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Localization of the Hasse-Schmidt Algebra

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Abstract. The behaviour of the Hasse-Schmidt algebra of higher derivations under localization is studied using André cohomology. Elementary techniques are used to describe the Hasse-Schmidt derivations on certain monomial rings in the nonmodular case. The localization conjecture is then verified for all monomial rings.

1 Introduction

The Hasse-Schmidt algebra is a higher-order analogue of the derivation algebra that enjoys particularly nice properties in prime characteristic. It arises naturally in many areas of commutative algebra: the study of singularity theory [5], differential operators [4], [7], tight closure [8] and modular invariant theory [6], [9]. Despite the interest inspired by these connections, several fundamental questions about the Hasse-Schmidt algebra remain open, including the localization question: Does formation of the Hasse-Schmidt algebra commute with localization?

Let R be an algebra of finite type over a field k. An (infinite order) *Hasse-Schmidt derivation* $\{\delta_n\}_{n=0}^{\infty}$ is a sequence of k-module endomorphisms of R with $\delta_0 = \mathrm{id}_R$ that satisfy the *product rule*

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

Similarly, an *m-th order* Hasse-Schmidt derivation is just a finite collection $\{\delta_n\}_{n=0}^{m}$ of k-linear endomorphisms of R satisfying the previous relations. The divided powers operators $\delta_n = \frac{1}{n!}\frac{\partial^n}{\partial x_i^n}$ on the polynomial ring $R = k[x_1, \ldots, x_d]$ are the simplest example of a Hasse-Schmidt derivation. The *Hasse-Schmidt algebra* $\mathrm{HS}(R/k)$ (or just $\mathrm{HS}(R)$ when k is understood) is the R-algebra generated by all components δ_n of all Hasse-Schmidt derivations. The R-algebra structure on $\mathrm{HS}(R)$ comes from identifying R with the multiplication maps in $\mathrm{End}_k(R)$.

Another way to describe the Hasse-Schmidt derivations involves k-algebra homomorphisms $R \to R[[t]]$. Each Hasse-Schmidt derivation $\{\delta_n\}$ is identified with a k-algebra map, $D = \sum_{n=0}^{\infty} \delta_n t^n \colon R \to R[[t]]$ that reduces to the identity on R modulo (t). Similarly, m-th order Hasse-Schmidt derivations are identified with deformations $D_m = \sum_{n=0}^{m} \delta_n t^n \colon R \to R[t]/(t^{m+1})$ of the identity on R.

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Brown and Kuan [2] investigated the localization of Hasse-Schmidt derivatives. Given a Hasse-Schmidt derivative $\{\delta_n\}$ and $r \in R$, let $r\{\delta_n\} = \{r^n\delta_n\}$. It can be shown that every Hasse-Schmidt derivation on R induces a unique Hasse-Schmidt derivation on each localization $S^{-1}R$ (see [7], [8]); however, it is not true that every Hasse-Schmidt derivation on the localization arises in this way. For example, if $R = k[x]$ and $S = \{1, g, g^2, \dots\}$ then this would require that the powers of g appearing in the denominators of $\delta_n(x)$ grow linearly in n. Though the Hasse-Schmidt derivations do not localize well, the Hasse-Schmidt algebra might behave better.

The Localization Conjecture for $\mathrm{HS}(R/k)$ If R is a k-algebra of finite type and S is a multiplicatively closed set in R, then $\mathrm{HS}(R) \otimes_R S^{-1}R$ is naturally isomorphic to $\mathrm{HS}(S^{-1}R/k)$.

In characteristic zero, $\mathrm{HS}(R/k)$ is just the derivation algebra $\mathrm{Der}(R/k)$ [7] and when R is smooth over k, then $\mathrm{HS}(R/k)$ equals the ring of differential operators $D(R/k)$ [8]. So the localization conjecture holds trivially unless R is singular and k is a field of prime characteristic.

The next section describes the obstruction (living in André cohomology [1]) to lifting a finite-order Hasse-Schmidt derivation to a Hasse-Schmidt derivation of higher order. Unfortunately, the obstruction lives in a cohomology module that seldom vanishes. The last section describes the Hasse-Schmidt derivations for schemes defined by monomial ideals. This extends results of Brumatti and Simis [3] on the derivations of such algebras. The localization conjecture for Hasse-Schmidt algebras is then verified for monomial rings.

2 Lifting Truncated Derivations

We describe a homological approach to the localization conjecture based on [5]. Let $\{\delta_n\}$ be a Hasse-Schmidt derivation on $S^{-1}R$. Fix an integer $m > 0$. We aim to show that $\delta_m \in \mathrm{HS}(R) \otimes_R S^{-1}R$. Multiplying the derivation by an element of S, we may assume that the maps δ_n with $n \le m$ map R into R. Now we ask whether the m-th order Hasse-Schmidt derivation extends to an infinite Hasse-Schmidt derivation on R.

The truncated derivation determines a map $D_m \colon R \to R[t]/(t^{m+1})$. Because $R[[t]]$ is the inverse limit of $R[t]/(t^n)$, it suffices to find step-by-step extensions $D_n \colon R \to R[t]/(t^{n+1})$ $(n > m)$ such that $D_m(a) \equiv D_n(a) \mod t^{m+1}$. Now consider the exact sequence

$$0 \longrightarrow R \xrightarrow{t^n} R[t]/(t^{n+1}) \xrightarrow{p} R[t]/(t^n) \longrightarrow 0$$

where $p(x) = x \mod t^n$. Taking the pullback by the map $D_{n-1} \colon R \to R[t]/(t^n)$ gives the exact sequence

$$(*) \qquad 0 \longrightarrow R \longrightarrow T \xrightarrow{\pi_2} R \longrightarrow 0$$

with

$$T = \{(u, r) \in R[t]/(t^{n+1}) \times R : p(u) = D_{n-1}(r)\}.$$

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This extension of k-algebras splits (determines the trivial extension) if and only if D_{n-1} lifts to D_n. Therefore the obstruction to lifting D_{n-1} is the cohomology class represented by $(*)$ in the module $H^1(k,R,R)$. This cohomology module agrees with Grothendieck's $\mathrm{Exalcom}_k(R,R)$, but we do not discuss this further.

One might naively hope that $H^1(k,R,R)=0$ for many algebras, but this is seldom the case for nonsmooth R, as the next example shows.

Example The cohomology module $H^1(k,R,R)$ need not vanish. If $R = k[x_1,\dots,x_d]/(f)$, then $H^1(k,R,R) = R/(f_1,\dots,f_d)$, where f_i is the i-th partial derivative of f. For instance, if $R=k[x,y]/(xy)$, then $H^1(k,R,R)=k\neq 0$.

The example also shows that a direct approach using cohomology will be difficult. For instance, using Corollaries 4.59 and 5.27 of André [1], we find that

$$H^1(k,S^{-1}R,S^{-1}R)\cong S^{-1}H^1(k,R,R).$$

Since we are given a truncated Hasse-Schmidt derivation on $S^{-1}R$ that extends to an infinite Hasse-Schmidt derivation, the associated class of $(*)$ in $S^{-1}H^1(k,R,R)$ is zero. However, this is not enough to guarantee that the class of $(*)$ is zero in $H^1(k,R,R)$; indeed, this module has support on the singular locus of R. For instance, localizing by any nontrivial multiplicative set S in the example gives $S^{-1}H^1(k,R,R) = 0$. So a direct approach using localization results on cohomology appears to be fruitless. The next section collects some preliminary results, after which, we give a direct proof of the localization conjecture for quotients of polynomial rings by monomial ideals.

3 Preliminary Lemmata

Lemma 3.1 *Let I be an ideal of a commutative ring R and x and y be elements of R. Then $\big(I:(I:x)\big)\big(I:(I:y)\big)\subset I:(I:xy)$.*

Proof Note $I:(I:y)\subset (I:x):(I:xy)$. Then:

$$\begin{aligned}\big(I:(I:x)\big)\big(I:(I:y)\big) &\subset \big(I:(I:x)\big)\big((I:x):(I:xy)\big)\\ &\subset I:(I:xy).\end{aligned}$$

■

When $\mathbf{a}=(a_1,\dots,a_d)\in\mathbb{N}^d$, then define the order $|\mathbf{a}|=a_1+\cdots+a_d$, the factorial $\mathbf{a}! = a_1!\,a_2!\cdots a_d!$, and the associated monomial $x^{\mathbf{a}} = x_1^{a_1}\cdots x_d^{a_d}$. For two multiexponents $\mathbf{a}$ and $\mathbf{b}$ we say that $\mathbf{a}\le\mathbf{b}$ if $x^{\mathbf{a}}$ divides $x^{\mathbf{b}}$ and we set $\binom{\mathbf{b}}{\mathbf{a}} = \binom{b_1}{a_1}\binom{b_2}{a_2}\cdots\binom{b_d}{a_d}$.

Lemma 3.2 *Let $\{\delta_n\}$ be a Hasse-Schmidt derivation on $R=k[x_1,\dots,x_d]/I$. Each δ_n is a differential operator of order less than or equal to n and for $n>0$, $\delta_n = \sum_{|\mathbf{a}|\le n} P_{\mathbf{a}}\frac{1}{\mathbf{a}!}\frac{\partial^{|\mathbf{a}|}}{\partial x^{\mathbf{a}}}$, where*

$$P_{\mathbf{a}} = \sum\prod_{j=1}^{d}\prod_{k=1}^{a_j}\delta_{i_{jk}}(x_j)$$

and the sum is over all tuples $(i_{11}, \ldots, i_{da_d})$ of order n, none of whose entries are zero. In particular, the constant term of δ_n is zero.

Proof First use induction to see that δ_n is a differential operator of order less than or equal to n. This is clear for $n = 0$ and the inductive step follows from

$$[\delta_n, f] = \sum_{k=1}^{n} \delta_k(f)\delta_{n-k}.$$

So for $n > 0$ we can write $\delta_n = \sum_{|\mathbf{a}|\le n} Q_{\mathbf{a}} \frac{1}{\mathbf{a}!}\frac{\partial^{|\mathbf{a}|}}{\partial x^{\mathbf{a}}}$. We aim to show that $Q_{\mathbf{a}} = P_{\mathbf{a}}$. This follows from comparing the expressions given by applying $\sum_{|\mathbf{a}|\le n} Q_{\mathbf{a}} \frac{1}{\mathbf{a}!}\frac{\partial^{|\mathbf{a}|}}{\partial x^{\mathbf{a}}}$ and δ_n to monomials $x^{\mathbf{b}}$:

$$\left(\sum_{|\mathbf{a}|\le n} Q_{\mathbf{a}} \frac{1}{\mathbf{a}!}\frac{\partial^{|\mathbf{a}|}}{\partial x^{\mathbf{a}}}\right)(x^{\mathbf{b}}) = \sum_{\mathbf{a}\le\mathbf{b}} Q_{\mathbf{a}} \binom{\mathbf{b}}{\mathbf{a}} x^{\mathbf{b}-\mathbf{a}}$$

$$\delta_n(x^{\mathbf{b}}) = \sum_{\mathbf{a}\le\mathbf{b}} \binom{\mathbf{b}}{\mathbf{a}} \Big(\sum \prod_{j=1}^{d}\prod_{k=1}^{a_j} d_{i_{jk}}(x_j)\Big)\, d_0(x^{\mathbf{b}-\mathbf{a}}),$$

where the interior summation is over all tuples $(i_{11}, \ldots, i_{da_d})$ of order n, none of whose entries are zero. The last statement of the Lemma can also be obtained from the product rule using induction on n. ■

4 Monomial Algebras

Let $I \subset k[x_1, \ldots, x_d]$ be a monomial ideal with minimal monomial generators $m_1, \ldots, m_t$. Assume that the characteristic of k is $p > 0$. Define the ideals

$$\begin{aligned} I_i &= (m_j\colon p \text{ does not divide the exponent of } x_i \text{ in } m_j) \\ &= \left(x_i \frac{\partial m_j}{\partial x_i} : j = 1, \ldots, t\right). \end{aligned}$$

When $I = R$ we adopt the convention that $I_i = R$ as well.

Theorem 4.1 *Let $\{\delta_n\}$ be a Hasse-Schmidt derivation of $R = k[x_1, \ldots, x_d]/(m_1, \ldots, m_t)$. Then $\{\delta_n\}$ induces a ring map $D = \sum d_n t^n\colon R \to R[[t]]$ with $d_0 = \mathrm{id}_R$ and $d_n(x_k) \in I : (I_k : x_k)$ for $n > 0$.*

Proof Assume that $\{\delta_n\}$ is a Hasse-Schmidt derivation. Fix $n > 0$. Then the algebra map $D_n\colon R \to R[t]/(t^{n+1})$ induced by $D_n(x_i) = \sum_{k=0}^{n} \delta_k(x_i)t^k$ is a well-defined ring homomorphism. Applying this map to any generator m_u gives

$$D_n(m_u) = m_u + \cdots + \nabla m_u \cdot \langle \delta_n(x_1), \ldots, \delta_n(x_d)\rangle t^n \in IR[t]/(t^{n+1}).$$

Since I is a monomial ideal, this implies $\delta_n(x_k)\frac{\partial m_u}{\partial x_k} \in I$ for all generators m_u. That is, $\delta_n(x_k) \in I : (\frac{\partial m_u}{\partial x_k} : u = 1, \ldots, t)$. However, $(\frac{\partial m_u}{\partial x_k} : u = 1, \ldots, t) = I_k : x_k$, so the result holds for each index n. ■

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In the nonmodular case, the converse to this theorem is also true.

Theorem 4.2 *Suppose that for each variable x_i and each monomial generator m_u, the exponent of x_i in m_u is not divisible by the prime p and consider the ring $R = k[x_1, \ldots, x_d]/(m_1, \ldots, m_t)$ of characteristic p. Then $I_k = I$ for all k and each choice of images $\delta_n(x_k) \in I : (I : x_k)$ for $n > 0$ induces a Hasse-Schmidt derivation $\{\delta_n\}$ on R.*

Proof We need to check that the induced algebra map $\sum_{n=0}^{\infty} \delta_n t^n \colon R \to R[[t]]$ is well-defined. It suffices to show that $\delta_n(m_u) \in I$ for all n and all $u = 1, \ldots, t$. Using Lemma 3.2, $\delta_n(m_u) = \sum_{|\mathbf{a}| \le n} P_{\mathbf{a}} \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}}$, where

$$P_{\mathbf{a}} = \sum \prod_{j=1}^{d} \prod_{k=1}^{a_j} \delta_{i_{jk}}(x_j)$$

and the sum is over all tuples $(i_{11}, \ldots, i_{da_d})$ of order n, none of whose entries are zero. Each $\delta_{i_{jk}}(x_j) \in I : (I : x_j)$ so by Lemma 3.1, $P_{\mathbf{a}} \in I : (I : x^{\mathbf{a}})$. But $\frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}}$ is either 0 or a multiple of $m_u / x^{\mathbf{a}}$; in either case, $\frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} \in I : x^{\mathbf{a}}$. So $P_{\mathbf{a}} \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} \in I$, as desired. ∎

Example The conditions in Theorems 4.1 and 4.2 are not equivalent. For instance, when $I = (x^p, x^2 y^3, x y^4, y^5)$ in $k[x, y]$ with $\operatorname{char} k = p > 5$, then $I : (I_k : x) \neq I : (I : x)$.

In the nonmodular case, these two theorems give necessary and sufficient conditions for a collection of maps $\{\delta_n\}$ on a monomial ring to be a Hasse-Schmidt derivation. In particular, they characterize the Hasse-Schmidt derivations of Stanley-Reisner rings. Two remarks are pertinent here:

1. Every derivation in the nonmodular case is integrable, that is, it can be extended to an infinite Hasse-Schmidt derivation. Integrable derivations play a key role in the study of singularities in prime characteristic via derivational methods [5].
2. Since the conditions $d_n(x_k) \in I : (I : x_k)$ localize well, these theorems establish the localization conjecture in the nonmodular case. We now give an argument that verifies the conjecture for all monomial rings.

Theorem 4.3 *The localization conjecture holds for all monomial rings $R = k[x_1, \ldots, x_d]/(m_1, \ldots, m_t)$.*

Proof Assume that $D = \sum_{n=0}^{\infty} \delta_n t^n \colon S^{-1}R \to S^{-1}R[[t]]$ defines a ring homomorphism. Then from Lemma 3.2, we see that

$$\sum \prod_{j=1}^{d} \prod_{k=1}^{a_j} \delta_{i_{jk}}(x_j) \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} \in (m_1, \ldots, m_t) S^{-1}R.$$

Pick $\tilde{\delta}_n(x_j) \in R$ such that there is an $s_n \in S$ with $\delta_n(x_j) = \tilde{\delta}_n(x_j)/s_n$. Since the ideal is monomial, we see that there exist $s_{\mathbf{a}}$ and $\tilde{s}_{\mathbf{a}}$ in S such that

$$s_{\mathbf{a}} \prod_{j=1}^{d} \prod_{k=1}^{a_j} \delta_{i_{jk}}(x_j) \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} = \tilde{s}_{\mathbf{a}} \prod_{j=1}^{d} \prod_{k=1}^{a_j} \tilde{\delta}_{i_{jk}}(x_j) \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} \in (m_1, \dots, m_t)R.$$

Now since the ring R is Noetherian, the ideal $\left(\prod_{j=1}^{d} \prod_{k=1}^{a_j} \tilde{\delta}_{i_{jk}}(x_j) \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}}\right)_{\mathbf{a}}$ is finitely generated. It follows that there exists a single $s \in S$ such that $s \prod_{j=1}^{d} \prod_{k=1}^{a_j} \tilde{\delta}_{i_{jk}}(x_j) \frac{1}{\mathbf{a}!} \frac{\partial^{\mathbf{a}} m_u}{\partial x^{\mathbf{a}}} \in (m_1, \dots, m_t)R$ for all $\mathbf{a}$. This implies that the collection $s\{\tilde{\delta}_n\}$ is a Hasse-Schmidt derivation on R. In particular, $s^n \tilde{\delta}_n \in \mathrm{HS}(R)$ and so $\tilde{\delta}_n \in \mathrm{HS}(R) \otimes_R S^{-1}R$. Since $\tilde{\delta}_n$ and δ_n differ by a factor in S, $\delta_n \in \mathrm{HS}(R) \otimes_R S^{-1}R$, as desired. ■

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