(\mathbf{T}^2)$, to show ϕ_f does not have topologically quasi-discrete spectrum it will suffice to check that there is no $h \in C(\mathbf{T}^2)$ with $|h| = 1$ satisfying that $h \circ \phi_f = au^k h$, where $|a| = 1$ and k is a non-zero integer. (If $k = 0$, then h is just an eigenfunction.)

So we assume that for some $k \neq 0$ there is a solution $h \in C(\mathbf{T}^2)$ such that $h \circ \phi_f = au^k h$ and $|h| = 1$. By the Homotopy Lifting Theorem we can write h as

$$h(x, y) = x^m y^n e^{2\pi i S(x,y)}$$

where m, n are integers and S is a real valued continuous function on $\mathbf{T}^2$. Then since $h \circ \phi_f = au^k h$, we see that $n = k$ and that

$$e^{2\pi i\{S(\phi_f(x,y)) - S(x,y) + kf(x)\}} = ae^{-2\pi i m\theta}.$$

Since the right hand side is constant, we obtain that

$$S\big(\phi_f(x, y)\big) - S(x, y) + kf(x) = c$$

where c is a real constant number. In the same way as in the proof of Lemma 2, we see that

$$\int_{\mathbf{T}} f(x)\,dx = \frac{c}{k}.$$

Furthermore for any $x \in \mathbf{T}$ let

$$g(x) = \frac{1}{k}\int_{\mathbf{T}} S(x, y)\,dy.$$

Then g is a real valued continuous function and

$$g(e^{2\pi i\theta}x) - g(x) + f(x) = \frac{c}{k}.$$

Since $\frac{c}{k} = \int_{\mathbf{T}} f(z)\,dz$, we obtain that

$$g(x) - g(e^{2\pi i\theta}x) = f(x) - \int_{\mathbf{T}} f(z)\,dz.$$

By Theorem 3 ϕ_f is topologically conjugate to ϕ_0. This is a contradiction. Therefore ϕ_f does not have topologically quasi-discrete spectrum. ∎

COROLLARY 5. *Let f and ϕ_f, ϕ_0 be as above. Then ϕ_f has topologically quasi-discrete spectrum if and only if ϕ_f is topologically conjugate to ϕ_0.*

PROOF. This is immediate by Lemma 4. ∎

For $j = 1, 2$ let ϕ_j and $A(\phi_j)$ be as in Section 1. If ϕ_1 and ϕ_2 have topologically quasi-discrete spectrum, $A(\phi_1) \cong A(\phi_2)$ by Corollary 5.

It is natural that we consider the following question: Let ϕ_0 and ϕ_f be as in Section 1. Let $A(\phi_0) = C(\mathbf{T}^2) \times_{\phi_0} \mathbf{Z}$ and $A(\phi_f) = C(\mathbf{T}^2) \times_{\phi_f} \mathbf{Z}$ be the associated crossed product C^*-algebras. Is there a Furstenberg transformation ϕ_f satisfying that $A(\phi_f)$ is not isomorphic to $A(\phi_0)$?

In the next section we will see that many Furstenberg transformations are not conjugate to Anzai transformations.

4. **Furstenberg transformations without quasi-discrete spectrum.** In [10] Rouhani constructed a Furstenberg transformation which does not have topologically quasi-discrete spectrum but is uniquely ergodic for a Liouville number θ.

In this section we will construct a Furstenberg transformation ϕ_f which does not have topologically quasi-discrete spectrum but is uniquely ergodic for any irrational number θ.

Since θ is irrational, we can choose a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that

$$|e^{2\pi i n_j \theta} - 1| < \frac{1}{j} \quad \text{for } j \geq 1.$$

Let $\{a_n\}_{n=-\infty}^{\infty}$ be the sequence defined by

$$a_n = \begin{cases} \frac{1}{j}(1 - e^{2\pi i n_j \theta}) & \text{if } n = n_j \\ \frac{1}{j}(1 - e^{-2\pi i n_j \theta)} & \text{if } n = -n_j \\ 0 & \text{elsewhere.} \end{cases}$$

For any $x \in \mathbf{T}$ let $f(x) = \sum_{n=-\infty}^{\infty} a_n x^n$. Then for $n = \pm n_j$.

$$|a_n| = \frac{1}{j}|1 - e^{2\pi i n_j \theta}| < \frac{1}{j^2}.$$

Hence the series $\sum_{n=-\infty}^{\infty} a_n x^n$ converges uniformly and f is a real valued continuous function on $\mathbf{T}$. We note that $\int_{\mathbf{T}} f(z)\,dz = 0$ since $a_0 = 0$.

LEMMA 6. *Let $\{n_j\}_{j=1}^{\infty}$, $\{a_n\}_{n=-\infty}^{\infty}$ and f be as above. We consider the equation*

$$g(x) - g(e^{2\pi i \theta} x) = f(x) \quad (x \in \mathbf{T}).$$

Then the above equation has a real valued $L^2(\mathbf{T})$-solution g but no real valued $C(\mathbf{T})$-solution.

PROOF. Let $\{b_n\}_{n=-\infty}^{\infty}$ be the sequence defined by

$$b_n = \begin{cases} \frac{1}{j} & \text{if } n = \pm n_j \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in \mathbf{T}$ let $g(x) = \sum_{n=-\infty}^{\infty} b_n x^n$. Then the series $\sum_{n=-\infty}^{\infty} b_n x^n$ converges with respect to the L^2-norm. Hence g is a real valued function in $L^2(\mathbf{T})$. And by a direct computation

$$g(x) - g(e^{2\pi i\theta}x) = f(x) \quad (\text{a.e. } x \in \mathbf{T}).$$

Furthermore in the same way as in the proof of Rouhani [10, Lemma 2.3] the above equation has no real valued $C(\mathbf{T})$-solution. ∎

THEOREM 7. *Let f be as in Lemma 6. Let ϕ_f be the homeomorphism on $\mathbf{T}^2$ defined by*

$$\phi_f(x, y) = (e^{2\pi i\theta}x, e^{2\pi i f(x)}xy)$$

for any $x, y \in \mathbf{T}$. Then ϕ_f does not have topologically quasi-discrete spectrum but is uniquely ergodic.

PROOF. It is clear that ϕ_f is uniquely ergodic by Rouhani [10, Proposition 2.5] and Lemma 6. Moreover by Theorem 3, Corollary 5 and Lemma 6, we see that ϕ_f does not have topologically quasi-discrete spectrum. Therefore we obtain the conclusion. ∎

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