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TOTALLY UMBILICAL SUBMANIFOLDS IN IRREDUCIBLE SYMMETRIC SPACES

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Abstract

A submanifold of a Riemannian manifold is called a totally umbilical submanifold if its first and second fundamental forms are proportional. In this paper we prove the following best possible result.

THEOREM. *There is no totally umbilical submanifold of codimension less than rank of M in any irreducible symmetric space M.*

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1. Introduction

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold $M (n \geqslant 2)$ with the first fundamental form g. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on N and M, respectively. The second fundamental form h of the immersion is defined by the equation

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y, \tag{1.1}$$

where X and Y are vector fields tangent to N. The submanifold N is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{1.2}$$

for all vector fields X, Y tangent to N, where $H = 1/n$ (trace h) is the *mean curvature vector* of N in M. The length of H is called the *mean curvature* of N in M. A totally umbilical submanifold with vanishing mean curvature is called a *totally geodesic submanifold*.

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In Chen (1980), the following results were proved.

PROPOSITION 1. *Let N be a totally umbilical submanifold in a symmetric space M. If* co-dim $N <$ rank $M - 1$, *then N has constant mean curvature and N is either totally geodesic or of constant sectional curvature.*

PROPOSITION 2. *If N is a totally umbilical submanifold in an irreducible symmetric space M, then* co-dim $N \geqslant$ rank $M - 1$.

It is known that the Riemannian product $M = \mathbf{R} \times S^n$ of a real line $\mathbf{R}$ and an n-sphere S^n is a rank 2 symmetric space which admits a totally umbilical hypersurface with nonconstant mean curvature. Thus the estimate of the codimension of N in Proposition 1 is best possible.

It is also known that the real Grassmann manifold $M = SO(p+q)/SO(p) \times SO(q)$ $(p \geqslant q \geqslant 1)$ is an irreducible symmetric space of rank q which admits a totally umbilical (in fact, totally geodesic) submanifold with codimension equal to rank M. In this note we shall prove that there is no totally umbilical submanifold N in an irreducible symmetric space M with codimension equal to rank $M - 1$. By combining this result with Proposition 2, we obtain the following fundamental result.

MAIN THEOREM. *Let N be a totally umbilical submanifold in an irreducible symmetric space M. Then* co-dim $N \geqslant$ rank M.

From the examples of real Grassmann manifolds, we see that the estimate of codimension in Main Theorem is best possible.

2. Basic formulas

Let N be an n-dimensional submanifold of a Riemannian manifold M. For a vector field ξ normal to N we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.1}$$

where $-A_\xi X$ and $D_X\xi$ are the tangential and normal components of $\tilde{\nabla}_X \xi$, respectively. A normal vector field ξ is said to be *parallel* if $D\xi = 0$ identically.

Let R and $\tilde{R}$ be the curvature tensors associated with ∇ and $\tilde{\nabla}$, respectively. For example, $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. For the second fundamental form

h, we define the covariant derivative in $(TN) \oplus (T^{\perp}N)$, to be

$$(2.2)\qquad (\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where TN and $T^{\perp}N$ denote the tangent and normal bundles of N, respectively.

We put $R(X, Y; Z, W) = g(R(X, Y)Z, W)$. For vector fields X, Y, Z, W tangent to N, the equations of Gauss and Codazzi take the forms:

$$(2.3)\qquad R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

$$(2.4)\qquad (\tilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z),$$

where $\perp$ in (2.4) denotes the normal component.

Let X and Y be orthonormal vectors tangent to N. The sectional curvature $K(X \wedge Y)$ of the plane $X \wedge Y$ is given by

$$(2.5)\qquad K(X \wedge Y) = R(X, Y; Y, X).$$

If N is a totally umbilical submanifold in a Riemannian manifold M, (2.4) gives

$$(2.6)\qquad \tilde{R}(X, Y; Z, H) = \tfrac{1}{2}\{g(Y, Z)X\alpha^2 - g(X, Z)Y\alpha^2\},$$

where $\alpha^2 = g(H, H)$.

3. Constancy of mean curvature

An isometry s of a Riemannian manifold is said to be involutive if its iterate s^2 is the identity map. A Riemannian manifold M is a *symmetric space* if, at each point p of M, there exists an involutive isometry s_p of M such that p is an isolated fixed point of s_p.

We denote by G the closure of the group of isometries generated by $\{s_p \mid p \in M\}$ in the compact-open topology. Then G acts transitively on M; hence the typical isotropy subgroup K, say at 0, is compact and $M = G/K$.

Let σ_0 be the involutive automorphism of G given by $\sigma_0(x) = s_0 \cdot x \cdot s_0$, $x \in G$. Then σ_0 fixes K and it induces an involutive automorphism of the Lie algebra $\mathfrak{g}$ of G. The Cartan decomposition of $\mathfrak{g}$ is given by

$$(3.1)\qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{m},$$

where $\mathfrak{k}$ and $\mathfrak{m}$ are the eigenspaces of σ_0 with eigenvalues 1 and -1, respectively. It is known that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ can be identified with the tangent space T_0M of M at 0. Moreover, we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

The following lemmas of E. Cartan are well-known (see Helgason (1978)).

LEMMA 3. *The curvature tensor R of M at* 0 *satisfies*

$$\tilde{R}(X, Y)Z = -[[X, Y], Z] \tag{3.2}$$

for X, Y, $Z \in \mathfrak{m}$.

LEMMA 4. *Let B be a totally geodesic submanifold of M through* 0. *Then B is flat if and only if* $[\pi, \pi] = 0$ *where* $\pi = T_0B \subset T_0M = \mathfrak{m}$.

We recall the following result of Chen-Nagano (see Chen (1980)).

LEMMA 5. *Every totally geodesic submanifold B of an irreducible symmetric space M satisfies*

$$\text{co-dim } B \geqslant \text{rank } M. \tag{3.3}$$

We give the following.

PROPOSITION 6. *Let N be a totally umbilical submanifold in a symmetric space M. If* co-dim $N \leqslant$ rank $M - 1$ *and M is Einsteinian, then the mean curvature of N is constant.*

PROOF. Let N be a totally umbilical submanifold in a symmetric space M. If co-dim $N <$ rank $M - 1$, N has constant mean curvature (Proposition 1). Therefore, we only need to consider the case where co-dim $N =$ rank $M - 1$.

For any fixed point 0 in N, let B be a maximal flat totally geodesic submanifold of M through 0 such that $H(0) \in T_0B$. We have dim $T_0B =$ rank M. Since B is a flat totally geodesic submanifold of M, Lemma 4 implies that $[U, V] = 0$ for all $U, V \in T_0B \subset \mathfrak{m}$.

Since dim $N =$ dim $M -$ rank $M + 1$, dim $T_0N \cap T_0B \geqslant 1$. Let X_0 be a unit vector in $T_0N \cap T_0B$. Then $[X_0, H] = 0$. Thus (2.6) and Lemma 3 give

$$0 = \tilde{R}(Y, X_0; X_0, H) = \tfrac{1}{2} Y\alpha^2 \tag{3.4}$$

for any vector Y in $\{Z \in T_0N \mid g(X_0, Z) = 0\}$. If dim $T_0N \cap T_0B \geqslant 2$, this implies that $W\alpha^2 = 0$ for all $W \in T_0N$.

If dim $T_0N \cap T_0B = 1$, then $T_0N \cup T_0B$ spans T_0M. Hence,

$$T_0N + T_0B = T_0M. \tag{3.5}$$

For any vector $\eta \in T_0M$ we put

$$\eta = \eta' + \eta^B \tag{3.6}$$

where $\eta' \in T_0N$ and $\eta^B \in T_0B$ with $g(X_0, \eta') = 0$. We have

$$[H, \eta^B] = [X_0, \eta^B] = 0. \tag{3.7}$$

Combining this with Lemma 3 we obtain

$$\tilde{R}(X_0, \eta^B; \eta^t, H) = \tilde{R}(X_0, \eta; \eta^B, H) = 0 \tag{3.8}$$

because $X_0, H, \eta^B \in T_0 B \subset \mathfrak{m}$.

Let $E_1, \ldots, E_n$ be an orthonormal basis of $T_0 N$ with $E_n = X_0$ and $\eta_1, \ldots, \eta_{m-n}$ an orthonormal basis of $T_0^\perp N$ with η_{m-n} parallel to H. Equations (2.6) and (3.8) give

$$\tilde{R}(E_n, \eta_i; \eta_i, H) = \tfrac{1}{2} g(\eta_i^t, \eta_i^t) E_n \alpha^2, \qquad i = 1, \ldots, m-n-1, \tag{3.9}$$

$$\tilde{R}(E_n, E_j, E_j, H) = \tfrac{1}{2} E_n \alpha^2, \qquad j \neq n. \tag{3.10}$$

Therefore, the Ricci tensor $\tilde{S}$ of M satisfies

$$\tilde{S}(E_n, H) = \tfrac{1}{2}\left\{(n-1) + \sum_{i=1}^{m-n-1} |\eta_i^t|^2\right\} X_0 \alpha^2.$$

If M is Einsteinian, this implies

$$X_0 \alpha^2 = 0. \tag{3.11}$$

(3.4) and (3.11) gives $W\alpha^2 = 0$ for all W in $T_0 N$. Since this is true for arbitrary point in N, α^2 is constant.

4. Proof of Main Theorem

Let M be an irreducible symmetric space. Then M is Einsteinian. If N is totally umbilical in M with co-dim $N \leqslant$ rank $M - 1$, N has constant mean curvature α (Proposition 6). Moreover, the mean curvature α *is a nonzero constant* (Lemma 5).

If the mean curvature vector H is parallel, N is an extrinsic sphere in M. Theorem 2 of Chen (1979) implies that M admits a totally geodesic submanifold $\bar{N}$ of constant sectional curvature of dimension equal to $1 + \dim N$. This is impossible (Lemma 5). Consequently, the mean curvature vector of N is *not parallel.*

If $\dim N = 2$, $\dim M \leqslant$ rank $M + 1$. From the list of irreducible symmetric spaces (see Helgason (1978)), M is 2-dimensional. This is a contradiction. Thus $\dim N > 2$.

Case (a). If M is of compact type, Theorem 4 of Chen (1980) shows that $\dim N < \frac{1}{2} \dim M$. Thus, from the assumption of codimension, we get

$$\dim M \leqslant 2 \operatorname{rank} M - 1.$$

This contradicts the classification of irreducible symmetric spaces.

Case (b). If M is of non-compact type, M is nonpositively curved, that is, the sectional curvature $\tilde{K}$ of M satisfies

$$\tilde{K} \leqslant 0. \tag{4.1}$$

Since N is totally umbilical submanifold of constant mean curvature $\alpha \neq 0$ in a symmetric space, N is one of the following spaces (see the proof of Theorem 4 in Chen (1980)):

(1) a space of constant sectional curvature c,

(2) a local product of two spaces $N_1(c)$ and $N_2(-c)$ of constant sectional curvatures c and $-c$, ($c \neq 0$) respectively, or

(3) a local product of a curve and a space $N_2(c)$ of constant sectional curvature $c \neq 0$.

Since N has constant mean curvature, (2.6) implies

$$\tilde{R}(X, Y; Z, H) = 0 \tag{4.2}$$

for any vector fields X, Y and Z tangent to N. Taking the derivative of (4.2) with respect to a tangent vector U in TN, we have

$$\begin{aligned}\alpha^2\tilde{R}(X, Y; Z, U) = g(U, X)\tilde{R}(H, Y; Z, H) - g(U, Y)\tilde{R}(H, X; Z, H)\\ +g(Y, Z)g(D_XH, D_UH) - g(X, Z)g(D_YH, D_UH).\end{aligned}$$

Let $X = U$, $Y = Z$ be orthonormal vectors tangent to N. This implies

$$\alpha^2\tilde{K}(H \wedge Y) = \alpha^2\tilde{K}(X \wedge Y) - |D_XH|^2. \tag{4.3}$$

For an arbitrary fixed point 0 in N let B be a maximal flat totally geodesic submanifold of M through 0 such that $H(0) \in T_0B$. Because co-dim $N \leqslant$ rank M $- 1$, we have

$$\dim T_0N \cap T_0B \geqslant 1.$$

Let Y_0 be a unit vector in $T_0N \cap T_0B$. We have $\tilde{K}(Y_0 \wedge H) = 0$. Thus, (4.3) gives

$$\alpha^2\tilde{K}(X \wedge Y_0) = |D_XH|^2. \tag{4.4}$$

Comparing (4.1) and (4.4) we obtain

$$D_XH = 0 \quad \text{for } X \in \{Z \in T_0N \,|\, g(Z, Y_0) = 0\}. \tag{4.5}$$

Case (b.1). If N is of constant sectional curvature c, (4.3) implies that $|D_XH|$ is independent of the choice of the unit vector X in T_0N. Thus, (4.5) gives $D_XH = 0$ for all $X \in T_0N$. Since this argument applies to an arbitrary in N, H is parallel. This gives a contradiction.

Case (b.2). If N is the local product of two spaces $N_1(c)$ and $N_2(-c)$ of constant sectional curvatures c and $-c$, $c \neq 0$, respectively, equation (4.3) proves that $|D_XH|$ is the same for all unit vectors X in T_0N_1 and $|D_ZH|$ is the same for all unit vectors Z in $T_0 N_2$. Since both N_1 and N_2 are of dimensions $\geqslant 2$, (4.5) shows that $D_U H = 0$ for all U in $T_0 N$. Because this is true for arbitrary point in N, H is parallel. This gives a contradiction.

Case (b.3). If N is the local product of a curve and a space $N_2(c)$ of constant sectional curvature c. Then (4.3) and (4.5) imply that $|D_X H|$ is independent of the choice of unit vector X in $T_0 N_2$ and in fact

$$D_X H = 0 \quad \text{for} \quad X \in T_0 N_2. \tag{4.1}$$

Since $N_2 = N_2(c)$ is totally geodesic in N and N is totally umbilical in M, N_2 is totally umbilical in M. Moreover, the mean curvature vector of N_2 in M is in fact the restriction of H on N_2. Moreover, it can be easily proved that the normal connection D^2 of N_2 in M satisfies $D^2_X H = D_X H = 0$. Since this is true for arbitrary point in N, N_2 is an extrinsic sphere in M. Theorem 2 of Chen (1979) then implies that M admits a totally geodesic submanifold of dimension equal to $1 + \dim N_2$. This contradicts Lemma 5 and our assumption.

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