Forum of Mathematics, Pi (2023), **Vol. 11**:e26 1–32
doi:10.1017/fmp.2023.26

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

On Thakur's basis conjecture for multiple zeta values in positive characteristic

Chieh-Yu Chang [1], Yen-Tsung Chen [2] and Yoshinori Mishiba [3]

[1]Department of Mathematics, National Tsing Hua University, Hsinchu City, 30042, Taiwan R.O.C.;
E-mail: cychang@math.nthu.edu.tw.
[2]Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; E-mail: ytchen.math@gmail.com.
[3]Mathematical Institute, Tohoku University, 6-3, Aoba, Aramaki, Aoba-ku, Sendai, 980-8578, Japan;
E-mail: mishiba@tohoku.ac.jp.

Received: 29 July 2022; **Revised:** 5 January 2023; **Accepted:** 5 September 2023

2020 Mathematics Subject Classification: *Primary* – 11J93, 11R58

Abstract

In this paper, we study multiple zeta values (abbreviated as MZV's) over function fields in positive characteristic. Our main result is to prove Thakur's basis conjecture, which plays the analogue of Hoffman's basis conjecture for real MZV's. As a consequence, we derive Todd's dimension conjecture, which is the analogue of Zagier's dimension conjecture for classical real MZV's.

Contents

1 Introduction **2**
1.1 Classical conjectures . . . 2
1.2 The main results . . . 3
1.3 Strategy of proofs . . . 4
1.4 Organization of the paper . . . 6
2 Two products on $\mathcal{H}$ **6**
2.1 Indices . . . 6
2.2 The maps $\mathscr{L}^{\mathrm{Li}}$ and $\mathscr{L}^{\zeta}$ . . . 7
2.3 Product formulae . . . 9
3 Generators **11**
3.1 The box-plus operator . . . 11
3.2 Generating sets . . . 11
4 Frobenius difference equations **13**
4.1 ABP-criterion . . . 13
4.2 Some lemmas . . . 14
4.3 Dimension of $\mathscr{X}_w$ . . . 16
5 Linear independence **21**
5.1 The key lemma . . . 22
5.2 Linear independence . . . 25
5.3 Proof of Theorem 1.5 . . . 25
5.4 Generating set of linear relations . . . 25

© The Author(s), 2023. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

A Proof of Theorem 3.6 **26**
A.1 Inequalities of depth 26
A.2 Binary relations 27
A.3 Maps between relations 28

1. Introduction

1.1. Classical conjectures

In this paper, we study multiple zeta values (abbreviated as MZV's) over function fields in positive characteristic introduced by Thakur [31]. Our motivation arises from Zagier's dimension conjecture and Hoffman's basis conjecture for classical real MZV's.

The special value of the Riemann ζ-function at positive integer $s \geq 2$ is the following series:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \in \mathbb{R}^{\times}.$$

Classical real MZV's are generalizations of the special ζ-values above. Their study was initiated by Euler on double zeta values and fully generalized by Zagier [36] in the 1990s. An *admissible index* is an r-tuple of positive integer $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{Z}_{>0}^r$ with $s_1 \geq 2$. The real MZV at $\mathfrak{s}$ is defined by the following multiple series:

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}^{\times}.$$

We call $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ and $\mathrm{dep}(\mathfrak{s}) := r$ the weight and depth of the presentation $\zeta(\mathfrak{s})$, respectively. Over the past decades, the study of real MZV's has attracted many researchers' attention as MZV's have many interesting and important connections with various topics. For example, MZV's occur as periods of mixed Tate motives by Terasoma [29], Goncharov [18] and Deligne-Goncharov [16], and MZV's of depth two have close connection with modular forms by Gangl-Kaneko-Zagier [17] etc. For more details and relevant references, we refer the reader to the books [5, 37, 7].

By the theory of regularized double shuffle relations [28, 23], there are rich $\mathbb{Q}$-linear relations among the same weight MZV's. One core problem on this topic is Zagier's following dimension conjecture:

Conjecture 1.1 (Zagier's dimension conjecture). *For an integer $w \geq 2$, we let $\mathfrak{Z}_w$ be the $\mathbb{Q}$-vector space spanned by real MZV's of weight w. We put $d_0 := 1$, $d_1 := 0$, $d_2 := 1$ and $d_w := d_{w-2} + d_{w-3}$ for integers $w \geq 3$. Then for each integer $w \geq 2$, we have*

$$\dim_{\mathbb{Q}} \mathfrak{Z}_w = d_w.$$

The best known result towards Zagier's dimension conjecture until now has been the *upper bound* result proved by Terasoma [29] and Goncharov [18] independently. Namely, they showed that $\dim_{\mathbb{Q}} \mathfrak{Z}_w \leq d_w$ for all integers $w \geq 2$. Furthermore, Hoffman [21] proposed the following conjectural basis for $\mathfrak{Z}_w$ for each $w \geq 2$.

Conjecture 1.2 (Hoffman's basis conjecture). *For an integer $w \geq 2$, we let $\mathcal{I}_w^{\mathrm{H}}$ be the set of admissible indices $\mathfrak{s} = (s_1, \ldots, s_r)$ with $s_i \in \{2, 3\}$ satisfying $\mathrm{wt}(\mathfrak{s}) = w$. Then the following set*

$$\mathcal{B}_w^{\mathrm{H}} := \{\zeta(\mathfrak{s}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{H}}\}$$

is a basis of the $\mathbb{Q}$-vector space $\mathfrak{Z}_w$.

We mention that Hoffman's basis conjecture implies Zagier's dimension conjecture. In [6], Brown proved Hoffman's basis conjecture for *motivic* MZV's. As there is a surjective map from motivic MZV's

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

to real MZV's, Brown's theorem implies that Hoffman's conjectural basis would be a generating set for $\mathfrak{Z}_w$, and as a consequence, the upper bound result of Terasoma and Goncharov would be derived. Our main results in this paper are to prove the analogues of Conjecture 1.1 and Conjecture 1.2 and give precise generators of all linear relations among MZV's in positive characteristic.

1.2. The main results

In the positive characteristic setting, we let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime number p. We let k be the field of fractions of A and $|\cdot|_\infty$ be the normalized absolute value on k at the infinite place ∞ of k for which $|\theta|_\infty = q$. Let $k_\infty := \mathbb{F}_q((1/\theta))$ be the completion of k with respect to $|\cdot|_\infty$. We then fix an algebraic closure $\overline{k_\infty}$ of k_∞ and still denote by $|\cdot|_\infty$ the extended absolute value on $\overline{k_\infty}$. We let $\mathbb{C}_\infty$ be the completion of $\overline{k_\infty}$ with respect to $|\cdot|_\infty$. Finally, we let $\overline{k}$ be the algebraic closure of k inside $\mathbb{C}_\infty$.

Let A_+ be the set of monic polynomials of A, which plays an analogous role to the set of positive integers now in the function field setting. In [31], Thakur introduced the following positive characteristic MZV's: for any r-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r_{>0}$,

$$\zeta_A(\mathfrak{s}) := \sum_{a_1, \cdots, a_r \in A_+} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty \tag{1.1}$$

with the restriction that $|a_1|_\infty > \cdots > |a_r|_\infty$. As our absolute value $|\cdot|_\infty$ is non-archimedean, the series $\zeta_A(\mathfrak{s})$ converges in k_∞. However, Thakur [32] showed that such series are in fact non-vanishing. The weight and the depth of the presentation $\zeta_A(\mathfrak{s})$ are defined to be $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ and $\mathrm{dep}(\mathfrak{s}) := r$, respectively. When $\mathrm{dep}(\mathfrak{s}) = 1$, these values are now called Carlitz zeta values, as their study was initiated by Carlitz in [8].

As from now on, we focus on MZV's in positive characteristic; in what follows, MZV's will be Thakur's (∞-adic) MZV's. In [34], Thakur established a product on MZV's, which is called q-shuffle relation in this paper. Namely, the product of two MZV's can be expressed as an $\mathbb{F}_p$-linear combination of MZV's whose weights are the same. It follows that the k-vector space $\mathcal{Z}$ spanned by all MZV's forms an algebra.

Note that $\overline{k}$-algebraic relations among MZV's are $\overline{k}$-linear relations among monomials of MZV's, which can be expressed as $\overline{k}$-linear relations among MZV's by q-shuffle relations mentioned above. In [9], the first named author of the present paper proved that all $\overline{k}$-linear relations come from k-linear relations among MZV's of the same weight. Therefore, determination of the dimension of the k-vector space of MZV's of weight w for $w \in \mathbb{Z}_{>0}$ is the central problem in this topic. Via numerical computation, Todd [30] provided the following analogue of Zagier's dimension conjecture.

Conjecture 1.3 (Todd's dimension conjecture). *For any positive integer w, let $\mathcal{Z}_w$ be the k-vector space spanned by all MZV's of weight w. Define*

$$d'_w = \begin{cases} 2^{w-1} & \text{if } 1 \le w < q, \\ 2^{q-1} - 1 & \text{if } w = q, \\ \sum_{i=1}^q d'_{w-i} & \text{if } w > q. \end{cases}$$

Then one has

$$\dim_k \mathcal{Z}_w = d'_w.$$

In analogy with Hoffman's basis conjecture, Thakur [35] proposed the following conjecture.

Conjecture 1.4 (Thakur's basis conjecture). *Let w be a positive integer and let $\mathcal{I}^{\mathrm{T}}_w$ be the set consisting of all $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{Z}^r_{>0}$ (varying positive integers r) for which*

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

- $\mathrm{wt}(\mathfrak{s}) = w$;
- $s_i \leq q$ *for all* $1 \leq i \leq r-1$;
- $s_r < q$.

Then the following set

$$\mathcal{B}_w^{\mathrm{T}} := \{\zeta_A(\mathfrak{s}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}}\}$$

is a basis of the k-vector space $\mathcal{Z}_w$.

We mention that for every positive integer w, the cardinality of $\mathcal{I}_w^{\mathrm{T}}$ is equal to d'_w given in Conjecture 1.3. It follows that as in the classical case, Thakur's basis conjecture implies Todd's dimension conjecture. The main result of this paper is to prove Thakur's basis conjecture.

Theorem 1.5. *For any positive integer w, Conjecture 1.4 is true. As a consequence, Conjecture 1.3 is also true.*

As we have determined the dimension of $\mathcal{Z}_w$ for each $w \in \mathbb{N}$, it is a natural question about how to describe all the k-linear relations among the MZV's of weight w. We establish a concrete and simple mechanism in Theorem 5.3 that (5.3) accounts for all the k-linear relations. Note that Theorem 5.3 basically verifies the $\mathscr{B}^*$-version of [30, Conjecture 5.1].

Remark 1.6. In [26], Ngo Dac showed that for each $w \geq 1$, $\mathcal{B}_w^{\mathrm{T}}$ is a generating set for the k-vector space $\mathcal{Z}_w$. As a consequence of Ngo Dac's result, one has the *upper bound result*:

$$\dim_w \mathcal{Z}_w \leq d'_w, \ \forall w \geq 1.$$

Theoretically, to prove Thakur's basis conjecture, it suffices to show that his conjectural basis is linearly independent over k. However, when developing our approaches in a unified framework, we reprove Ngo Dac's result mentioned above in Corollary 3.7, which also provides a generating set for the k-vector space spanned by the special values $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ for all indices $\mathfrak{s}$ with $\mathrm{wt}(\mathfrak{s}) = w$ and which is described in the next section.

Remark 1.7. As mentioned above, by [9], k-linear independence of MZV's implies $\overline{k}$-linear independence. It follows that for each positive integer w, $\mathcal{B}_w^{\mathrm{T}}$ is also a basis for the $\overline{k}$-vector space spanned by the MZV's of weight w.

Given a finite place v of k, we mention that v-adic MZV's $\zeta_A(\mathfrak{s})_v$ were introduced in [13] for $\mathfrak{s} \in \mathbb{Z}_{>0}^r$. By [13, Thm. 1.2.2], there is a natural k-linear map from (∞-adic) MZV's to v-adic MZV's, and it was further shown in [11] that the map is indeed an algebra homomorphism with kernel containing the ideal generated by $\zeta_A(q-1)$ since $\zeta_A(q-1)_v = 0$ by [19]. As a consequence of Theorem 1.5, we have the following upper bound result for v-adic MZV's.

Corollary 1.8. *Let v be a finite place of k. For a positive integer w, we let* $\mathcal{Z}_{v,w}$ *be the* $\overline{k}$*-vector space space by v-adic MZV's of weight w defined in [13]. Then we have*

$$\dim_k \mathcal{Z}_{v,w} \leq d'_w - d'_{w-(q-1)},$$

where $d'_0 := 1$ *and* $d'_n := 0$ *for* $n < 0$.

1.3. Strategy of proofs

The key ingredient of our proof of Theorem 1.5 is to switch the study of MZV's to that of the special values of Carlitz multiple polylogarithms (abbreviated as CMPL's) $\mathrm{Li}_{\mathfrak{s}}$ for $\mathfrak{s} \in \mathbb{Z}_{>0}^r$ defined by the first named author in [9]. For the definition of $\mathrm{Li}_{\mathfrak{s}}$, see (2.3). Note that CMPL's are higher depth generalization of the Carlitz polylogarithms initiated by Anderson-Thakur [3]. Based on the interpolation formula of

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Anderson-Thakur [3, 4], one knows from [9] that $\zeta_A(\mathfrak{s})$ can be expressed as a k-linear combination of CMPL's at some integral points, and in the particular case for $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}}$, one has the simple identity (2.7) that

$$\zeta_A(\mathfrak{s}) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}),$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}_{>0}^{\mathrm{dep}(\mathfrak{s})}$.

Given a positive integer w, we let $\mathcal{I}_w^{\mathrm{ND}}$ be the set consisting of all tuples of positive integers $\mathfrak{s} = (s_1, \ldots, s_r)$ with $\mathrm{wt}(\mathfrak{s}) = w$ and $q \nmid s_i$ for all $1 \le i \le r$, and note that $\mathcal{I}_w^{\mathrm{ND}}$ was used and studied in [26]. The overall arguments in the proof of Theorem 1.5 are divided into the following two parts:

(I) We show in Corollary 3.7 that the two sets $\mathcal{B}_w^{\mathrm{Li,T}} := \left\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}}\right\}$ and $\mathcal{B}_w^T$ are generating sets of the k-vector space $\mathcal{Z}_w$ for every positive integer w (the latter one was known by Ngo Dac [26]) and further show in Theorem 3.8 that the set $\left\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\right\}$ is a generating set for $\mathcal{Z}_w$.

(II) We prove in Theorem 5.2 that $\left\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\right\}$ is a linearly independent set over k. Since $|\mathcal{I}_w^{\mathrm{T}}| = |\mathcal{I}_w^{\mathrm{ND}}|$ (see Proposition 2.1), it follows that $\mathcal{B}_w^T$ is a k-basis of $\mathcal{Z}_w$.

Regarding the part (I) above, we try to abstract and unify our methods to have a wide scope. We consider a formal k-space $\mathcal{H}$ generated by indices, on which the harmonic product $*^{\mathrm{Li}}$ and q-shuffle product $*^{\zeta}$ are defined. We also consider the power/truncated sums maps arising from $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ and $\zeta(\mathfrak{s})$. We then define the maps $\mathscr{L}^{\mathrm{Li}}, \mathscr{L}^{\zeta} : \mathcal{H} \to k_\infty$ as the CMPL and MZV-realizations. These realizations preserve the harmonic product and q-shuffle product, respectively, illustrating the stuffle relations [9] for CMPL's and the q-shuffle relations [34, 15] for MZV's in a formal framework. See Theorem 2.7.

Then we adopt the ideas and methods rooted in [30, 26] to achieve Theorem 3.6, which enables us to show the results mentioned in (I). Since some essential arguments are from those in [26], we leave the detailed proofs in the appendix. However, under such abstraction, the maps $\mathscr{U}^{\bullet}$ given in Definition 3.4 are explicit. In particular, Theorem 3.6 (2) allows us to provide a concrete and effective way to express $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ (resp. $\zeta_A(\mathfrak{s})$) as linear combinations in terms of $\mathcal{B}_w^{\mathrm{Li,T}}$ (resp. $\mathcal{B}_w^T$) simultaneously. This is more concrete and explicit than those developed in [26] and enables us to establish a simple mechanism in Theorem 5.3 that generates all the k-linear relations among MZV's of the same weight (cf. the $\mathscr{B}^*$-version of [30, Conjecture 5.1]).

Concerning the second part (II), the primary tool that we use is the Anderson-Brownawell-Papanikolas criterion [2, Thm. 3.1.1] (abbreviated as ABP-criterion), which has very strong applications in transcendence theory in positive characteristic. We prove (II) by induction on w. We start with a linear equation

$$\sum_{\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}} \alpha_{\mathfrak{s}}(\theta)\, \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) = 0, \quad \text{for } \alpha_{\mathfrak{s}} = \alpha_{\mathfrak{s}}(t) \in \mathbb{F}_q(t), \ \forall \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}} \tag{1.2}$$

and aim to show that $\alpha_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}$. We outline our arguments as below.

(II-1) Using the period interpretation [9, (3.4.5)] for special values of CMPL's (inspired by [4] for MZV's), we can construct a system of difference equations $\widetilde{\psi}^{(-1)} = \widetilde{\Phi}\widetilde{\psi}$ fitting into the conditions of ABP-criterion for which (1.2) can be expressed as a k-linear identity among the entries of $\widetilde{\psi}(\theta)$. We then apply the arguments in the proof of [14, Thm. 2.5.2], and eventually we obtain the big system of Frobenius difference equations (E_w), which is essentially from [14, (3.1.3)].

(II-2) Denote by $\mathcal{X}_w$ the $\mathbb{F}_q(t)$-vector space of the solution space of (E_w). We then show in Theorem 4.8 that $\dim_{\mathbb{F}_q(t)} \mathcal{X}_w$ is either 1 if $(q-1) \mid w$ or 0 if $(q-1) \nmid w$.

(II-3) Using the trick of [9, 14] together with the induction hypothesis, we establish Lemma 5.1. By the first part of Lemma 5.1, we see that if $\alpha_{\mathfrak{s}} \neq 0$, then $\mathfrak{s}$ must be in $\mathcal{I}_w^{\mathrm{ND}_0}$, which is given in (4.1). Then we apply Lemma 5.1 (2) to conclude that there exists $(\varepsilon_{\mathfrak{s}}) \in \mathcal{X}_w$ for which $\alpha_{\mathfrak{s}} = \varepsilon_{\mathfrak{s}}$ for all $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}_0}$. Note that the description of indices in $\mathcal{I}_w^{\mathrm{ND}_0}$ arises from the simultaneously Eulerian phenomenon in [14, Cor. 4.2.3], which was first witnessed by Lara Rodríguez and Thakur in [25].

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

(II-4) When $(q-1) \nmid w$, we use the fact of this case that $\mathscr{X}_w = 0$ to conclude that $\varepsilon_{\mathfrak{s}} = 0$ for every $\mathfrak{s}$, and hence $\alpha_{\mathfrak{s}}(\theta) = 0$ for every $\mathfrak{s}$. When $(q-1) \mid w$, we apply Theorem 3.8 together with the fact of this case that $\dim_{\mathbb{F}_q(t)} \mathscr{X}_w = 1$. Having one trick further in the proof of Theorem 5.2 shows that $\alpha_{\mathfrak{s}}(\theta) = 0$ for every $\mathfrak{s}$.

Remark 1.9. The above is the overall strategy of our proof. However, to save some length without affecting logical arguments, we avoid many details in (II-1). Instead, we directly go to (1.2) in this paper.

1.4. Organization of the paper

In Section 2.1, we introduce the abstract k-vector space $\mathcal{H}$, on which the harmonic product $*^{\mathrm{Li}}$ and q-shuffle product $*^{\zeta}$ are defined. The purpose of Section 2.1 is to derive the product formulae, stated as Proposition 2.7, for the CMPL and MZV realizations $\mathscr{L}^{\mathrm{Li}}$ and $\mathscr{L}^{\zeta}$. In Section 3, we follow [30, 26] to transfer their $\mathscr{B}, \mathscr{C}, \mathscr{BC}$ maps into our formal framework in a concrete way. The primary result of this section is to establish Theorem 3.8, which is an application of Theorem 3.6, whose detailed proof is given in the appendix.

We study the specific system of Frobenius difference equations (E_w) mentioned in the (II) above in Section 4. We carefully analyze the solution space $\mathscr{X}_w$ of (E_w), and determine its dimension in Theorem 4.8. In Section 5, we establish the key Lemma 5.1, which is used to prove the k-linear independence of $\left\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\right\}$ in Theorem 5.2. With these results at hand, we prove Theorem 1.5 in Section 5.3 as a short conclusion. Finally, combining Theorems 3.6 and 1.5, we give a proof of Theorem 5.3 in Section 5.4. As mentioned above, the appendix consists of a detailed proof of Theorem 3.6.

Remark 1.10. When this paper was nearly in its final version, we announced our results of Theorem 1.5 to Thakur, and soon after, Ngo Dac sent his paper [22] to Thakur on the same day announcing that he and his coauthors show the same results for alternating MZV's. Our paper was finished a few days later than theirs. The key strategy of their proofs is in the same direction as ours. They switch the study of alternating MZV's to the k-vector space spanned by the following special values of CMPL's:

$$\left\{\mathrm{Li}_{\mathfrak{s}}(\gamma_1, \ldots, \gamma_r) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}},\ (\gamma_1, \ldots, \gamma_r) \in (\mathbb{F}_{q^{q-1}}^{\times})^r\right\}.$$

Note that any value in the set above is equal to an alternating MZV at $\mathfrak{s}$ up to an algebraic multiple (cf. [12, Prop. 2.12]). However, by the first author's result [9, Thm. 5.4.3] showing that CMPL's at algebraic points form a $\overline{k}$-graded algebra defined over k, one can remove the algebraic factor without affecting the study for the k-vector space of alternating MZV's. We find that some directions of our ideas and theirs are similar and the primary methods rooted in [2, 9, 30, 14, 26] are the same, but presentations and detailed arguments are different. However, we prove Theorem 5.3 explicitly describing all k-linear relations among MZV's, but there is no such a result proved in [22].

2. Two products on $\mathcal{H}$

2.1. Indices

By an index, we mean the empty set $\emptyset$ or an r-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r)$. In the former case, its depth and weight are defined to be $\mathrm{dep}(\emptyset) = 0$ and $\mathrm{wt}(\emptyset) = 0$. The depth and weight of the latter case are defined to be $\mathrm{dep}(\mathfrak{s}) = r$ and $\mathrm{wt}(\mathfrak{s}) = \sum_{i=1}^r s_i$, respectively. We denote by $\mathcal{I} := \bigsqcup_{r \geq 0} \mathbb{Z}_{>0}^r$ the set of indices, where $\mathbb{Z}_{>0}^0 := \{\emptyset\}$.

Throughout this paper, we adapt the following notations.

$$\begin{aligned}
\mathcal{I}_w &:= \{\mathfrak{n} \in \mathcal{I} \mid \mathrm{wt}(\mathfrak{n}) = w\} \quad (w \geq 0),\\
\mathcal{I}_{>0} &:= \{\mathfrak{n} \in \mathcal{I} \mid \mathrm{wt}(\mathfrak{n}) > 0\} = \mathcal{I} \setminus \{\emptyset\},\\
\mathcal{I}_w^{\mathrm{T}} &:= \{(s_1, \ldots, s_r) \in \mathcal{I}_w \mid s_i \leq q\ (1 \leq \forall i \leq r-1) \text{ and } s_r < q\} \quad (w > 0),
\end{aligned}$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

$$\mathcal{I}_w^{\mathrm{ND}} := \{(s_1, \ldots, s_r) \in \mathcal{I}_w \mid q \nmid s_i \ (1 \le \forall i \le r)\} \ \ (w > 0),$$
$$\mathcal{I}_0^{\mathrm{T}} := \mathcal{I}_0^{\mathrm{ND}} := \{\emptyset\}.$$

We mention that $\mathcal{I}_w^{\mathrm{T}}$ refers to the set of indexes arising from Thakur's basis, and $\mathcal{I}_w^{\mathrm{ND}}$ is the set studied in [26]. For any subset S of $\mathcal{I}$, we denote by $|S|$ the cardinality of S when no confusions arise.

Proposition 2.1. *For each weight $w \ge 0$, we have a bijection between $\mathcal{I}_w^{\mathrm{ND}}$ and $\mathcal{I}_w^{\mathrm{T}}$, and hence, $|\mathcal{I}_w^{\mathrm{ND}}| = |\mathcal{I}_w^{\mathrm{T}}|$.*

Proof. The desired result follows from the following correspondence:

$$(m_1 q + n_1, \ldots, m_r q + n_r) \longleftrightarrow (q^{\{m_1\}}, n_1, \ldots, q^{\{m_r\}}, n_r) \quad (m_i \ge 0,\ 1 \le n_i \le q-1),$$

where $q^{\{m\}}$ denotes the sequence $(q, \ldots, q) \in \mathbb{Z}_{>0}^m$. It is understood that in the case of $m = 0$, $q^{\{0\}}$ is referred to the empty index $\emptyset$. □

Given ℓ indices $\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell \in \mathcal{I}$ with $\mathfrak{s}_i = (s_{i1}, \ldots, s_{ir_i})$ $(1 \le i \le \ell)$, we define

$$(\mathfrak{s}_1, \ldots \mathfrak{s}_\ell) := (s_{11}, \ldots, s_{1r_1}, s_{21}, \ldots, s_{2r_2}, \ldots, s_{\ell 1}, \ldots, s_{\ell r_\ell}) \tag{2.1}$$

to be the index obtained by putting the given indices consecutively. For each nonempty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0}$, we define

$$\mathfrak{s}_+ := (s_1, \ldots, s_{r-1}) \text{ and } \mathfrak{s}_- := (s_2, \ldots, s_r). \tag{2.2}$$

In the depth one case, we note that $\mathfrak{s}_+ = \emptyset$ and $\mathfrak{s}_- = \emptyset$.

Let $\mathcal{H} = \bigoplus_{w \ge 0} \mathcal{H}_w$ be the k-vector space with basis $\mathcal{I}$ graded by weight and let $\mathcal{H}_{>0} := \bigoplus_{w>0} \mathcal{H}_w$. For each index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, the corresponding generator in $\mathcal{H}$ is denoted by $[\mathfrak{s}] = [s_1, \ldots, s_r]$ or $\mathfrak{s}$. For each $P = \sum_{\mathfrak{s} \in \mathcal{I}} a_{\mathfrak{s}} [\mathfrak{s}] \in \mathcal{H}$, the support of P is defined by

$$\mathrm{Supp}(P) := \{\mathfrak{s} \in \mathcal{I} \mid a_{\mathfrak{s}} \ne 0\}.$$

Definition 2.2. Given a positive integer ℓ, we define the following k-multilinear map

$$[-, -, \ldots, -] : \mathcal{H}^{\oplus \ell} \to \mathcal{H}$$

as follows. For any indices $\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell \in \mathcal{I}$, let $(\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell)$ be given in (2.1). We define

$$[\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell] := [(\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell)] \in \mathcal{H}.$$

Remark 2.3. As the map above is multilinear, for any $\mathfrak{s} \in \mathcal{I}$, we particularly have the following identity

$$[\mathfrak{s}, 0] = 0 \in \mathcal{H},$$

which will be used in the appendix. As $\emptyset$ is also an index by our definition, we mention that

$$[\mathfrak{s}, \emptyset] \ne [\mathfrak{s}, 0] = 0 \in \mathcal{H}.$$

2.2. The maps $\mathscr{L}^{\mathrm{Li}}$ and $\mathscr{L}^{\zeta}$

Recall that the Carlitz logarithm is given by

$$\log_C(z) := \sum_{d \ge 0} \frac{z^{q^d}}{L_d} \in k[[z]],$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

where $L_0 := 1$ and $L_d := (\theta - \theta^q) \cdots (\theta - \theta^{q^d})$ for $d \geq 1$ (see [20, 31]), and the nth Carlitz polylogarithm defined by Anderson-Thakur [3] is the following power series:

$$\mathrm{Li}_n(z) := \sum_{d \geq 0} \frac{z^{q^d}}{L_d^n} \in k[[z]].$$

For any index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0}$ of positive depth, the $\mathfrak{s}$th Carlitz multiple polylogarithm (abbreviated as CMPL) is given by the following series (see [9]):

$$\mathrm{Li}_{\mathfrak{s}}(z_1, \ldots, z_r) := \sum_{d_1 > \cdots > d_r \geq 0} \frac{z_1^{q^{d_1}} \cdots z_r^{q^{d_r}}}{L_{d_1}^{s_1} \cdots L_{d_r}^{s_r}} \in k[[z_1, \ldots, z_r]]. \tag{2.3}$$

For each $s \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}_{\geq 0}$, we set

$$S_d^{\mathrm{Li}}(s) := \frac{1}{L_d^s} \in k \quad \text{and} \quad S_d^{\zeta}(s) := \sum_{a \in A_{+,d}} \frac{1}{a^d} \in k,$$

where $A_{+,d}$ is the set of monic polynomials in A of degree d. We then define k-linear maps $\mathscr{L}_d^\bullet$, $\mathscr{L}_{<d}^\bullet$ and $\mathscr{L}^\bullet$ on $\mathcal{H}$.

Definition 2.4. Let $\bullet \in \{\mathrm{Li}, \zeta\}$ and $d \in \mathbb{Z}$.

(1) The map $\mathscr{L}_d^\bullet \colon \mathcal{H} \to k$ is the k-linear map defined by

$$\mathscr{L}_d^\bullet(\mathfrak{s}) := \begin{cases} \displaystyle\sum_{d = d_1 > \cdots > d_r \geq 0} S_{d_1}^\bullet(s_1) \cdots S_{d_r}^\bullet(s_r) & (\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0} \text{ and } d \geq \mathrm{dep}(\mathfrak{s}) - 1) \\ 1 & (\mathfrak{s} = \emptyset \text{ and } d = -1) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.4}$$

(2) The map $\mathscr{L}_{<d}^\bullet \colon \mathcal{H} \to k$ is the k-linear map defined by

$$\mathscr{L}_{<d}^\bullet(\mathfrak{s}) := \begin{cases} \displaystyle\sum_{d > d_1 > \cdots > d_r \geq 0} S_{d_1}^\bullet(s_1) \cdots S_{d_r}^\bullet(s_r) & (\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0} \text{ and } d \geq \mathrm{dep}(\mathfrak{s})) \\ 1 & (\mathfrak{s} = \emptyset \text{ and } d \geq 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.5}$$

(3) The map $\mathscr{L}^\bullet \colon \mathcal{H} \to k_\infty$ is the k-linear map defined by

$$\mathscr{L}^\bullet(\mathfrak{s}) := \begin{cases} \displaystyle\sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}^\bullet(s_1) \cdots S_{d_r}^\bullet(s_r) & (\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0}) \\ 1 & (\mathfrak{s} = \emptyset). \end{cases} \tag{2.6}$$

It follows that for each $P \in \mathcal{H}$, $\mathfrak{s} \in \mathcal{I}$, $s \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}$, we have

$$\begin{aligned} \mathscr{L}_d^\bullet(P) &= \mathscr{L}_{<d+1}^\bullet(P) - \mathscr{L}_{<d}^\bullet(P), \\ \mathscr{L}_{<d}^\bullet(P) &= \sum_{d' < d} \mathscr{L}_{d'}^\bullet(P), \\ \mathscr{L}^\bullet(P) &= \sum_{d \in \mathbb{Z}} \mathscr{L}_d^\bullet(P) = \lim_{d \to \infty} \mathscr{L}_{<d}^\bullet(P) \in k_\infty, \\ \mathscr{L}_d^\bullet(s) \mathscr{L}_{<d}^\bullet(P) &= \mathscr{L}_d^\bullet([s, P]), \\ \mathscr{L}^{\mathrm{Li}}(\mathfrak{s}) &= \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}), \\ \mathscr{L}^{\zeta}(\mathfrak{s}) &= \zeta_A(\mathfrak{s}), \end{aligned}$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

where $\mathbf{1}$ is simply referred to $(1,\dots,1)\in\mathbb{Z}_{>0}^{\mathrm{dep}(\mathfrak{s})}$ when it is clear from the context without confusion, and we set $\mathrm{Li}_{\emptyset}(\mathbf{1}):=\zeta_A(\emptyset):=1$.

Remark 2.5. For $1\le s\le q$ and $d\ge 0$, we have the following equality due to Carlitz (see also [31, Thm. 5.9.1]):

$$\frac{1}{L_d^s}=\sum_{a\in A_{+,d}}\frac{1}{a^s}.$$

Therefore, for each $\mathfrak{s}=(s_1,\dots,s_r)$ with $1\le s_i\le q$ and $d\in\mathbb{Z}$, we have

$$\mathscr{L}_d^{\mathrm{Li}}(\mathfrak{s})=\mathscr{L}_d^{\zeta}(\mathfrak{s}),\quad \mathscr{L}_{<d}^{\mathrm{Li}}(\mathfrak{s})=\mathscr{L}_{<d}^{\zeta}(\mathfrak{s}),$$

and

$$\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})=\mathscr{L}^{\mathrm{Li}}(\mathfrak{s})=\mathscr{L}^{\zeta}(\mathfrak{s})=\zeta_A(\mathfrak{s}). \tag{2.7}$$

2.3. Product formulae

The harmonic product on $\mathcal{H}$ is denoted by $*$ or $*^{\mathrm{Li}}$. Thus, $*$ is a k-bilinear map $\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ such that

$$\begin{aligned}&[\emptyset]*P=P*[\emptyset]=P,\\&[\mathfrak{s}]*[\mathfrak{n}]=[s_1,\mathfrak{s}_-*\mathfrak{n}]+[n_1,\mathfrak{s}*\mathfrak{n}_-]+[s_1+n_1,\mathfrak{s}_-*\mathfrak{n}_-]\end{aligned}$$

for each $P\in\mathcal{H}$, $\mathfrak{s}=(s_1,\mathfrak{s}_-),\mathfrak{n}=(n_1,\mathfrak{n}_-)\in\mathcal{I}_{>0}$. For each $\mathfrak{s},\mathfrak{n}\in\mathcal{I}_{>0}$, we put

$$D_{\mathfrak{s},\mathfrak{n}}^{\mathrm{Li}}:=0\in\mathcal{H}. \tag{2.8}$$

For each $s,n\ge 1$, H.-J. Chen showed in [15] that

$$\mathscr{L}_d^{\zeta}(s)\mathscr{L}_d^{\zeta}(n)=\mathscr{L}_d^{\zeta}(s+n)+\sum_{j=1}^{s+n-1}\Delta_{s,n}^{[j]}\mathscr{L}_d^{\zeta}(s+n-j,j), \tag{2.9}$$

where we set

$$\Delta_{s,n}^{[j]}:=\begin{cases}(-1)^{s-1}\binom{j-1}{s-1}+(-1)^{n-1}\binom{j-1}{n-1} & \text{if } (q-1)\mid j \text{ and } 1\le j<s+n\\ 0 & \text{otherwise}\end{cases}.$$

The q-shuffle product on $\mathcal{H}$ is denoted by $*^{\zeta}$. Thus, $*^{\zeta}$ is a k-bilinear map $\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ such that

$$\begin{aligned}&[\emptyset]*^{\zeta}P=P*^{\zeta}[\emptyset]=P,\\&[\mathfrak{s}]*^{\zeta}[\mathfrak{n}]=[s_1,\mathfrak{s}_-*^{\zeta}\mathfrak{n}]+[n_1,\mathfrak{s}*^{\zeta}\mathfrak{n}_-]+[s_1+n_1,\mathfrak{s}_-*^{\zeta}\mathfrak{n}_-]+D_{\mathfrak{s},\mathfrak{n}}^{\zeta}\end{aligned}$$

for each $P\in\mathcal{H}$, $\mathfrak{s}=(s_1,\mathfrak{s}_-),\mathfrak{n}=(n_1,\mathfrak{n}_-)\in\mathcal{I}_{>0}$, where we set

$$D_{\mathfrak{s},\mathfrak{n}}^{\zeta}:=\sum_{j=1}^{s_1+n_1-1}\Delta_{s_1,n_1}^{[j]}[s_1+n_1-j,(j)*^{\zeta}(\mathfrak{s}_-*^{\zeta}\mathfrak{n}_-)] \tag{2.10}$$

(see Yamamoto's definition in [35, p. 1006]). By the induction on $w+w'$, we can show that $P*^{\bullet}Q\in\mathcal{H}_{w+w'}$ for each $w,w'\in\mathbb{Z}_{\ge 0}$, $P\in\mathcal{H}_w$, $Q\in\mathcal{H}_{w'}$ and $\bullet\in\{\mathrm{Li},\zeta\}$. In particular, when $w,w'\in\mathbb{Z}_{\ge 1}$, we have $D_{\mathfrak{s},\mathfrak{n}}^{\zeta}\in\mathcal{H}_{w+w'}$ for each $\mathfrak{s}\in\mathcal{I}_w$ and $\mathfrak{n}\in\mathcal{I}_{w'}$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Remark 2.6. When $s+n \le q$, we observe that $\Delta_{s,n}^{[j]} = 0$ for all j. Thus, for $\mathfrak{s} = (s_1, \ldots), \mathfrak{n} = (n_1, \ldots) \in \mathcal{I}_{>0}$, if $s_1 + n_1 \le q$, then we have $D_{\mathfrak{s},\mathfrak{n}}^{\zeta} = 0$.

The following product formulae are crucial when proving Theorem 3.6, whose detailed proof is given in the appendix.

Proposition 2.7. *Let $\bullet \in \{\mathrm{Li}, \zeta\}$. Given any $P, Q \in \mathcal{H}$ and $\mathfrak{s} = (s_1, \mathfrak{s}_-), \mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_{>0}$, we have the following identities for each $d \in \mathbb{Z}$.*

(1) $\mathscr{L}^{\bullet}(P)\mathscr{L}^{\bullet}(Q) = \mathscr{L}^{\bullet}(P *^{\bullet} Q)$.
(2) $\mathscr{L}^{\bullet}_{<d}(P)\mathscr{L}^{\bullet}_{<d}(Q) = \mathscr{L}^{\bullet}_{<d}(P *^{\bullet} Q)$.
(3)

$$\begin{aligned}\mathscr{L}^{\bullet}_{d}(\mathfrak{s})\mathscr{L}^{\bullet}_{d}(\mathfrak{n}) &= \mathscr{L}^{\bullet}_{d}([s_1+n_1, \mathfrak{s}_- *^{\bullet} \mathfrak{n}_-]) + \mathscr{L}^{\bullet}_{d}(D^{\bullet}_{\mathfrak{s},\mathfrak{n}}) \\ &= \mathscr{L}^{\bullet}_{d}(\mathfrak{s} *^{\bullet} \mathfrak{n}) - \mathscr{L}^{\bullet}_{d}([s_1, \mathfrak{s}_- *^{\bullet} \mathfrak{n}]) - \mathscr{L}^{\bullet}_{d}([n_1, \mathfrak{s} *^{\bullet} \mathfrak{n}_-]).\end{aligned}$$

Proof. We first mention that (3) for d follows from the second one for the same d. Indeed, for each $\mathfrak{s} = (s_1, \ldots), \mathfrak{n} = (n_1, \ldots) \in \mathcal{I}_{>0}$, we have

$$\begin{aligned}\mathscr{L}^{\mathrm{Li}}_{d}(\mathfrak{s})\mathscr{L}^{\mathrm{Li}}_{d}(\mathfrak{n}) &= \mathscr{L}^{\mathrm{Li}}_{d}(s_1)\mathscr{L}^{\mathrm{Li}}_{d}(n_1)\mathscr{L}^{\mathrm{Li}}_{<d}(\mathfrak{s}_-)\mathscr{L}^{\mathrm{Li}}_{<d}(\mathfrak{n}_-) \\ &= \mathscr{L}^{\mathrm{Li}}_{d}(s_1+n_1)\mathscr{L}^{\mathrm{Li}}_{<d}(\mathfrak{s}_- *^{\mathrm{Li}} \mathfrak{n}_-) \\ &= \mathscr{L}^{\mathrm{Li}}_{d}([s_1+n_1, \mathfrak{s}_- *^{\mathrm{Li}} \mathfrak{n}_-]) + \mathscr{L}^{\mathrm{Li}}_{d}(D^{\mathrm{Li}}_{\mathfrak{s},\mathfrak{n}}),\end{aligned}$$

where the second equality comes from (2) and the definition of $\mathscr{L}^{\mathrm{Li}}_{d}$, and the third identity comes from (2.8). Similarly, we have

$$\begin{aligned}\mathscr{L}^{\zeta}_{d}(\mathfrak{s})\mathscr{L}^{\zeta}_{d}(\mathfrak{n}) &= \mathscr{L}^{\zeta}_{d}(s_1)\mathscr{L}^{\zeta}_{d}(n_1)\mathscr{L}^{\zeta}_{<d}(\mathfrak{s}_-)\mathscr{L}^{\zeta}_{<d}(\mathfrak{n}_-) \\ &= \left(\mathscr{L}^{\zeta}_{d}(s_1+n_1) + \sum_{j} \Delta^{j}_{s_1,n_1}\mathscr{L}^{\zeta}_{d}(s_1+n_1-j)\mathscr{L}^{\zeta}_{<d}(j)\right)\mathscr{L}^{\zeta}_{<d}(\mathfrak{s}_- *^{\zeta} \mathfrak{n}_-) \\ &= \mathscr{L}^{\zeta}_{d}([s_1+n_1, \mathfrak{s}_- *^{\zeta} \mathfrak{n}_-]) + \sum_{j} \Delta^{[j]}_{s_1,n_1}\mathscr{L}^{\zeta}_{d}(s_1+n_1-j)\mathscr{L}^{\zeta}_{<d}((j) *^{\zeta} (\mathfrak{s}_- *^{\zeta} \mathfrak{n}_-)) \\ &= \mathscr{L}^{\zeta}_{d}([s_1+n_1, \mathfrak{s}_- *^{\zeta} \mathfrak{n}_-]) + \mathscr{L}^{\zeta}_{d}(D^{\zeta}_{\mathfrak{s},\mathfrak{n}}),\end{aligned}$$

where the second equality comes from (2.9) and (2), the third identity comes from (2) and the fourth one comes from (2.10).

To prove the formula (2), we first mention that when $d < 0$, the formula holds as both sides of the identity are zero. We then prove the formula (2) by induction on d. Now, let d be any non-negative integer. By bilinearity, we may assume that $P = \mathfrak{s}, Q = \mathfrak{n} \in \mathcal{I}_{>0}$. Then we have

$$\begin{aligned}&\mathscr{L}^{\bullet}_{<d}(\mathfrak{s})\mathscr{L}^{\bullet}_{<d}(\mathfrak{n}) \\ &= \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}(s_1)\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{s}_-)\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{n}) + \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}(n_1)\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{s})\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{n}_-) + \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}(\mathfrak{s})\mathscr{L}^{\bullet}_{d_1}(\mathfrak{n}) \\ &= \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}(s_1)\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{s}_- *^{\bullet} \mathfrak{n}) + \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}(n_1)\mathscr{L}^{\bullet}_{<d_1}(\mathfrak{s} *^{\bullet} \mathfrak{n}_-) \\ &\quad + \sum_{d_1<d} \mathscr{L}^{\bullet}_{d_1}([s_1+n_1, \mathfrak{s}_- *^{\bullet} \mathfrak{n}_-] + D^{\bullet}_{\mathfrak{s},\mathfrak{n}}) \\ &= \mathscr{L}^{\bullet}_{<d}(\mathfrak{s} *^{\bullet} \mathfrak{n}),\end{aligned}$$

where the second equality comes from the induction hypothesis as well as (3) (as (2) holds for $d_1 < d$ by induction hypothesis). Finally, the formula (1) follows from (2) by taking the limit $d \to \infty$. □

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

3. Generators

The purpose of this section is to establish Theorem 3.6, which allows us to obtain the desired generating sets for $\mathcal{Z}_w$. To achieve it, we need to set up the box-plus operator on $\mathcal{H}^{\oplus 2}$ as well as the k-linear maps $\mathcal{U}^\bullet$ on $\mathcal{H}$.

3.1. The box-plus operator

Definition 3.1. Let $\boxplus\colon \mathcal{H}^{\oplus 2} \to \mathcal{H}$ be the k-linear map defined by

$$\emptyset \boxplus P = P \boxplus \emptyset := 0 \ \text{ and } \ \mathfrak{s} \boxplus \mathfrak{n} := (\mathfrak{s}_+, s_r + n_1, \mathfrak{n}_-)$$

for $P \in \mathcal{H}$ and $\mathfrak{s} = (\mathfrak{s}_+, s_r), \mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_{>0}$.

Remark 3.2. To avoid confusion, we mention that for each $n \geq 1$ and $P = \sum_{\mathfrak{s}\in\mathcal{I}} a_{\mathfrak{s}}[\mathfrak{s}] \in \mathcal{H}$,

$$\begin{aligned}
[n, P] &= \sum_{\mathfrak{s}=(s_1,\ldots,s_r)\in\mathcal{I}} a_{\mathfrak{s}}[n, s_1, \ldots, s_r] \in \mathcal{H}_{>0}, \\
(n) \boxplus P &= \sum_{\mathfrak{s}=(s_1,\ldots,s_r)\in\mathcal{I}_{>0}} a_{\mathfrak{s}}[n + s_1, s_2, \ldots, s_r] \in \mathcal{H}_{>0} \quad (\neq [n] + P).
\end{aligned}$$

Furthermore, in the depth one case for $\mathfrak{s} = (s)$ and $\mathfrak{n} = (n)$, we have $\mathfrak{s}_+ = \emptyset = \mathfrak{n}_-$, and hence in this case,

$$\mathfrak{s} \boxplus \mathfrak{n} = [s + n].$$

3.2. Generating sets

We define a k-linear endomorphism $\mathcal{U}^\bullet\colon \mathcal{H} \to \mathcal{H}$ as follows. Let $\mathfrak{s} \in \mathcal{I}$. We write

$$\mathfrak{s} = (s_1, \ldots) = (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, \mathfrak{s}') \tag{3.1}$$

with $\mathfrak{s}^{\mathrm{T}} \in \mathcal{I}^{\mathrm{T}}$, $m \geq 0$ and $\mathfrak{s}' = (s'_1, \ldots)$ with $s'_1 > q$ or $\mathfrak{s}' = \emptyset$. When $\mathfrak{s}' \neq \emptyset$, we set

$$\mathfrak{s}'' := (s'_1 - q, \mathfrak{s}'_-). \tag{3.2}$$

Example 3.3. Let $\mathfrak{s} = (q-1, q, q, q+1, 1)$. Then $\mathfrak{s}^{\mathrm{T}} = (q-1)$, $m = 2$ and $\mathfrak{s}' = (q+1, 1)$. Since $\mathfrak{s}' \neq \emptyset$, we also have $\mathfrak{s}'' = (1, 1)$.

We set

$$\alpha_q^\bullet(P) := [1, (q-1) *^\bullet P] \tag{3.3}$$

for each $P \in \mathcal{H}$. For $m = 0$, $\alpha_q^{\bullet,0}$ is defined to be the identity map on $\mathcal{H}$, and for $m \in \mathbb{Z}_{>0}$, $\alpha_q^{\bullet,m}$ is defined to be the mth iteration of $\alpha_q^\bullet$.

Definition 3.4. For $\bullet \in \{\mathrm{Li}, \zeta\}$, we define the k-linear map $\mathcal{U}^\bullet : \mathcal{H} \to \mathcal{H}$ given by

$$\mathcal{U}^\bullet(\mathfrak{s}) := \begin{cases} -[\mathfrak{s}^{\mathrm{T}}, q^{\{m+1\}}, \mathfrak{s}''] + L_1^{m+1}[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m+1}(\mathfrak{s}'')] \\ \qquad\qquad + L_1^{m+1}(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m+1}(\mathfrak{s}'')) - [\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, D^\bullet_{q,\mathfrak{s}''}] & (\mathfrak{s}' \neq \emptyset) \\ L_1^m[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m}(\emptyset)] + L_1^m(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m}(\emptyset)) & (\mathfrak{s}' = \emptyset) \end{cases}$$

Remark 3.5. From the definition, one sees that $\mathcal{U}^\bullet(\mathcal{H}_w) \subset \mathcal{H}_w$. We also mention that when $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}$, then $\mathcal{U}^\bullet(\mathfrak{s}) = \mathfrak{s}$ since in this case, we have $m = 0$ and $\mathfrak{s}' = \emptyset$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Theorem 3.6. *Let* $\bullet \in \{\mathrm{Li}, \zeta\}$. *For each* $P \in \mathcal{H}$, *the following statements hold.*

(1) $\mathscr{L}^\bullet(\mathscr{U}^\bullet(P)) = \mathscr{L}^\bullet(P)$.

(2) *There exists an explicit integer* $e \geq 0$ *such that* $\mathrm{Supp}((\mathscr{U}^\bullet)^e(P)) \subset \mathcal{I}^{\mathrm{T}}$, *where* $(\mathscr{U}^\bullet)^0$ *is defined to be the identity map and for* $e \in \mathbb{Z}_{>0}$, $(\mathscr{U}^\bullet)^e$ *is defined to be the eth iteration of* $\mathscr{U}^\bullet$.

Note that our $\mathscr{U}^\bullet$ is concrete and explicit and simultaneously deals with the two products $*^{\mathrm{Li}}$ and $*^{\zeta}$. Our strategy for proving Theorem 3.6 arises from [26], and so we leave the detailed proof to the appendix. However, we give an outline of the proof of Theorem 3.6 (1) as follows. Let $\bullet \in \{\mathrm{Li}, \zeta\}$.

(I) We first define the space of *binary relations* $\mathcal{P}^\bullet \subset \mathcal{H}^{\oplus 2}$ in (A.1) and note that for $(P, Q) \in \mathcal{P}^\bullet$, we have

$$\mathscr{L}^\bullet(P, Q) := \mathscr{L}^\bullet(P) + \mathscr{L}^\bullet(Q) = 0.$$

(II) For any $\mathfrak{s} \in \mathcal{I}_{>0}$ and integer $m \geq 0$, we define in Sec. A.3 the following maps

$$\mathscr{B}^\bullet_{\mathfrak{s}}, \mathscr{C}^\bullet_{\mathfrak{s}}, \mathscr{BC}^{\bullet,m}_q : \mathcal{H}^{\oplus 2}_{>0} \to \mathcal{H}^{\oplus 2}_{>0},$$

and we further show in Proposition A.4 that they preserve $\mathcal{P}^\bullet$.

(III) In order to give a clear outline, we drop the subscripts to avoid heavy notation as all details are given in the proof of Theorem A.5. We start with $R_1 \in \mathcal{P}^\bullet_q$, and consider $\mathscr{D}^\bullet(R_1)$ for choosing a suitable $\mathscr{D}^\bullet$ arising from one of $\mathscr{B}^\bullet, \mathscr{C}^\bullet, \mathscr{BC}^\bullet$ as well as their composites (see (A.4), (A.5), (A.6) and (A.7)). Based on (II), we have $\mathscr{D}^\bullet(R_1) \in \mathcal{P}^\bullet$ and hence $\mathscr{L}^\bullet(\mathscr{D}^\bullet(R_1)) = 0$. In this case, the expansion of $\mathscr{L}^\bullet(\mathscr{D}^\bullet(R_1))$ is equal to $\mathscr{L}^\bullet(\mathfrak{s}) - \mathscr{L}^\bullet(\mathscr{U}^\bullet(\mathfrak{s}))$, and hence $\mathscr{L}^\bullet(\mathfrak{s}) = \mathscr{L}^\bullet(\mathscr{U}^\bullet(\mathfrak{s}))$ as desired.

As a consequence of Theorem 3.6, we obtain the following important equality.

Corollary 3.7. *Let* $\bullet \in \{\mathrm{Li}, \zeta\}$. *For any* $w \geq 0$, *we let* $\mathcal{Z}^\bullet_w$ *be the k-vector space spanned by the elements* $\mathscr{L}^\bullet(\mathfrak{s}) \in k_\infty$ *for* $\mathfrak{s} \in \mathcal{I}_w$. *Then*

$$\left\{\mathscr{L}^\bullet(\mathfrak{s}) | \mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w\right\}$$

is a generating set for $\mathcal{Z}^\bullet_w$. *In particular, we have*

$$\mathcal{Z}^{\mathrm{Li}}_w = \mathcal{Z}^{\zeta}_w \ (= \mathcal{Z}_w).$$

Proof. The first assertion comes from Theorem 3.6. We note that for $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w$, we have

$$\mathscr{L}^{\mathrm{Li}}(\mathfrak{s}) := \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) = \zeta_A(\mathfrak{s}) =: \mathscr{L}^{\zeta}(\mathfrak{s}),$$

whence obtaining $\mathcal{Z}^{\mathrm{Li}}_w = \mathcal{Z}^{\zeta}_w$. □

Theorem 3.8. *The k-vector space* $\mathcal{Z}_w$ *is spanned by the elements* $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ *for* $\mathfrak{s} \in \mathcal{I}^{\mathrm{ND}}_w$.

Proof. We set $d'_w := |\mathcal{I}^{\mathrm{T}}_w| = |\mathcal{I}^{\mathrm{ND}}_w|$. By the definition of $\mathscr{U}^{\mathrm{Li}}$ and Theorem 3.6 (2), there exists $e \gg 0$ such that $(\mathscr{U}^{\mathrm{Li}})^e(\mathcal{H}_{w/\mathbb{F}_p[L_1]}) \subset \mathcal{H}^{\mathrm{T}}_{w/\mathbb{F}_p[L_1]}$, where $\mathcal{H}_{w/\mathbb{F}_p[L_1]}$ (resp. $\mathcal{H}^{\mathrm{T}}_{w/\mathbb{F}_p[L_1]}$) is the $\mathbb{F}_p[L_1]$-module spanned by $\mathfrak{s} \in \mathcal{I}_w$ (resp. $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w$) in $\mathcal{H}_w$. We note that according to Remark 3.5, $(\mathscr{U}^{\mathrm{Li}})^e$ is independent of e for $e \gg 0$.

To show the desired result, it suffices to prove that the determinant of the matrix

$$U = (u_{\mathfrak{s},\mathfrak{n}})_{\mathfrak{s} \in \mathcal{I}^{\mathrm{ND}}_w, \mathfrak{n} \in \mathcal{I}^{\mathrm{T}}_w} \in \mathrm{Mat}_{d_w}(\mathbb{F}_p[L_1])$$

arising from

$$(\mathscr{U}^{\mathrm{Li}})^e(\mathfrak{s}) = \sum_{\mathfrak{n} \in \mathcal{I}^{\mathrm{T}}_w} u_{\mathfrak{s},\mathfrak{n}}[\mathfrak{n}] \quad (\mathfrak{s} \in \mathcal{I}^{\mathrm{ND}}_w)$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

is non-zero. We mention that $\mathcal{I}_w^{\mathrm{ND}}$ is equal to $\mathcal{I}_w^{\mathrm{T}}$ when $w \leq q$, and in this case, the result is valid. So in what follows, we assume that $\mathcal{I}_w^{\mathrm{ND}} \neq \mathcal{I}_w^{\mathrm{T}}$.

Taking any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_w^{\mathrm{ND}}$ but $\mathfrak{s} \notin \mathcal{I}_w^{\mathrm{T}}$, based on (3.1) and the definition of $\mathcal{I}_w^{\mathrm{ND}}$, we write $\mathfrak{s} = (\mathfrak{s}^{\mathrm{T}}, q^{\{0\}}, \mathfrak{s}')$ with $\mathfrak{s}^{\mathrm{T}} \in \mathcal{I}^{\mathrm{T}}$ ($q^{\{0\}} := \emptyset$), and $\mathfrak{s}' \neq \emptyset$. Note that $\mathfrak{s}'$ is of the form: $\mathfrak{s}' = (s_1', \ldots)$ with $s_1' > q$. By definition (2.8), $D^{\mathrm{Li}}_{q,\mathfrak{s}''} = 0$. Thus,

$$\mathcal{U}^{\mathrm{Li}}(\mathfrak{s}) = -[\mathfrak{s}^{\mathrm{T}}, q, \mathfrak{s}''] + L_1[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\mathrm{Li}}(\mathfrak{s}'')] + L_1(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\mathrm{Li}}(\mathfrak{s}'')), \tag{3.4}$$

where $\mathfrak{s}'' := (s_1' - q, \mathfrak{s}_-')$. Thus, for each $\mathfrak{s} = (m_1 q + n_1, \ldots, m_r q + n_r) \in \mathcal{I}_w^{\mathrm{ND}}$ with $m_i \geq 0$ (m_i are not all zero since $\mathfrak{s} \notin \mathcal{I}_w^{\mathrm{T}}$) and $1 \leq n_i \leq q-1$, we have

$$\begin{aligned}(\mathcal{U}^{\mathrm{Li}})^e(\mathfrak{s}) &= (\mathcal{U}^{\mathrm{Li}})^{e+m_1+\cdots+m_r}(\mathfrak{s}) \\ &\in (\mathcal{U}^{\mathrm{Li}})^e\left((-1)^{m_1+\cdots+m_r}[q^{\{m_1\}}, n_1, \ldots, q^{\{m_r\}}, n_r] + L_1 \cdot \mathcal{H}_{w/\mathbb{F}_p[L_1]}\right) \\ &\subset (-1)^{m_1+\cdots+m_r}[q^{\{m_1\}}, n_1, \ldots, q^{\{m_r\}}, n_r] + L_1 \cdot \mathcal{H}^{\mathrm{T}}_{w/\mathbb{F}_p[L_1]}.\end{aligned}$$

Note that the first identity above comes from the fact that $(\mathcal{U}^{\mathrm{Li}})^e(\mathfrak{s}) \in \mathcal{H}^{\mathrm{T}}_{w/\mathbb{F}_p[L_1]}$, which is fixed by $(\mathcal{U}^{\mathrm{Li}})^{m_1+\cdots+m_r}$ by Remark 3.5. The second belonging follows from (3.4), whence we have the last inclusion because of $(q^{\{m_1\}}, n_1, \ldots, q^{\{m_r\}}, n_r) \in \mathcal{I}_w^{\mathrm{T}}$ and $(\mathcal{U}^{\mathrm{Li}})^e(\mathcal{H}_{w/\mathbb{F}_p[L_1]}) \subset \mathcal{H}^{\mathrm{T}}_{w/\mathbb{F}_p[L_1]}$. Since

$$(m_1 q + n_1, \ldots, m_r q + n_r) \mapsto (q^{\{m_1\}}, n_1, \ldots, q^{\{m_r\}}, n_r)$$

gives a bijection between $\mathcal{I}_w^{\mathrm{ND}}$ and $\mathcal{I}_w^{\mathrm{T}}$, we have $\det(U \bmod L_1) = \pm 1$ in $\mathbb{F}_p$. In particular, $\det(U) \neq 0$. □

4. Frobenius difference equations

In this section, we focus on a specific system (E_M) of Frobenius difference equations arising from our study of MZV's, and the major result is Theorem 4.8 giving the precise dimension of the solution space of (E_M).

4.1. ABP-criterion

Let t be a new variable and $\mathbb{C}_\infty((t))$ be the field of Laurent series in t over $\mathbb{C}_\infty$. For any integer n, we define the following *n-fold Frobenious twisting*:

$$\tau^n := \left(f \mapsto f^{(n)}\right) : \mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t)),$$

where $f^{(n)} := \sum a_i^{q^n} t^i$ for $f = \sum a_i t^i \in \mathbb{C}_\infty((t))$. We further extend τ^n to the action on $\mathrm{Mat}_{r\times s}(\mathbb{C}_\infty((t)))$ by entrywise action.

Note that as an automorphism of the field $\mathbb{C}_\infty((t))$, τ^n stabilizes several subrings such as the power series ring $\mathbb{C}_\infty[[t]]$, the Tate algebra

$$\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]] \,|\, f \text{ converges on } |t|_\infty \leq 1\}$$

and the polynomial rings $\mathbb{C}_\infty[t], \overline{k}[t]$ as well as the subfields $\mathbb{C}_\infty(t)$ and $\overline{k}(t)$. When $n = -1$, we denote by $\sigma := \tau^{-1}$. For $n \in \{1, -1\}$, the following fixed elements are particularly used in this paper:

$$\mathbb{T}^\tau = \overline{k}[t]^\tau = \mathbb{F}_q[t] = \mathbb{T}^\sigma = \overline{k}[t]^\sigma \text{ and } \overline{k}(t)^\tau = \mathbb{F}_q(t) = \overline{k}(t)^\sigma.$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Following [2], we denote by $\mathcal{E} \subset \mathbb{C}_\infty[[t]]$ the subring consisting of power series $f = \sum_{i=0}^{\infty} a_i t^i$ with algebraic coefficients ($a_i \in \overline{k}$, $\forall i \in \mathbb{Z}_{\geq 0}$) for which

$$\lim_{i\to\infty} \sqrt[i]{|a_i|_\infty} = 0 \text{ and } [k_\infty(a_0, a_1, \ldots) : k_\infty] < \infty.$$

It follows that such any power series in $\mathcal{E}$ converges on whole $\mathbb{C}_\infty$, and we have

$$\mathcal{E}^\tau = \mathbb{F}_q[t] = \mathcal{E}^\sigma.$$

The primary tool that we use in this paper to show linear independence results is the following ABP-criterion

Theorem 4.1 [2, Theorem 3.1.1]. *Let $\Phi \in \mathrm{Mat}_\ell(\overline{k}[t])$ be a matrix satisfying that as a polynomial in $\overline{k}[t]$, $\det \Phi$ vanishes only (if at all) at $t = \theta$. Suppose that a column vector $\psi = \psi(t) \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ satisfies the following difference equation*

$$\psi^{(-1)} = \Phi\psi.$$

We denote by $\psi(\theta) \in \mathrm{Mat}_{\ell\times 1}(\mathbb{C}_\infty)$ the evaluation of ψ at $t = \theta$. Then for any row vector $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{k})$ for which

$$\rho\psi(\theta) = 0,$$

there exists a row vector $\mathrm{P} \in \mathrm{Mat}_{1\times\ell}(\overline{k}[t])$ such that

$$\mathrm{P}\psi = 0, \text{ and } \mathrm{P}(\theta) = \rho.$$

The spirit of ABP-criterion asserts any $\overline{k}$-linear relation among the entries of $\psi(\theta)$ can be lifted to a $\overline{k}[t]$-linear relation among the entries of ψ. We mention that the condition on Φ in the theorem above arises from *dual t-motives* and refer the reader to [2, 27] and [1] also.

4.2. Some lemmas

For each $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ and $M \subset \mathcal{I}$, we set

$$\mathcal{I}(\mathfrak{s}) := \{\emptyset, (s_1), (s_1, s_2), \ldots, (s_1, \ldots, s_r)\} \text{ and } \mathcal{I}(M) := \bigcup_{\mathfrak{s}\in M} \mathcal{I}(\mathfrak{s}).$$

Throughout this paper, we denote by R the following $\mathbb{F}_q(t)$-algebra:

$$\mathrm{R} := \{f/g \mid f \in \overline{k}[t],\ g \in \mathbb{F}_q[t] \setminus \{0\}\} \subset \overline{k}(t).$$

For any finite nonempty set J, it is understood that R^J refers to the $\mathbb{F}_q(t)$-vector space $\bigoplus_{x\in J} \mathrm{R}$ and any element of R^J is written in the form $(r_x)_{x\in J}$ with $r_x \in \mathrm{R}$.

Given any nonempty subset $M \subset \mathcal{I}_w$ for some $w \geq 0$, we consider the system of Frobenius equations

$$\varepsilon_{\mathfrak{s}}^{(1)} = \varepsilon_{\mathfrak{s}}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} + \sum_{\substack{s'>0 \\ (\mathfrak{s},s')\in\mathcal{I}(M)}} \varepsilon_{(\mathfrak{s},s')}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} \quad (\mathfrak{s} \in \mathcal{I}(M)) \tag{E_M}$$

with $(\varepsilon_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{I}(M)} \in \mathrm{R}^{\mathcal{I}(M)}$. We mention that ($\mathrm{E}_M$) is basically the case of $Q_i = 1$ from [14, (3.1.3)], which is derived from the study of establishing a criterion for the *zeta-like* MZV's. Note that our $\varepsilon_{\mathfrak{s}}$ is essentially referring to $\delta_i^{(-1)}$ used in [14, (3.1.3)] (with $Q_i = 1$ in [14]), but we follow [26] to express (E_M) with subscripts parameterized by indices. See also [26, (6.3), (6.4)] in the depth two case.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Remark 4.2. Note that if $w = 0$, we have $M = \mathcal{I}_0 = \{\emptyset\}$ and so $\mathcal{J}(M) = \{\emptyset\}$. In this special case, the equation (E_M) becomes $\varepsilon_\emptyset^{(1)} = \varepsilon_\emptyset$.

Lemma 4.3. *Let $F = \sum_{i=0}^n f_i t^i, G \in A[t]$ be polynomials over A such that $n = \deg_t F \geq 1$ and $f_n \in \mathbb{F}_q^\times$. If $\varepsilon \in \mathsf{R}$ satisfies $\varepsilon^{(1)} = \varepsilon F + G$, then we have $\varepsilon \in \mathbb{F}_q(t)[\theta]$.*

Proof. Our method is inspired by H.-J. Chen's formulation in the proof of [24, Thm. 2 (a)] (see Step I of the proof of [10, Thm. 6.1.1] also). By multiplying the denominator of ε on the equation, without loss of generality we assume that $\varepsilon \in \overline{k}[t] \setminus \{0\}$. We put $m := \deg_t \varepsilon \geq 0$. Since $\deg_t \varepsilon^{(1)} = \deg_t \varepsilon < \deg_t(\varepsilon F)$, we have $\deg_t G = m + n$. We set

$$\varepsilon = \sum_{i=0}^{m} a_i t^i \ (a_i \in \overline{k}) \ \text{ and } \ G = \sum_{i=0}^{m+n} g_i t^i \ (g_i \in A).$$

Moreover, we set $a_i := 0$ (resp. $f_i := 0$) when i does not satisfy $0 \leq i \leq m$ (resp. $0 \leq i \leq n$). Then we have

$$\sum_{i=0}^{m} a_i^q t^i = \sum_{j=0}^{m} a_j t^j \sum_{\ell=0}^{n} f_\ell t^\ell + \sum_{i=0}^{m+n} g_i t^i = \sum_{i=0}^{m+n} \sum_{j+\ell=i} a_j f_\ell t^i + \sum_{i=0}^{m+n} g_i t^i.$$

By comparing the coefficients of t^i for $m + 1 \leq i \leq m + n$, we have

$$a_{i-n} f_n = -g_i - \sum_{j>i-n} a_j f_{i-j}.$$

Then we can show that $a_m, a_{m-1}, \ldots, a_{m+1-n} \in A$ inductively.

Suppose that $m + 1 - n \geq 1 (\Leftrightarrow m \geq n)$. Then by comparing the coefficients of t^i for $n \leq i \leq m$, we have

$$a_{i-n} f_n = a_i^q - g_i - \sum_{j>i-n} a_j f_{i-j}.$$

Then we can show that $a_{m-n}, a_{m-n-1}, \ldots, a_0 \in A$ inductively. Therefore, in any case, we have $\varepsilon \in A[t]$. □

The next lemma can be viewed as an improvement of Step II of [10, Thm. 6.1.1] and [24, Thm. 2 (b)] which gives a better upper bound for the degree of solutions $(\varepsilon_\mathfrak{s})_{\mathfrak{s} \in \mathcal{J}(M)} \in \mathsf{R}^{\mathcal{J}(M)}$ satisfying (E_M).

Lemma 4.4. *Let $w \geq 0$ and $\emptyset \neq M \subset \mathcal{I}_w$. If $(\varepsilon_\mathfrak{s})_{\mathfrak{s} \in \mathcal{J}(M)} \in \mathsf{R}^{\mathcal{J}(M)}$ satisfies (E_M)*

$$\varepsilon_\mathfrak{s}^{(1)} = \varepsilon_\mathfrak{s}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} + \sum_{\substack{s'>0 \\ (\mathfrak{s},s') \in \mathcal{J}(M)}} \varepsilon_{(\mathfrak{s},s')}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} \ \ (\mathfrak{s} \in \mathcal{J}(M)),$$

then we have $\varepsilon_\mathfrak{s} \in \mathbb{F}_q(t)[\theta]$ and $\deg_\theta \varepsilon_\mathfrak{s} \leq \dfrac{w - \mathrm{wt}(\mathfrak{s})}{q-1}$ for each $\mathfrak{s} \in \mathcal{J}(M)$.

Proof. When $\mathfrak{s} \in M$, then we have $\varepsilon_\mathfrak{s} \in \mathbb{F}_q(t)$ since in this case, $\varepsilon_\mathfrak{s}^{(1)} = \varepsilon_\mathfrak{s}$. Thus, the statements are clearly valid. By Lemma 4.3 and the induction on $w - \mathrm{wt}(\mathfrak{s})$, we have $\varepsilon_\mathfrak{s} \in \mathbb{F}_q(t)[\theta]$ for all $\mathfrak{s} \in \mathcal{J}(M)$. We now consider $\mathfrak{s} \in \mathcal{J}(M) \setminus M$ and suppose that $\deg_\theta \varepsilon_{(\mathfrak{s},s')} \leq \dfrac{w - \mathrm{wt}(\mathfrak{s}, s)}{q-1}$ for all $s' > 0$ with $(\mathfrak{s}, s') \in \mathcal{J}(M)$.

Suppose on the contrary that $\deg_\theta \varepsilon_\mathfrak{s} > \dfrac{w - \mathrm{wt}(\mathfrak{s})}{q-1}$. Then we have $\deg_\theta \varepsilon_\mathfrak{s}^{(1)} > \dfrac{q(w - \mathrm{wt}(\mathfrak{s}))}{q-1}$. However, the induction hypothesis implies

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

$$\deg_\theta(\varepsilon_{(\mathfrak{s},s')}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})}) \le \frac{w-\mathrm{wt}(\mathfrak{s},s')}{q-1} + w - \mathrm{wt}(\mathfrak{s}) < \frac{q(w-\mathrm{wt}(\mathfrak{s}))}{q-1}.$$

It follows that $\deg_\theta \varepsilon_{\mathfrak{s}}^{(1)} = \deg_\theta(\varepsilon_{\mathfrak{s}}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})})$, and hence $\deg_\theta \varepsilon_{\mathfrak{s}} = \dfrac{w-\mathrm{wt}(\mathfrak{s})}{q-1}$, which is a contradiction. □

4.3. Dimension of $\mathcal{X}_w$

For $w \in \mathbb{Z}_{w\ge 0}$, we define $\mathcal{I}_w^{\mathrm{ND}_0} \subset \mathcal{I}_w^{\mathrm{ND}}$ as follows:

$$\mathcal{I}_w^{\mathrm{ND}_0} = \begin{cases} \{\emptyset\} & \text{if } w = 0 \\ \{(s_1,\dots,s_r) \in \mathcal{I}_w^{\mathrm{ND}} | s_2,\dots,s_r \text{ are divisible by } q-1\} & \text{if } w > 0 \end{cases} \tag{4.1}$$

Note that the description of indices in $\mathcal{I}_w^{\mathrm{ND}_0}$ comes from the simultaneously Eulerian phenomenon in [14, Cor. 4.2.3]. For each $w \ge 0$, we consider the system of Frobenius equations (E_M) for $M = \mathcal{I}_w^{\mathrm{ND}_0}$:

$$\varepsilon_{\mathfrak{s}}^{(1)} = \varepsilon_{\mathfrak{s}}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} + \sum_{\substack{s'>0 \\ (\mathfrak{s},s')\in\mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})}} \varepsilon_{(\mathfrak{s},s')}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} \quad (\mathfrak{s} \in \mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})) \tag{E_w}$$

with $(\varepsilon_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})} \in \mathrm{R}^{\mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})}$. We emphasize that $(E_w) = (E_{\mathcal{I}_w^{\mathrm{ND}_0}})$ but $(E_w) \ne (E_{\{w\}})$.

Example 4.5. We give some explicit examples for the system of Frobenius equations (E_w). The simplest example is the case $w = 0$. We note that $\mathcal{I}_0^{\mathrm{ND}_0} = \mathcal{J}(\mathcal{I}_0^{\mathrm{ND}_0}) = \{\emptyset\}$. Thus, ($\mathrm{E}_w$) consists of a single equation

$$\varepsilon_\emptyset^{(1)} = \varepsilon_\emptyset.$$

The second example is the case $w = q$. In this case, $\mathcal{I}_q^{\mathrm{ND}_0} = \{(1, q-1)\}$ and

$$\mathcal{J}(\mathcal{I}_q^{\mathrm{ND}_0}) = \{\emptyset, (1), (1, q-1)\}.$$

Thus, (E_w) consists of three equations

$$\begin{cases} \varepsilon_{(1,q-1)}^{(1)} &= \varepsilon_{(1,q-1)} \\ \varepsilon_{(1)}^{(1)} &= \varepsilon_{(1)}(t-\theta)^{q-1} + \varepsilon_{(1,q-1)}(t-\theta)^{q-1} \\ \varepsilon_\emptyset^{(1)} &= \varepsilon_\emptyset(t-\theta)^q + \varepsilon_{(1)}(t-\theta)^q. \end{cases}$$

The third example is the case $w = 2q-2$ with $q \ge 3$. Since $\mathcal{I}_{2q-2}^{\mathrm{ND}_0} = \{(q-1, q-1), (2q-2)\}$, we have

$$\mathcal{J}(\mathcal{I}_{2q-2}^{\mathrm{ND}_0}) = \{\emptyset, (q-1), (q-1, q-1), (2q-2)\}.$$

Then (E_w) consists of four equations

$$\begin{cases} \varepsilon_{(2q-2)}^{(1)} &= \varepsilon_{(2q-2)} \\ \varepsilon_{(q-1,q-1)}^{(1)} &= \varepsilon_{(q-1,q-1)} \\ \varepsilon_{(q-1)}^{(1)} &= \varepsilon_{(q-1)}(t-\theta)^{q-1} + \varepsilon_{(q-1,q-1)}(t-\theta)^{q-1} \\ \varepsilon_\emptyset^{(1)} &= \varepsilon_\emptyset(t-\theta)^{2q-2} + \varepsilon_{(q-1)}(t-\theta)^{2q-2} + \varepsilon_{(2q-2)}(t-\theta)^{2q-2}. \end{cases}$$

Definition 4.6. For each $w \in \mathbb{Z}_{\ge 0}$, we let $\mathcal{X}_w$ be the set of R-valued solutions of (E_w).

Remark 4.7. Since $f^{(1)} = f$ for any $f \in \mathbb{F}_q(t) \subset \mathrm{R}$, we see that $\mathcal{X}_w$ forms an $\mathbb{F}_q(t)$-vector subspace of $\mathrm{R}^{\mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})}$.

The main result of this section is the following dimension formula for $\mathcal{X}_w$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Theorem 4.8. *For any $w \in \mathbb{Z}_{\geq 0}$, we have*

$$\dim_{\mathbb{F}_q(t)} \mathcal{X}_w = \begin{cases} 1 & \text{if } (q-1) \mid w \\ 0 & \text{if } (q-1) \nmid w \end{cases}.$$

When $w = \ell(q-1)$ for some $\ell \in \mathbb{Z}_{\geq 0}$, we can choose a generator $(\eta_{\ell;\mathfrak{s}})_{\mathfrak{s} \in \mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})} \in \mathcal{X}_w$ such that

$$\eta_{\ell;\emptyset} = \sum_{i=0}^{\ell} b_{\ell i}(t-\theta)^i, \quad b_{\ell i} \in T\mathbb{F}_p[T]_{(T)} \ (0 \leq i < \ell) \ \text{ and } \ b_{\ell\ell} = 1,$$

where $T := t - t^q$ and $\mathbb{F}_p[T]_{(T)} \subset \mathbb{F}_p(t)$ is the localization of $\mathbb{F}_p[T]$ at the prime ideal (T).

Proof. Note that by Lemma 4.4, we have $\mathcal{X}_w \subset \mathbb{F}_q(t)[\theta]^{\mathcal{J}(\mathcal{I}_w^{\mathrm{ND}_0})}$. We write

$$w = \ell(q-1) + s \ \ (\ell \geq 0, \ \ 0 \leq s < q-1),$$
$$\ell = mq + n \ \ (m \geq 0, \ \ 0 \leq n < q).$$

First, we assume that $s = 0$ and hence $w = \ell(q-1)$ with $\ell \geq 0$. When $\ell = 0$, we have $\mathcal{J}(\mathcal{I}_0^{\mathrm{ND}_0}) = \{\emptyset\}$ and the equation is $\varepsilon_\emptyset^{(1)} = \varepsilon_\emptyset$. Thus, $\mathcal{X}_0 = \mathbb{F}_q(t)$, and we have a generator $\eta_{0;\emptyset} = 1$. Let $\ell \geq 1$ and suppose that the desired properties hold for weight $\ell'(q-1)$ with $\ell' < \ell$. The system of Frobenius equations $(\mathrm{E}_{\ell(q-1)})$ becomes

$$\varepsilon_\emptyset^{(1)} = \varepsilon_\emptyset(t-\theta)^{\ell(q-1)} + \sum_{\substack{0 \leq j < \ell \\ j \not\equiv \ell \bmod q}} \varepsilon_{((\ell-j)(q-1))}(t-\theta)^{\ell(q-1)} \tag{4.2}$$

and

$$\begin{aligned} \varepsilon_{((\ell-j)(q-1),\mathfrak{s})}^{(1)} &= \varepsilon_{((\ell-j)(q-1),\mathfrak{s})}(t-\theta)^{j(q-1)-\mathrm{wt}(\mathfrak{s})} \\ &\quad + \sum_{\substack{s'>0 \\ (\mathfrak{s},s') \in \mathcal{J}(\mathcal{I}_{j(q-1)}^{\mathrm{ND}_0})}} \varepsilon_{((\ell-j)(q-1),\mathfrak{s},s')}(t-\theta)^{j(q-1)-\mathrm{wt}(\mathfrak{s})} \end{aligned}$$

for $0 \leq j < \ell$ with $j \not\equiv \ell \bmod q$ and $\mathfrak{s} \in \mathcal{J}(\mathcal{I}_{j(q-1)}^{\mathrm{ND}_0})$. Since $(\varepsilon_{((\ell-j)(q-1),\mathfrak{s})})_{\mathfrak{s} \in \mathcal{J}(\mathcal{I}_{j(q-1)}^{\mathrm{ND}_0})} \in \mathcal{X}_{j(q-1)}$ for each j, the induction hypothesis implies that we have

$$\varepsilon_{((\ell-j)(q-1))} = f_j \sum_{i=0}^{j} b_{ji}(t-\theta)^i, \quad f_j \in \mathbb{F}_q(t), \quad b_{ji} \in T\mathbb{F}_p[T]_{(T)} \ \text{ and } \ b_{jj} = 1.$$

By Lemma 4.4, we have $\deg_\theta \varepsilon_\emptyset \leq \ell$. We write $\varepsilon_\emptyset = \sum_{i=0}^{\ell} a_i(t-\theta)^i$ with $a_i \in \mathbb{F}_q(t)$ and plug it into (4.2). Then we obtain

$$\sum_{j=0}^{\ell} a_j(t-\theta^q)^j = \sum_{i=0}^{\ell} a_i(t-\theta)^{\ell(q-1)+i} + \sum_{\substack{0 \leq j < \ell \\ j \not\equiv \ell \bmod q}} f_j \sum_{i=0}^{j} b_{ji}(t-\theta)^{\ell(q-1)+i}.$$

Note that the equation above can be expressed as

$$\begin{aligned} &\sum_{i=0}^{\ell-1}\sum_{j=i}^{\ell-1} \binom{j}{i} T^{j-i} a_j (t-\theta)^{iq} - \sum_{i=0}^{\ell-1} a_i(t-\theta)^{\ell(q-1)+i} - \sum_{i=0}^{\ell-1} \sum_{\substack{i \leq j < \ell \\ j \not\equiv \ell \bmod q}} b_{ji} f_j (t-\theta)^{\ell(q-1)+i} \\ &\qquad = -a_\ell \sum_{i=0}^{\ell-1} \binom{\ell}{i} T^{\ell-i}(t-\theta)^{iq}. \end{aligned}$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Note further that

$$iq < w = (\ell - m)q - n \iff i < \ell - m - \frac{n}{q}.$$

By comparing the coefficients of $(t-\theta)^{\nu}$ for $\nu = iq$ $(0 \le i < \ell - m)$ or $\ell(q-1) \le \nu < \ell q$, this gives a system of linear equations with $(2\ell - m)$-equations and $(2\ell - m + 1)$-variables. We write $\mathbf{a} := (a_0, \dots, a_{\ell-1})^{\mathrm{tr}}$ and $\mathbf{f} := (f_0, \dots, f_{\ell-1})^{\mathrm{tr}}$ (excluding $f_n, f_{q+n}, \dots, f_{(m-1)q+n}$). Then the system mentioned above can be expressed as

$$U \begin{pmatrix} \mathbf{a} \\ \mathbf{f} \end{pmatrix} = a_\ell \mathbf{b}, \quad U \in \mathrm{Mat}_{2\ell-m}(\mathbb{F}_p[T]_{(T)}) \text{ and } \mathbf{b} \in (T\mathbb{F}_p[T]_{(T)})^{2\ell-m}. \tag{4.3}$$

As our goal of this case $w = \ell(q-1)$ is to show $\dim_{\mathbb{F}_q(t)} \mathscr{X}_w = 1$, it is to show that the existence of $\begin{pmatrix} \mathbf{a} \\ \mathbf{f} \end{pmatrix}$ is unique up to $\mathbb{F}_q(t)$-scalar multiple. Therefore, from (4.3), it suffices to show that $\det(U \bmod T) \neq 0$ in $\mathbb{F}_p$. Indeed, we have

$$(U \bmod T) + \begin{pmatrix} O & O \\ \mathrm{Id}_\ell & O \end{pmatrix}$$

$$= \begin{array}{c} \overbrace{\qquad\qquad}^{\ell} \quad \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{\ell-m} \\ \left(\begin{array}{ccc|ccccccccccc}
1 & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
\hline
 & & & -1 & & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
 & & & & & -1 & & & & & & & \\
\hline
 & & 1 & & & & 0 & & & & & & \\
\hline
 & & & & & & -1 & & & & & & \\
 & & & & & & & \ddots & & & & & \\
 & & & & & & & & -1 & & & & \\
\hline
 & & 1 & & & & & & & 0 & & & \\
\hline
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & \ddots \\
\hline
 & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & -1 \\
\hline
 & & 1 & & & & & & & & & 0 & \\
\hline
 & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & -1
\end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell - m \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} q-1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} q-1 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} q-1 \end{array} \end{array}.$$

The matrix $(U \bmod T) + \begin{pmatrix} O & O \\ \mathrm{Id}_\ell & O \end{pmatrix}$ can be described as follows:

(i) We first set up the lower right submatrix by beginning with a matrix of diagonal blocks, the first one being $-\mathrm{Id}_n$ and the remaining m of them being equal to $-\mathrm{Id}_{q-1}$. Then we insert zero rows between the blocks to obtain our submatrix of size $\ell \times (\ell - m)$.

(ii) We then put the identity matrix $\mathrm{Id}_{\ell-m}$ on the upper left corner.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

(iii) Finally, we define the $((\ell-m)+1)$st column to the ℓth column, each of length $2\ell-m$. In the $((\ell-m)+i)$th column, all entries are zero except for the entry 1 occurring in the same row as the ith row of zeroes inserted into the matrix constructed in step (i) (i.e., the row of zeroes lying above the ith diagonal block equal to $-\mathrm{Id}_{q-1}$ in step (i)). Thus, we have our $2(\ell-m)\times 2(\ell-m)$ matrix $(U \bmod T)+\begin{pmatrix} O & O \\ \mathrm{Id}_\ell & O\end{pmatrix}$.

For example, let $q=3$, $w=14=7(3-1)$ ($\ell=7=2\cdot 3+1$, $m=2$, $n=1$). Then we have

$$U \bmod T = \left(\begin{array}{ccccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1
\end{array}\right)$$

To evaluate $\det(U \bmod T)$, we appeal to elementary row operations which do not change the value of the determinant. Adding each of the top $\ell-m$ rows to the corresponding rows in the lower left matrix kills all entries there coming from $-\mathrm{Id}_\ell$ except the m bottom ones, which lie in the $((\ell-m)+1)$st one to the ℓth column. But each -1 in these last m rows can be killed by adding the appropriate row from step (iii) which has all zero entries except 1 in a unique column between $\ell-m+1$ and ℓ. Doing this leaves only the non-zero entries in those m columns the ones inserted in step (iii). We end up with a matrix which has a unique ± 1 in each row and column. So the elementary product expansion evaluates the determinant as ± 1. Note that one can also use elementary column operations to show the desired identity.

Next, we assume that $1\le s<q-1$ and put $w:=\ell(q-1)+s$. The system of Frobenius equations (E_w) is

$$\varepsilon_\emptyset^{(1)} = \varepsilon_\emptyset (t-\theta)^w + \sum_{\substack{0\le j\le \ell \\ j\not\equiv \ell-s \bmod q}} \varepsilon_{((\ell-j)(q-1)+s)}(t-\theta)^w \tag{4.4}$$

and

$$\begin{aligned}\varepsilon^{(1)}_{((\ell-j)(q-1)+s,\mathfrak{s})} &= \varepsilon_{((\ell-j)(q-1)+s,\mathfrak{s})}(t-\theta)^{j(q-1)-\mathrm{wt}(\mathfrak{s})} \\ &\quad + \sum_{\substack{s'>0 \\ (\mathfrak{s},s')\in\mathscr{I}(\mathcal{I}^{\mathrm{ND}_0}_{j(q-1)})}} \varepsilon_{((\ell-j)(q-1)+s,\mathfrak{s},s')}(t-\theta)^{j(q-1)-\mathrm{wt}(\mathfrak{s})}\end{aligned}$$

for $0\le j\le \ell$ with $j\not\equiv \ell-s \bmod q$ and $\mathfrak{s}\in\mathscr{I}(\mathcal{I}^{\mathrm{ND}_0}_{j(q-1)})$. Since $(\varepsilon_{((\ell-j)(q-1)+s,\mathfrak{s})})_{\mathfrak{s}\in\mathscr{I}(\mathcal{I}^{\mathrm{ND}_0}_{j(q-1)})}\in\mathscr{X}_{j(q-1)}$ for each j, we have

$$\varepsilon_{((\ell-j)(q-1)+s)} = f_j\sum_{i=0}^{j} b_{ji}(t-\theta)^i,\quad f_j\in\mathbb{F}_q(t),\quad b_{ji}\in T\mathbb{F}_p[T]_{(T)}\ \text{ and }\ b_{jj}=1.$$

By Lemma 4.4, we have $\deg_\theta \varepsilon_\emptyset \le \ell$. We write $\varepsilon_\emptyset=\sum_{i=0}^{\ell} a_i(t-\theta)^i$ with $a_i\in\mathbb{F}_q(t)$. Then (4.4) becomes

$$\sum_{j=0}^{\ell} a_j(t-\theta^q)^j = \sum_{i=0}^{\ell} a_i(t-\theta)^{w+i} + \sum_{\substack{0\le j\le\ell \\ j\not\equiv \ell-s \bmod q}} f_j\sum_{i=0}^{j} b_{ji}(t-\theta)^{w+i}.$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

This identity can be written as

$$\sum_{i=0}^{\ell}\sum_{j=i}^{\ell}\binom{j}{i}T^{j-i}a_j(t-\theta)^{iq}-\sum_{i=0}^{\ell}a_i(t-\theta)^{w+i}-\sum_{i=0}^{\ell}\sum_{\substack{i\le j\le \ell\\ j\not\equiv \ell-s \bmod q}}b_{ji}f_j(t-\theta)^{w+i}=0.$$

We note that

$$iq<w=(\ell-m)q+(s-n)\iff i<\ell-m+\frac{s-n}{q}.$$

When $s\le n$, by comparing the coefficients of $(t-\theta)^{\nu}$ for $\nu=iq$ $(0\le i<\ell-m)$ or $w\le\nu\le w+\ell$, this gives a system of linear equations with $(2\ell-m+1)$-equations and $(2\ell-m+1)$-variables. We write $\mathbf{a}:=(a_0,\dots,a_\ell)^{\mathrm{tr}}$ and $\mathbf{f}:=(f_0,\dots,f_\ell)^{\mathrm{tr}}$ (excluding $f_{n-s},f_{q+n-s},\dots,f_{\ell-s}$). Then the system can be written as

$$U\begin{pmatrix}\mathbf{a}\\ \mathbf{f}\end{pmatrix}=\mathbf{0}\ \text{ and }\ U\in\mathrm{Mat}_{2\ell-m+1}(\mathbb{F}_p[T]_{(T)}).$$

Since we aim to show that the solution space $\mathscr{X}_w$ is trivial in this case, it suffices to show that $\det(U \bmod T)\neq 0$ in $\mathbb{F}_p$. Indeed, we have

$$(U \bmod T)+\begin{pmatrix}O & O\\ \mathrm{Id}_{\ell+1} & O\end{pmatrix}$$

$$=\begin{array}{c}\overbrace{\qquad\qquad}^{\ell+1}\ \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{\ell-m}\\ \left(\begin{array}{c|cccccccccccccccccc}
1 & & & & & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & & \\
\hline
 & & & -1 & & & & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & & & & \\
 & & & & & -1 & & & & & & & & & & & & \\
 & 1 & & & & & 0 & & & & & & & & & & & \\
 & & & & & & & -1 & & & & & & & & & & \\
 & & & & & & & & \ddots & & & & & & & & & \\
 & & & & & & & & & -1 & & & & & & & & \\
 & & 1 & & & & & & & & 0 & & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & & & & -1 \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & & & & & -1
\end{array}\right)
\begin{array}{l}\left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\}\ell-m\\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\}n-s\\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\}q-1\\ \vdots\\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\}q-1\\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\}s\end{array}\end{array}$$

and hence $\det(U \bmod T)=\pm 1$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

When $s > n$, by comparing the coefficients of $(t-\theta)^\nu$ for $\nu = iq$ $(0 \le i \le \ell - m)$ or $w \le \nu \le w + \ell$, this gives a system of linear equations with $(2\ell - m + 2)$-equations and $(2\ell - m + 2)$-variables. We write $\mathbf{a} := (a_0, \ldots, a_\ell)^{\mathrm{tr}}$ and $\mathbf{f} := (f_0, \ldots, f_\ell)^{\mathrm{tr}}$ (excluding $f_{q+n-s}, f_{2q+n-s}, \ldots, f_{\ell-s}$). Then the system can be written as

$$U \begin{pmatrix} \mathbf{a} \\ \mathbf{f} \end{pmatrix} = \mathbf{0} \ \text{ and } \ U \in \mathrm{Mat}_{2\ell-m+2}(\mathbb{F}_p[T]_{(T)}).$$

It is enough to show that $\det(U \bmod T) \neq 0$ in $\mathbb{F}_p$. Indeed, we have

$$
(U \bmod T) + \begin{pmatrix} O & O \\ \mathrm{Id}_{\ell+1} & O \end{pmatrix}
$$

$$
= \overbrace{}^{\ell+1}\ \overbrace{}^{\ell-m+1}
$$

$$
= \left(\begin{array}{ccc|ccccccccccc}
1 & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\ \hline
 & & & -1 & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & \\
 & & & & & -1 & & & & & & & & \\
 & 1 & & & & & 0 & & & & & & & \\
 & & & & & & -1 & & & & & & & \\
 & & & & & & & \ddots & & & & & & \\
 & & & & & & & & -1 & & & & & \\
 & 1 & & & & & & & & 0 & & & & \\
 & & & & & & & & & & \ddots & & & \\
 & \ddots & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & -1 \\
 & & 1 & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & & & -1
\end{array}\right)
\begin{array}{l}
\left.\right\}\ell-m+1 \\
\left.\right\}q+n-s \\
\left.\right\}q-1 \\
\vdots \\
\left.\right\}q-1 \\
\left.\right\}s
\end{array}
$$

and hence $\det(U \bmod T) = \pm 1$. □

5. Linear independence

In this section, we aim to show that $\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) \in k_\infty \mid \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\}$ is a k-linearly independent set. To begin with, we adopt the following setting. Let $(-\theta)^{\frac{1}{q-1}} \in \overline{k}^\times$ be a fixed $(q-1)$st root of $-\theta$. Following [2], we consider the following power series:

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{k_\infty}[[t]].$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Note that it satisfies the Frobenius difference equation

$$\Omega^{(-1)}(t) = (t-\theta)\Omega(t).$$

In addition, we have $\Omega(t) \in \mathcal{E}$ and $\widetilde{\pi} := 1/\Omega(\theta)$ is the fundamental period of the Carlitz module (see [2, Sec. 3.1.2]). We also set $\mathbb{L}_0 := 1$ and $\mathbb{L}_d := (t-\theta^q)\cdots(t-\theta^{q^d})$ for $d \geq 1$. Put $\mathcal{L}(\emptyset) := 1$ and for each index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}_{>0}$, we define the following deformation series:

$$\mathcal{L}(\mathfrak{s}) := \Omega^{\mathrm{wt}(\mathfrak{s})} \sum_{d_1 > \cdots > d_r \geq 0} \frac{1}{\mathbb{L}_{d_1}^{s_1} \cdots \mathbb{L}_{d_r}^{s_r}} \in \mathbb{T},$$

which was first introduced by Papanikolas [27] for $\mathrm{dep}(\mathfrak{s}) = 1$. It is shown in [9, Lemma 5.3.1] that $\mathcal{L}(\mathfrak{s}) \in \mathcal{E}$. Moreover, by [14, Proposition 2.3.3], we have

$$\mathcal{L}(\mathfrak{s})|_{t=\theta^{q^j}} = (\mathcal{L}(\mathfrak{s})|_{t=\theta})^{q^j} = (\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})/\widetilde{\pi}^{\mathrm{wt}(\mathfrak{s})})^{q^j}$$

for all $j \geq 0$ (see also [9, Lemma 5.3.5]). Let $P = \sum_{\mathfrak{s}\in\mathcal{I}} a_{\mathfrak{s}}[\mathfrak{s}] \in \mathcal{H}(a_{\mathfrak{s}} \in k)$. For the convenience of later use, we set

$$\mathrm{Li}_P(\mathbf{1}) := \sum_{\mathfrak{s}} a_{\mathfrak{s}} \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}).$$

5.1. The key lemma

The fundamental system of Frobenius difference equations that $\{\mathcal{L}(\mathbf{n})\}_{\mathbf{n}\in\mathcal{J}(\mathfrak{s})}$ satisfy is given in [9, (5.3.3),(5.3.4)] as well as [14, (2.3.4), (2.3.7)] with all $Q_i = 1$ there, and it plays a crucial role in the proof of the following Lemma when applying ABP-criterion. We further mention that the first formulation of the following Lemma arises from the ideas in the proof of [14, Thm. 2.5.2], and the second one is an extension of part of Step 3 in the proof of [26, Thm. 6], which dealt with $w \leq 2q-2$.

Lemma 5.1. *Let $w \geq 0$ and $\emptyset \neq M \subset \mathcal{I}_w$. Let $P = \sum_{\mathfrak{s}\in\mathcal{I}} a_{\mathfrak{s}}[\mathfrak{s}] \in \mathcal{H}(a_{\mathfrak{s}} \in k)$ and suppose that*

- $\mathrm{Supp}(P) \subset M$,
- $\mathrm{Li}_P(\mathbf{1})$ *is Eulerian; that is,* $\mathrm{Li}_P(\mathbf{1}) = \sum_{\mathfrak{s}} a_{\mathfrak{s}} \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) \in k \cdot \widetilde{\pi}^w$,
- $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})(\mathfrak{s} \in \mathcal{J}(M) \cap \mathcal{I}_\ell)$ *are linearly independent over k for each* $0 \leq \ell < w$.

Then the following hold:

(1) *We have*

$$\sum_{\mathfrak{s}'\in\mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}} a_{(\mathfrak{s},\mathfrak{s}')} \mathrm{Li}_{\mathfrak{s}'}(\mathbf{1}) = 0 \ \text{ or } \ (q-1) \mid (w - \mathrm{wt}(\mathfrak{s}))$$

for each $\mathfrak{s} \in \mathcal{J}(M)$. In particular, if $\mathrm{Li}_{\mathfrak{s}'}(\mathbf{1})(\mathfrak{s}' \in \mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}, (\mathfrak{s},\mathfrak{s}') \in \mathrm{Supp}(P))$ *are linearly independent over k for each $\mathfrak{s} \in \mathcal{J}(\mathrm{Supp}(P)) \setminus \{\emptyset\}$, then*

$$\mathrm{Supp}(P) \subset \{(s_1, \ldots, s_r) \in \mathcal{I}_w \mid (q-1) \mid s_2, \ldots, s_r\}.$$

(2) *The system of Frobenius equations (E_M) has a solution $(\varepsilon_{\mathfrak{s}})_{\mathfrak{s}\in\mathcal{J}(M)} \in \mathbb{F}_q(t)[\theta]^{\mathcal{J}(M)}$ such that $\varepsilon_{\mathfrak{s}} = a_{\mathfrak{s}}|_{\theta=t}$ for each $\mathfrak{s} \in M$.*

Proof. In what follows, our essential arguments are rooted in the ideas of the proof of [14]. Let $\alpha_{\mathfrak{s}} = \alpha_{\mathfrak{s}}(t) := a_{\mathfrak{s}}|_{\theta=t} \in \mathbb{F}_q(t)$, $c := \mathrm{Li}_P(\mathbf{1})/\widetilde{\pi}^w \in k$ and $\mathcal{J}(M)' := \mathcal{J}(M) \setminus M$. We may assume that $w > 0$, $P \neq 0$ and $\alpha_{\mathfrak{s}} \in \mathbb{F}_q[t](\mathfrak{s} \in \mathcal{I})$. In particular, $\mathcal{J}(M)' \neq \emptyset$. We define matrices $\Phi' =$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

$(\Phi'_{\mathfrak{n},\mathfrak{s}})_{\mathfrak{n},\mathfrak{s}\in\mathscr{I}(M)'} \in \mathrm{Mat}_{|\mathscr{I}(M)'|}(\overline{k}[t])$ and $\Psi' = (\Psi'_{\mathfrak{n},\mathfrak{s}})_{\mathfrak{n},\mathfrak{s}\in\mathscr{I}(M)'} \in \mathrm{GL}_{|\mathscr{I}(M)'|}(\mathbb{T})$ indexed by the set $\mathscr{I}(M)'$. They are defined by

$$\Phi'_{\mathfrak{n},\mathfrak{s}} := \begin{cases} (t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} & (\mathfrak{s}=\mathfrak{n} \text{ or } \mathfrak{s}=\mathfrak{n}_+) \\ 0 & (\text{otherwise}) \end{cases} \text{ and } \Psi'_{\mathfrak{n},\mathfrak{s}} := \begin{cases} \mathcal{L}(\mathfrak{s}')\Omega^{w-\mathrm{wt}(\mathfrak{n})} & (\mathfrak{n}=(\mathfrak{s},\mathfrak{s}')) \\ 0 & (\text{otherwise}) \end{cases}.$$

In particular, we have $\Psi'_{\mathfrak{n},\emptyset} = \mathcal{L}(\mathfrak{n})\Omega^{w-\mathrm{wt}(\mathfrak{n})}$. We also define row vectors $\mathbf{v} = (v_{\mathfrak{s}})_{\mathfrak{s}\in\mathscr{I}(M)'} \in \mathrm{Mat}_{1\times|\mathscr{I}(M)'|}(\overline{k}[t])$ and $\mathbf{f} = (f_{\mathfrak{s}})_{\mathfrak{s}\in\mathscr{I}(M)'} \in \mathrm{Mat}_{1\times|\mathscr{I}(M)'|}(\mathbb{T})$ by

$$v_{\mathfrak{s}} := \begin{cases} \alpha_{(\mathfrak{s},w-\mathrm{wt}(\mathfrak{s}))}(t-\theta)^{w-\mathrm{wt}(\mathfrak{s})} & ((\mathfrak{s},w-\mathrm{wt}(\mathfrak{s}))\in M) \\ 0 & (\text{otherwise}) \end{cases} \text{ and } f_{\mathfrak{s}} := \sum_{\mathfrak{s}'\in\mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}} \alpha_{(\mathfrak{s},\mathfrak{s}')}\mathcal{L}(\mathfrak{s}').$$

In particular, we have $f_\emptyset := \sum_{\mathfrak{s}'\in\mathcal{I}_w} \alpha_{\mathfrak{s}'}\mathcal{L}(\mathfrak{s}') = \mathcal{L}(P)$. Then we set

$$\Phi := \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \mathrm{Mat}_{|\mathscr{I}(M)'|+1}(\overline{k}[t]) \text{ and } \Psi := \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \mathrm{GL}_{|\mathscr{I}(M)'|+1}(\mathbb{T}).$$

We further set

$$\widetilde{\Phi} := \begin{pmatrix} 1 & 0 \\ 0 & \Phi \end{pmatrix} \in \mathrm{Mat}_{|\mathscr{I}(M)'|+2}(\overline{k}[t]) \text{ and } \widetilde{\psi} := \begin{pmatrix} 1 \\ (\Psi'_{\mathfrak{n},\emptyset})_{\mathfrak{n}\in\mathscr{I}(M)'} \\ f_\emptyset \end{pmatrix} \in \mathrm{Mat}_{(|\mathscr{I}(M)'|+2)\times 1}(\mathbb{T}).$$

Then using [14, (2.3.4), (2.3.7)], one checks that

$$\Psi^{(-1)} = \Phi\Psi \text{ and } \widetilde{\psi}^{(-1)} = \widetilde{\Phi}\widetilde{\psi}.$$

We mention that we follow [26] to use double indices to indicate the entries of Φ here, and putting such f_ϕ into a system of Frobenius difference equations was first used by the first named author in [10], and later used in [12, 26]. Such f_ϕ naturally appears in the period matrix of the fiber coproduct of rigid analytically trivial dual t-motives in [13].

By [2, Theorem 3.1.1], there exist $g \in \overline{k}(t)$ and $\mathbf{g} = (g_{\mathfrak{s}})_{\mathfrak{s}\in\mathscr{I}(M)'} \in \mathrm{Mat}_{1\times|\mathscr{I}(M)'|}(\overline{k}(t))$ such that

$$\widetilde{\mathbf{g}}\widetilde{\psi} = 0, \quad g \text{ and } \mathbf{g} \text{ are regular at } t=\theta \text{ and } \widetilde{\mathbf{g}}|_{t=\theta} = (-c,0,\dots,0,1),$$

where $\widetilde{\mathbf{g}} := (g,\mathbf{g},1)$.

We set $B = B(t)$ and $B_{\mathfrak{s}} = B_{\mathfrak{s}}(t)$ in $\overline{k}(t)$ by

$$(B,(B_{\mathfrak{s}})_{\mathfrak{s}\in\mathscr{I}(M)'},0) := \widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^{(-1)}\widetilde{\Phi} \in \mathrm{Mat}_{1\times(|\mathscr{I}(M)'|+2)}(\overline{k}(t)).$$

We claim that $\widetilde{\mathbf{g}}^{(-1)}\widetilde{\Phi} = \widetilde{\mathbf{g}}$; that is, $B = B_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathscr{I}(M)'$. Indeed,

$$B + \sum_{\mathfrak{s}\in\mathscr{I}(M)'} B_{\mathfrak{s}}\mathcal{L}(\mathfrak{s})\Omega^{w-\mathrm{wt}(\mathfrak{s})} = (B,(B_{\mathfrak{s}})_{\mathfrak{s}\in\mathscr{I}(M)'},0)\widetilde{\psi} = (\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^{(-1)}\widetilde{\Phi})\widetilde{\psi} = \widetilde{\mathbf{g}}\widetilde{\psi} - (\widetilde{\mathbf{g}}\widetilde{\psi})^{(-1)} = 0$$

and $\mathscr{I}(M)' = \bigsqcup_{\ell=0}^{w-1} \mathscr{I}(M)\cap\mathcal{I}_\ell$ imply the equality

$$B + \sum_{\mathfrak{s}\in\mathscr{I}(M)\cap\mathcal{I}_{w-1}} B_{\mathfrak{s}}\mathcal{L}(\mathfrak{s})\Omega + \sum_{\mathfrak{s}\in\mathscr{I}(M)\cap\mathcal{I}_{w-2}} B_{\mathfrak{s}}\mathcal{L}(\mathfrak{s})\Omega^2 + \cdots + \sum_{\mathfrak{s}\in\mathscr{I}(M)\cap\mathcal{I}_0} B_{\mathfrak{s}}\mathcal{L}(\mathfrak{s})\Omega^w = 0. \tag{5.1}$$

We note that

- B and $B_{\mathfrak{s}}(\mathfrak{s}\in\mathscr{I}(M)')$ are regular at $t=\theta^{q^i}$ for $j \gg 0$,
- Ω and $\mathcal{L}(\mathfrak{s})(\mathfrak{s}\in\mathcal{I})$ are entire,

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

- $\Omega(\theta^{q^j}) = 0$ for $j \geq 1$,
- $\mathcal{L}(\mathfrak{s})|_{t=\theta^{q^j}} = (\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})/\widetilde{\pi}^{\mathrm{wt}(\mathfrak{s})})^{q^j}$ for $\mathfrak{s} \in \mathcal{I}$ and $j \geq 0$.

Thus, evaluating at $t = \theta^{q^j}$ for $j \gg 0$ in (5.1) implies $B(\theta^{q^j}) = 0$. Since B is rational, we have $B = 0$. Then dividing (5.1) by Ω and evaluating at $t = \theta^{q^j}$ for $j \gg 0$ imply

$$\sum_{\mathfrak{s} \in \mathcal{J}(M) \cap \mathcal{I}_{w-1}} B_{\mathfrak{s}}(\theta^{q^j})(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})/\widetilde{\pi}^{w-1})^{q^j} = 0.$$

This is equivalent to

$$\sum_{\mathfrak{s} \in \mathcal{J}(M) \cap \mathcal{I}_{w-1}} B_{\mathfrak{s}}(\theta^{q^j})^{q^{-j}} \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) = 0.$$

By the assumption of linear independence and [9, Theorem 2.2.1], we have $B_{\mathfrak{s}}(\theta^{q^j}) = 0$. Thus, we have $B_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathcal{J}(M) \cap \mathcal{I}_{w-1}$. Repeating this process, we have $B_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathcal{J}(M)'$. We note that the claim implies

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \mathbf{g} & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \mathbf{g} & 1 \end{pmatrix}. \tag{5.2}$$

(1) We set

$$X := \begin{pmatrix} \mathrm{Id} & 0 \\ \mathbf{g} & 1 \end{pmatrix} \Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{g}\Psi' + \mathbf{f} & 1 \end{pmatrix} \in \mathrm{GL}_{|\mathcal{J}(M)'|+1}(\mathrm{Frac}(\mathbb{T})).$$

By using (5.2), we can verify that $X^{(-1)} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} X$. Thus, by [27, §4.1.6],

$$\mathrm{GL}_{|\mathcal{J}(M)'|+1}(\mathbb{F}_q(t)) \ni \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix}^{-1} X = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathbf{g}\Psi' + \mathbf{f} & 1 \end{pmatrix} =: \begin{pmatrix} \mathrm{Id} & 0 \\ (h_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{J}(M)'} & 1 \end{pmatrix}.$$

Then evaluating at $t = \theta^{q^j}$ for $j \gg 0$ in $\mathbf{g}\Psi' + \mathbf{f} = (h_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{J}(M)'}$, we have $f_{\mathfrak{s}}(\theta^{q^j}) = h_{\mathfrak{s}}(\theta^{q^j}) = h_{\mathfrak{s}}(\theta)^{q^j}$. Since

$$f_{\mathfrak{s}}(\theta^{q^j}) = \sum_{\mathfrak{s}' \in \mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}} \alpha_{(\mathfrak{s},\mathfrak{s}')}(\theta^{q^j}) \mathcal{L}(\mathfrak{s}')|_{t=\theta^{q^j}} = \sum_{\mathfrak{s}' \in \mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}} \alpha_{(\mathfrak{s},\mathfrak{s}')}(\theta)^{q^j} (\mathrm{Li}_{\mathfrak{s}'}(\mathbf{1})/\widetilde{\pi}^{w-\mathrm{wt}(\mathfrak{s})})^{q^j} = f_{\mathfrak{s}}(\theta)^{q^j},$$

we have $f_{\mathfrak{s}}(\theta) = h_{\mathfrak{s}}(\theta) \in k$ for each $\mathfrak{s} \in \mathcal{J}(M)'$. Thus, we have

$$\sum_{\mathfrak{s}' \in \mathcal{I}_{w-\mathrm{wt}(\mathfrak{s})}} a_{(\mathfrak{s},\mathfrak{s}')} \mathrm{Li}_{\mathfrak{s}'}(\mathbf{1}) \in k \cdot \widetilde{\pi}^{w-\mathrm{wt}(\mathfrak{s})}$$

for each $\mathfrak{s} \in \mathcal{J}(M)'$. We note that this holds whenever $\mathfrak{s} \in M$. Then the first statement of (1) follows from this relation because $\widetilde{\pi} \in (-\theta)^{\frac{1}{q-1}} \cdot k_\infty^\times$.

Next we prove the second statement of (1). Let $(\mathfrak{s}, \mathfrak{s}'') \in \mathrm{Supp}(P)$ with $\mathfrak{s} \neq \emptyset$. Then by the first statement of (1) and the assumption on the linear independence of the second statement of (1), we have $(q-1) \mid (w - \mathrm{wt}(\mathfrak{s})) = \mathrm{wt}(\mathfrak{s}'')$.

(2) By (5.2) and [14, Proposition 2.2.1], there exists $\alpha \in \mathbb{F}_q[t] \setminus \{0\}$ such that $\alpha\mathbf{g} \in \mathrm{Mat}_{1\times|\mathcal{J}(M)'|}(\overline{k}[t])$. If we set $\varepsilon_{\mathfrak{s}} := g_{\mathfrak{s}}^{(-1)} (\mathfrak{s} \in \mathcal{J}(M)')$ and $\varepsilon_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \in \mathbb{F}_q(t) (\mathfrak{s} \in M)$, then

$$((\varepsilon_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{J}(M)'}, 1) \begin{pmatrix} \Phi' \\ \mathbf{v} \end{pmatrix} = (\varepsilon_{\mathfrak{s}}^{(1)})_{\mathfrak{s} \in \mathcal{J}(M)'} \text{ and } \varepsilon_{\mathfrak{s}}^{(1)} = \varepsilon_{\mathfrak{s}} \ (\mathfrak{s} \in M)$$

give the desired equations. □

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

5.2. *Linear independence*

With the crucial properties of Lemma 5.1 established, we are able to show the following linear independence result.

Theorem 5.2. *For each $w \geq 0$, the elements*

$$\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) \text{ with } \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}$$

form a basis of $\mathcal{Z}_w$.

Proof. (cf. [26, p. 388] for the special case of k-linear independence of $\zeta_A(w)$ and $\zeta_A(w-(q-1), q-1)$ with $w \leq 2q-2$.) By Theorem 3.8, $\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\}$ is a generating set for $\mathcal{Z}_w$. Thus, it is enough to show that the given elements are linearly independent over k. This is proved by induction on w. When $w = 0$, we have $\mathcal{I}_0^{\mathrm{ND}} = \{\emptyset\}$, and $\mathrm{Li}_{\emptyset}(\mathbf{1}) = 1$ is linearly independent over k.

Let $w \geq 1$ and assume that the elements $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ with $\mathfrak{s} \in \mathcal{I}_{w'}^{\mathrm{ND}}$ are linearly independent over k for $w' < w$. Let $P = \sum_{\mathfrak{s} \in \mathcal{I}} \alpha_{\mathfrak{s}}(\theta)[\mathfrak{s}] \in \mathcal{H}(\alpha_{\mathfrak{s}} = \alpha_{\mathfrak{s}}(t) \in \mathbb{F}_q(t))$ be a linear relation among $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$'s over k such that $\mathrm{Supp}(P) \subset \mathcal{I}_w^{\mathrm{ND}}$. By Lemma 5.1 (1) for $M = \mathrm{Supp}(P)$ and the induction hypothesis, we have $\mathrm{Supp}(P) \subset \mathcal{I}_w^{\mathrm{ND}_0}$. By Lemma 5.1 (2) for $M = \mathcal{I}_w^{\mathrm{ND}_0}$, there exists a solution $(\varepsilon_{\mathfrak{s}}) \in \mathcal{X}_w$ such that $\varepsilon_{\mathfrak{s}} = \alpha_{\mathfrak{s}}$ for all $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}_0}$.

When $(q-1) \nmid w$, Theorem 4.8 implies that $\alpha_{\mathfrak{s}} = 0$ for all $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}_0}$. When $(q-1) \mid w$, we have $\widetilde{\pi}^w \in \mathcal{Z}_w$ and hence Theorem 3.8 implies that there exists $P_w = \sum_{\mathfrak{s} \in \mathcal{I}} \beta_{\mathfrak{s}}(\theta)[\mathfrak{s}] \in \mathcal{H}_w(\beta_{\mathfrak{s}}(t) \in \mathbb{F}_q(t))$ such that $\mathrm{Supp}(P_w) \subset \mathcal{I}_w^{\mathrm{ND}}$ and $\mathrm{Li}_{P_w}(\mathbf{1}) = \widetilde{\pi}^w$. It is clear that $P_w \neq 0$. By Lemma 5.1 (1) for $M = \mathrm{Supp}(P_w)$, we have $\mathrm{Supp}(P_w) \subset \mathcal{I}_w^{\mathrm{ND}_0}$. By Lemma 5.1 (2) for $M = \mathcal{I}_w^{\mathrm{ND}_0}$, there exists a solution $(\varepsilon'_{\mathfrak{s}}) \in \mathcal{X}_w$ such that $\varepsilon'_{\mathfrak{s}} = \beta_{\mathfrak{s}}$ for all $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}_0}$. Then by Theorem 4.8, there exists an $\alpha \in \mathbb{F}_q(t)$ for which $(\varepsilon_{\mathfrak{s}}) = \alpha(\varepsilon'_{\mathfrak{s}})$, and hence we have $P = \alpha(\theta)P_w$. Then we have

$$0 = \mathrm{Li}_P(\mathbf{1}) = \mathrm{Li}_{\alpha(\theta)P_w}(\mathbf{1}) = \alpha(\theta)\widetilde{\pi}^w.$$

Thus, $\alpha = 0$ and $P = 0$. □

5.3. *Proof of Theorem 1.5*

With the fundamental results established, we can now give a short proof of Theorem 1.5. Given $w \in \mathbb{Z}_{>0}$, we claim that $\mathcal{B}_w^{\mathrm{T}}$ is a basis of the k-vector space $\mathcal{Z}_w$. By Theorem 5.2, $\{\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) | \mathfrak{s} \in \mathcal{I}_w^{\mathrm{ND}}\}$ is a k-basis of $\mathcal{Z}_w$. Since $|\mathcal{I}_w^{\mathrm{ND}}| = |\mathcal{I}_w^{\mathrm{T}}|$ by Proposition 2.1, and $\mathcal{B}_w^{\mathrm{T}}$ is a generating set of $\mathcal{Z}_w$ by Corollary 3.7, we have that

$$|\mathcal{B}_w^{\mathrm{T}}| \geq \dim_k \mathcal{Z}_w = |\mathcal{I}_w^{\mathrm{ND}}| = |\mathcal{I}_w^{\mathrm{T}}| \geq |\mathcal{B}_w^{\mathrm{T}}|,$$

whence the desired result follows.

5.4. *Generating set of linear relations*

Recall the k-linear map $\mathcal{U}^{\zeta}$ defined in Definition 3.4, and Theorem 3.6 asserts that for $\mathfrak{s} \in \mathcal{I}_w \setminus \mathcal{I}_w^{\mathrm{T}}$,

$$\mathcal{L}^{\zeta}\left([\mathfrak{s}] - \mathcal{U}^{\zeta}(\mathfrak{s})\right) = 0. \tag{5.3}$$

We prove in the following theorem, which verifies the $\mathcal{B}^*$-version of [30, Conjecture 5.1], that these relations account for all k-linear relations among MZV's of the same weight.

Theorem 5.3. *Let w be a positive integer. Then all the k-linear relations among the MZV's of weight w are generated by (5.3) for $\mathfrak{s} \in \mathcal{I}_w \setminus \mathcal{I}_w^{\mathrm{T}}$. In other words, if we denote by $\mathcal{L}_w^{\zeta} := \mathcal{L}^{\zeta}|_{\mathcal{H}_w} := ([\mathfrak{s}] \mapsto \zeta_A(\mathfrak{s}))$:*

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

$\mathcal{H}_w \twoheadrightarrow \mathcal{Z}_w$ *and put*

$$\mathscr{R}_w := \mathrm{Span}_k \left\{ [\mathfrak{s}] - \mathscr{U}^\zeta(\mathfrak{s}) \mid \mathfrak{s} \in \mathcal{I}_w \setminus \mathcal{I}_w^{\mathrm{T}} \right\} \subset \mathcal{H}_w,$$

then

$$\mathrm{Ker}\, \mathscr{L}_w^\zeta = \mathscr{R}_w.$$

Moreover, we have

$$\mathrm{Ker}\, \mathscr{L}^\zeta = \bigoplus_{w \in \mathbb{N}} \mathscr{R}_w.$$

Proof. Theorem 3.6 implies the inclusion $\mathrm{Ker}\, \mathscr{L}_w^\zeta \supset \mathscr{R}_w$, and Remark 3.5 implies

$$\mathscr{R}_w = \mathrm{Span}_k \left\{ P - \mathscr{U}^\zeta(P) \mid P \in \mathcal{H}_w \right\}.$$

We fix a positive integer e given in Theorem 3.6. Then for each $P \in \mathcal{H}_w$, we have

$$P \equiv \mathscr{U}^\zeta(P) \equiv \cdots \equiv (\mathscr{U}^\zeta)^e(P) \bmod \mathscr{R}_w.$$

By these relations and Theorem 3.6, the quotient space $\mathcal{H}_w/\mathscr{R}_w$ is spanned by the image of $\mathcal{I}_w^{\mathrm{T}}$. It follows that

$$|\mathcal{I}_w^{\mathrm{T}}| \geq \dim_k \mathcal{H}_w/\mathscr{R}_w \geq \dim_k \mathcal{H}_w/\mathrm{Ker}\, \mathscr{L}_w^\zeta = \dim_k \mathcal{Z}_w = |\mathcal{I}_w^{\mathrm{T}}|,$$

where the last equality comes from Theorem 1.5. So we have $\mathrm{Ker}\, \mathscr{L}_w^\zeta = \mathscr{R}_w$. By [9, Thm. 2.2.1], the last assertion follows. □

Remark 5.4. According to [9, Thm. 2.2.1], all $\overline{k}$-linear relations among MZV's are decent to k-linear relations, and hence, (5.3) accounts for all $\overline{k}$-linear relations among MZV's.

A. Proof of Theorem 3.6

The aim of this appendix is to give a detailed proof of Theorem 3.6.

A.1. Inequalities of depth

Proposition A.1. *Let* $\bullet \in \{\mathrm{Li}, \zeta\}$ *and* $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ *be indices. Then for each* $\mathfrak{u} \in \mathrm{Supp}(\mathfrak{s} *^\bullet \mathfrak{n})$*, we have*

$$\max\{\mathrm{dep}(\mathfrak{s}), \mathrm{dep}(\mathfrak{n})\} \leq \mathrm{dep}(\mathfrak{u}) \leq \mathrm{dep}(\mathfrak{s}) + \mathrm{dep}(\mathfrak{n}).$$

Proof. Put $r := \mathrm{dep}(\mathfrak{s})$ and $\ell := \mathrm{dep}(\mathfrak{n})$. We prove the inequalities by induction on $r + \ell$. Note that in the case of $r = 0$ or $\ell = 0$ (namely, $\mathfrak{s} = \emptyset$ or $\mathfrak{n} = \emptyset$), the result follows from the definition of $*^\bullet$.

Suppose that $r \geq 1$ and $\ell \geq 1$. Since the binary operation $*^\bullet$ is commutative, without loss of generality we may assume that $r \geq \ell \geq 1$. We first consider the case that $r > \ell$. By the induction hypothesis, we have:

- when $\mathfrak{u} \in \mathrm{Supp}([s_1, \mathfrak{s}_- *^\bullet \mathfrak{n}])$, then $1 + (r-1) \leq \mathrm{dep}(\mathfrak{u}) \leq 1 + (r-1) + \ell$;
- when $\mathfrak{u} \in \mathrm{Supp}([n_1, \mathfrak{s} *^\bullet \mathfrak{n}_-])$, then $1 + r \leq \mathrm{dep}(\mathfrak{u}) \leq 1 + r + (\ell - 1)$;
- when $\mathfrak{u} \in \mathrm{Supp}([s_1 + n_1, \mathfrak{s}_- *^\bullet \mathfrak{n}_-])$, then $1 + (r-1) \leq \mathrm{dep}(\mathfrak{u}) \leq 1 + (r-1) + (\ell - 1)$;
- when $\mathfrak{u} \in \mathrm{Supp}(D^\zeta_{\mathfrak{s},\mathfrak{n}})$, then there exist j, $\mathfrak{u}'$ and $\mathfrak{u}''$ such that $1 \leq j < s_1 + n_1$, $\mathfrak{u}'' \in \mathrm{Supp}(\mathfrak{s}_- *^\zeta \mathfrak{n}_-)$ and $\mathfrak{u}' \in \mathrm{Supp}((j) *^\zeta \mathfrak{u}'')$ with $\mathfrak{u} = (s_1 + n_1 - j, \mathfrak{u}')$. It follows that $r - 1 \leq \mathrm{dep}(\mathfrak{u}'') \leq (r-1) + (\ell - 1)$, and hence, $r - 1 \leq \mathrm{dep}(\mathfrak{u}') \leq 1 + (r-1) + (\ell - 1)$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

In any case, we have $r \leq \mathrm{dep}(\mathfrak{u}) \leq r + \ell$. In the case of $r = \ell$, we can verify that $r \leq \mathrm{dep}(\mathfrak{u}) \leq 2r$ by a similar argument as above. □

Corollary A.2. *Let $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ be indices.*

(1) *Suppose that both $\mathfrak{s}$ and $\mathfrak{n}$ are nonempty indices. Let $D^{\zeta}_{\mathfrak{s},\mathfrak{n}}$ be defined in (2.10). Then for each $\mathfrak{u} \in \mathrm{Supp}(D^{\zeta}_{\mathfrak{s},\mathfrak{n}})$, we have*

$$\max\{\mathrm{dep}(\mathfrak{s}), \mathrm{dep}(\mathfrak{n})\} \leq \mathrm{dep}(\mathfrak{u}) \leq \mathrm{dep}(\mathfrak{s}) + \mathrm{dep}(\mathfrak{n}).$$

(2) *For any integer $m \geq 0$, we let $\alpha_q^{\bullet,m}$ be the m-th iteration of $\alpha_q^{\bullet}$ defined in (3.3). Then for each $\mathfrak{u} \in \mathrm{Supp}(\alpha_q^{\bullet,m}(\mathfrak{s}))$, we have*

$$\mathrm{dep}(\mathfrak{u}) \geq \begin{cases} 1 + m & (\mathfrak{s} = \emptyset,\ m \geq 1) \\ \mathrm{dep}(\mathfrak{s}) + m & (otherwise) \end{cases} \quad \textit{and} \quad \mathrm{dep}(\mathfrak{u}) \leq \mathrm{dep}(\mathfrak{s}) + 2m.$$

Proof. This is a direct consequence of Proposition A.1. □

A.2. Binary relations

Let $\bullet \in \{\mathrm{Li}, \zeta\}$. For each $d \in \mathbb{Z}$ and $R = (P, Q) \in \mathcal{H}^{\oplus 2}$, we define

$$\mathscr{L}^{\bullet}(R) := \mathscr{L}^{\bullet}(P) + \mathscr{L}^{\bullet}(Q) \ \text{ and } \ \mathscr{L}^{\bullet}_d(R) := \mathscr{L}^{\bullet}_d(P) + \mathscr{L}^{\bullet}_{d+1}(Q).$$

We set

$$\mathcal{P}^{\bullet} := \{(P, Q) \in \mathcal{H}^{\oplus 2} \mid \mathscr{L}^{\bullet}_d(P) + \mathscr{L}^{\bullet}_{d+1}(Q) = 0 \text{ forall } d \in \mathbb{Z}\} \tag{A.1}$$

and

$$\mathcal{P}^{\bullet}_w := \mathcal{P}^{\bullet} \cap \mathcal{H}^{\oplus 2}_w \tag{A.2}$$

for $w \geq 0$. Elements in $\mathcal{P}^{\bullet}$ are called *binary relations*. We note that each binary relation $R = (P, Q) \in \mathcal{P}^{\bullet}$ induces a k-linear relation $\mathscr{L}^{\bullet}(R) = \mathscr{L}^{\bullet}(P) + \mathscr{L}^{\bullet}(Q) = 0$. Indeed, we have

$$\mathscr{L}^{\bullet}(P) + \mathscr{L}^{\bullet}(Q) = \sum_{d \in \mathbb{Z}} \mathscr{L}^{\bullet}_d(P) + \sum_{d \in \mathbb{Z}} \mathscr{L}^{\bullet}_{d+1}(Q) = \sum_{d \in \mathbb{Z}} (\mathscr{L}^{\bullet}_d(P) + \mathscr{L}^{\bullet}_{d+1}(Q)) = 0.$$

The above ideas were introduced by Todd [30].

Example A.3. Thakur proved the identity [33, Thm. 5]

$$\mathscr{L}^{\zeta}_d(q) - L_1 \mathscr{L}^{\zeta}_{d+1}(1, q-1) = 0 \quad (d \in \mathbb{Z}).$$

Note that by Remark 2.5 we can replace 'ζ' by 'Li' in the above equation. It follows that for $\bullet \in \{\mathrm{Li}, \zeta\}$, we have a binary relation

$$R_1 := ([q],\ -L_1[1, q-1]) \in \mathcal{P}^{\bullet}_q; \tag{A.3}$$

namely, for each $d \in \mathbb{Z}$, the following equation holds:

$$\mathscr{L}^{\bullet}_d(q) - L_1 \mathscr{L}^{\bullet}_{d+1}(1, q-1) = 0.$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

A.3. Maps between relations

In what follows, the essential ideas we use are rooted in [30, 26]. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}_{>0}$ be a nonempty index. We recall that $\mathfrak{s}_+ := (s_1, \dots, s_{r-1})$ and $\mathfrak{s}_- := (s_2, \dots, s_r)$, and the operator $\boxplus : \mathcal{H}^{\oplus 2} \to \mathcal{H}$ is defined in Definition 3.1, and the notations $D^{\mathrm{Li}}_{\mathfrak{s},\mathfrak{n}}$ in (2.8) and $D^{\zeta}_{\mathfrak{s},\mathfrak{n}}$ in (2.10). For $\bullet \in \{\mathrm{Li}, \zeta\}$, we define the maps $\mathscr{B}^{\bullet}_{\mathfrak{s}}, \mathscr{C}^{\bullet}_{\mathfrak{s}} : \mathcal{H}^{\oplus 2}_{>0} \to \mathcal{H}^{\oplus 2}_{>0}$ as follows: For each $R = (P, Q) = (\sum_{\mathfrak{n} \in \mathcal{I}_{>0}} a_{\mathfrak{n}}[\mathfrak{n}],\ \sum_{\mathfrak{n} \in \mathcal{I}_{>0}} b_{\mathfrak{n}}[\mathfrak{n}]) \in \mathcal{H}^{\oplus 2}_{>0}$, we set

$$\begin{aligned}
\mathscr{B}^{\bullet}_{\mathfrak{s}}(R) &:= \Big([\mathfrak{s}, P] + [\mathfrak{s}, Q] + (\mathfrak{s} \boxplus Q) + [\mathfrak{s}_+, D^{\bullet}_{s_r, Q}],\ 0\Big), \\
\mathscr{C}^{\bullet}_{\mathfrak{s}}(R) &:= \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_{>0}} \big(a_{\mathfrak{n}}[n_1 + s_1, \mathfrak{n}_- *^{\bullet} \mathfrak{s}_-] + a_{\mathfrak{n}} D^{\bullet}_{\mathfrak{n},\mathfrak{s}},\ b_{\mathfrak{n}}[n_1, \mathfrak{n}_- *^{\bullet} \mathfrak{s}]\big),
\end{aligned}$$

where

$$D^{\bullet}_{s,Q} := \sum_{\mathfrak{n} \in \mathcal{I}_{>0}} b_{\mathfrak{n}} D^{\bullet}_{s,\mathfrak{n}}.$$

For any integer $m \geq 0$, we further define the map $\mathscr{B}\mathscr{C}^{\bullet,m}_q : \mathcal{H}^{\oplus 2}_{>0} \to \mathcal{H}^{\oplus 2}_{>0}$ given by

$$\mathscr{B}\mathscr{C}^{\bullet,m}_q(R) := \Big([q^{\{m\}}, P],\ L_1^m \alpha^{\bullet,m}_q(Q)\Big).$$

It is clear that these maps are k-linear endomorphisms on $\mathcal{H}^{\oplus 2}_{>0}$.

Proposition A.4. *Let $\bullet \in \{\mathrm{Li}, \zeta\}$. For each $\mathfrak{s} \in \mathcal{I}_{>0}$, and integers m, w satisfying $m \geq 0, w > 0$, the maps $\mathscr{B}^{\bullet}_{\mathfrak{s}}$, $\mathscr{C}^{\bullet}_{\mathfrak{s}}$ and $\mathscr{B}\mathscr{C}^{\bullet,m}_q$ satisfy*

$$\mathscr{B}^{\bullet}_{\mathfrak{s}}(\mathcal{P}^{\bullet}_w) \subset \mathcal{P}^{\bullet}_{w+\mathrm{wt}(\mathfrak{s})}, \quad \mathscr{C}^{\bullet}_{\mathfrak{s}}(\mathcal{P}^{\bullet}_w) \subset \mathcal{P}^{\bullet}_{w+\mathrm{wt}(\mathfrak{s})} \quad \text{and} \quad \mathscr{B}\mathscr{C}^{\bullet,m}_q(\mathcal{P}^{\bullet}_w) \subset \mathcal{P}^{\bullet}_{w+mq},$$

where $\mathcal{P}^{\bullet}_w$ is defined in (A.2).

Proof. When $R \in \mathcal{P}^{\bullet}_w$, it is clear that

$$\mathscr{B}^{\bullet}_{\mathfrak{s}}(R) \in \mathcal{H}^{\oplus 2}_{w+\mathrm{wt}(\mathfrak{s})}, \quad \mathscr{C}^{\bullet}_{\mathfrak{s}}(R) \in \mathcal{H}^{\oplus 2}_{w+\mathrm{wt}(\mathfrak{s})} \quad \text{and} \quad \mathscr{B}\mathscr{C}^{\bullet}_{\mathfrak{s}}(R) \in \mathcal{H}^{\oplus 2}_{w+\mathrm{wt}(\mathfrak{s})}.$$

Thus, it suffices to show that $\mathscr{B}^{\bullet}_{\mathfrak{s}}(R), \mathscr{C}^{\bullet}_{\mathfrak{s}}(R), \mathscr{B}\mathscr{C}^{\bullet,m}_q(R) \in \mathcal{P}^{\bullet}$.

For each $R = (\sum_{\mathfrak{n} \in \mathcal{I}_w} a_{\mathfrak{n}}[\mathfrak{n}],\ \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}}[\mathfrak{n}]) \in \mathcal{P}^{\bullet}_w$, the corresponding equalities

$$\sum_{\mathfrak{n} \in \mathcal{I}_w} a_{\mathfrak{n}} \mathscr{L}^{\bullet}_i(\mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_{i+1}(\mathfrak{n}) = 0 \quad (i \in \mathbb{Z})$$

hold. Thus, for each $s \geq 1$ and $d \in \mathbb{Z}$, we have

$$\begin{aligned}
0 &= \mathscr{L}^{\bullet}_d(s) \sum_{i<d} \left(\sum_{\mathfrak{n} \in \mathcal{I}_w} a_{\mathfrak{n}} \mathscr{L}^{\bullet}_i(\mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_{i+1}(\mathfrak{n}) \right) \\
&= \sum_{\mathfrak{n} \in \mathcal{I}_w} a_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s) \mathscr{L}^{\bullet}_{<d}(\mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s) \mathscr{L}^{\bullet}_{<d}(\mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s) \mathscr{L}^{\bullet}_d(\mathfrak{n}) \\
&= \sum_{\mathfrak{n} \in \mathcal{I}_w} a_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s, \mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s, \mathfrak{n}) + \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(s + n_1, \mathfrak{n}_-) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_{\mathfrak{n}} \mathscr{L}^{\bullet}_d(D^{\bullet}_{s,\mathfrak{n}}) \\
&= \mathscr{L}^{\bullet}_d(\mathscr{B}^{\bullet}_s(R)),
\end{aligned}$$

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

where the third equality comes from Proposition 2.7. This means that $\mathscr{B}_s^\bullet(R) \in \mathcal{P}^\bullet$. Note that for each $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}_{>0}$, we have $\mathscr{B}_\mathfrak{s}^\bullet(R) \in \mathcal{P}^\bullet$ since from the definition of $\mathscr{B}_\mathfrak{s}^\bullet$ we have

$$\mathscr{B}_\mathfrak{s}^\bullet = \mathscr{B}_{s_1}^\bullet \circ \cdots \circ \mathscr{B}_{s_r}^\bullet.$$

Similarly, we have

$$\begin{aligned}
0 &= \left(\sum_{\mathfrak{n} \in \mathcal{I}_w} a_\mathfrak{n} \mathscr{L}_d^\bullet(\mathfrak{n}) + \sum_{\mathfrak{n} \in \mathcal{I}_w} b_\mathfrak{n} \mathscr{L}_{d+1}^\bullet(\mathfrak{n}) \right) \mathscr{L}_{<d+1}^\bullet(\mathfrak{s}) \\
&= \sum_{\mathfrak{n} \in \mathcal{I}_w} a_\mathfrak{n} \mathscr{L}_d^\bullet(\mathfrak{n}) \mathscr{L}_d^\bullet(\mathfrak{s}) + \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} a_\mathfrak{n} \mathscr{L}_d^\bullet(n_1) \mathscr{L}_{<d}^\bullet(\mathfrak{n}_-) \mathscr{L}_{<d}^\bullet(\mathfrak{s}) \\
&\quad + \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} b_\mathfrak{n} \mathscr{L}_{d+1}^\bullet(n_1) \mathscr{L}_{<d+1}^\bullet(\mathfrak{n}_-) \mathscr{L}_{<d+1}^\bullet(\mathfrak{s}) \\
&= \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} a_\mathfrak{n} \left(\mathscr{L}_d^\bullet([n_1 + s_1, \mathfrak{n}_- *^\bullet \mathfrak{s}_-]) + \mathscr{L}_d^\bullet(D_{\mathfrak{n},\mathfrak{s}}^\bullet) \right) + \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} a_\mathfrak{n} \mathscr{L}_d^\bullet([n_1, \mathfrak{n}_- *^\bullet \mathfrak{s}]) \\
&\quad + \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} b_\mathfrak{n} \mathscr{L}_{d+1}^\bullet([n_1, \mathfrak{n}_- *^\bullet \mathfrak{s}]) \\
&= \mathscr{L}_d^\bullet(\mathscr{C}_\mathfrak{s}^\bullet(R)),
\end{aligned}$$

where the third equality comes from Proposition 2.7. Thus, we have $\mathscr{C}_\mathfrak{s}^\bullet(R) \in \mathcal{P}^\bullet$.

Finally, we have

$$\begin{aligned}
\mathcal{P}^\bullet &\ni \mathscr{B}_q^\bullet(R) - \sum_{\mathfrak{n} \in \mathcal{I}_w} b_\mathfrak{n} \mathscr{C}_\mathfrak{n}^\bullet(R_1) \\
&= \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} \left(a_\mathfrak{n}[q, \mathfrak{n}] + b_\mathfrak{n}[q, \mathfrak{n}] + b_\mathfrak{n}[q + n_1, \mathfrak{n}_-] + b_\mathfrak{n} D_{q,\mathfrak{n}}^\bullet,\ 0 \right) \\
&\quad - \sum_{\mathfrak{n} = (n_1, \mathfrak{n}_-) \in \mathcal{I}_w} b_\mathfrak{n} \left(\left([q + n_1, \mathfrak{n}_-] + [q, \mathfrak{n}] + D_{q,\mathfrak{n}}^\bullet \right),\ -L_1[1, (q-1) *^\bullet \mathfrak{n}] \right) \\
&= \sum_{\mathfrak{n} \in \mathcal{I}_w} \left(a_\mathfrak{n}[q, \mathfrak{n}],\ b_\mathfrak{n} L_1 \alpha_q^\bullet(\mathfrak{n}) \right) \\
&= \mathscr{B}\mathscr{C}_q^\bullet(R),
\end{aligned}$$

where the second equality comes from the definition of $\alpha_q^\bullet$ given in (3.3). Since

$$\mathscr{B}\mathscr{C}_q^{\bullet,m} = \mathscr{B}\mathscr{C}_q^\bullet \circ \cdots \circ \mathscr{B}\mathscr{C}_q^\bullet \quad (m\text{th iterate of } \mathscr{B}\mathscr{C}_q^\bullet),$$

it shows that $\mathscr{B}\mathscr{C}_q^{\bullet,m}(R) \in \mathcal{P}^\bullet$. □

Let $\mathfrak{s} \in \mathcal{I}$. We write $\mathfrak{s} = (s_1, \dots) = (\mathfrak{s}^\mathrm{T}, q^{\{m\}}, \mathfrak{s}')$ with $\mathfrak{s}^\mathrm{T} \in \mathcal{I}^\mathrm{T}$, $m \geq 0$ and $\mathfrak{s}' = (s_1', \dots)$ with $s_1' > q$ or $\mathfrak{s}' = \emptyset$, and set $\mathrm{Init}(\mathfrak{s}) := (\mathfrak{s}^\mathrm{T}, q^{\{m\}})$ and $\ell_1 := \mathrm{dep}(\mathrm{Init}(\mathfrak{s}))$. When $\mathfrak{s}' \neq \emptyset$, we set

$$\mathfrak{s}'' := (s_1' - q, \mathfrak{s}_-') = (s_{\ell_1+1} - q, \mathfrak{s}_-').$$

We recall $\mathscr{U}^\bullet(\mathfrak{s})$ given in Definition 3.4 by

$$\mathscr{U}^\bullet(\mathfrak{s}) := \begin{cases} -[\mathfrak{s}^\mathrm{T}, q^{\{m+1\}}, \mathfrak{s}''] + L_1^{m+1}[\mathfrak{s}^\mathrm{T}, \alpha_q^{\bullet,m+1}(\mathfrak{s}'')] \\ \qquad\qquad + L_1^{m+1}(\mathfrak{s}^\mathrm{T} \boxplus \alpha_q^{\bullet,m+1}(\mathfrak{s}'')) - [\mathfrak{s}^\mathrm{T}, q^{\{m\}}, D_{q,\mathfrak{s}''}^\bullet] & (\mathfrak{s}' \neq \emptyset) \\ L_1^m[\mathfrak{s}^\mathrm{T}, \alpha_q^{\bullet,m}(\emptyset)] + L_1^m(\mathfrak{s}^\mathrm{T} \boxplus \alpha_q^{\bullet,m}(\emptyset)) & (\mathfrak{s}' = \emptyset) \end{cases}.$$

It is clear that $\mathscr{U}^\bullet(\mathfrak{s}) \in \mathcal{H}_{\mathrm{wt}(\mathfrak{s})}$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Theorem A.5 (Theorem 3.6 (1)). *For each $P \in \mathcal{H}$, we have $\mathscr{L}^\bullet(\mathscr{U}^\bullet(P)) = \mathscr{L}^\bullet(P)$.*

Proof. We may assume that $P = \mathfrak{s} \in \mathcal{I}$ as above. By the definition of $\mathscr{U}^\bullet$, it suffices to show the desired result in the case $\mathfrak{s} \notin \mathcal{I}^{\mathrm{T}}$. We first consider the case that $\mathfrak{s}' \neq \emptyset$. Recall that R_1 is given in (A.3) and $\alpha_q^\bullet$ is defined in (3.3). Note that by definition, we have

$$\begin{aligned}\mathscr{B}\mathscr{C}_q^{\bullet,m}\left((\mathscr{C}_{\mathfrak{s}''}^\bullet(R_1))\right) &= \mathscr{B}\mathscr{C}_q^{\bullet,m}\left([\mathfrak{s}'] + [q, \mathfrak{s}''] + D_{q,\mathfrak{s}''}^\bullet,\ -L_1\alpha_q^\bullet(\mathfrak{s}'')\right)\\ &= \left([q^{\{m\}}, \mathfrak{s}'] + [q^{\{m+1\}}, \mathfrak{s}''] + [q^{\{m\}}, D_{q,\mathfrak{s}''}^\bullet],\ -L_1^{m+1}\alpha_q^{\bullet,m+1}(\mathfrak{s}'')\right).\end{aligned}$$

If $\mathfrak{s}^{\mathrm{T}} = \emptyset$, we can express

$$\mathscr{B}\mathscr{C}_q^{\bullet,m}(\mathscr{C}_{\mathfrak{s}''}^\bullet(R_1)) = \left([\mathfrak{s}] + [\mathfrak{s}^{\mathrm{T}}, q^{\{m+1\}}, \mathfrak{s}''] + [\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, D_{q,\mathfrak{s}''}^\bullet],\ -L_1^{m+1}[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m+1}(\mathfrak{s}'')]\right). \tag{A.4}$$

If $\mathfrak{s}^{\mathrm{T}} \neq \emptyset$, we have

$$\begin{aligned}&\mathscr{B}_{\mathfrak{s}^{\mathrm{T}}}^\bullet(\mathscr{B}\mathscr{C}_q^{\bullet,m}(\mathscr{C}_{\mathfrak{s}''}^\bullet(R_1))) \qquad (\text{A.5})\\ &= \left([\mathfrak{s}] + [\mathfrak{s}^{\mathrm{T}}, q^{\{m+1\}}, \mathfrak{s}''] + [\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, D_{q,\mathfrak{s}''}^\bullet] - L_1^{m+1}[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m+1}(\mathfrak{s}'')] - L_1^{m+1}[\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m+1}(\mathfrak{s}'')],\ 0\right),\end{aligned}$$

where we use Remark 2.6 and the fact $[\mathfrak{s}_+^{\mathrm{T}}, 0] = 0 \in \mathcal{H}$ from Remark 2.3.

Next, we suppose that $\mathfrak{s}' = \emptyset$. Then $m \geq 1$ and we have

$$\mathscr{B}\mathscr{C}_q^{\bullet,m-1}(R_1) = \left([q^{\{m\}}],\ L_1^{m-1}\alpha_q^{\bullet,m-1}(-L_1[1, q-1])\right) = \left([q^{\{m\}}],\ -L_1^m\alpha_q^{\bullet,m}(\emptyset)\right).$$

If $\mathfrak{s}^{\mathrm{T}} = \emptyset$, then we have

$$\mathscr{B}\mathscr{C}_q^{\bullet,m-1}(R_1) = \left([\mathfrak{s}],\ -L_1^m[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m}(\emptyset)]\right), \tag{A.6}$$

and if $\mathfrak{s}^{\mathrm{T}} \neq \emptyset$, then by Remark 2.6 and $[\mathfrak{s}_+^{\mathrm{T}}, 0] = 0 \in \mathcal{H}$, we have

$$\mathscr{B}_{\mathfrak{s}^{\mathrm{T}}}^\bullet(\mathscr{B}\mathscr{C}_q^{\bullet,m-1}(R_1)) = \left([\mathfrak{s}] - L_1^m[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m}(\emptyset)] - L_1^m(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m}(\emptyset)),\ 0\right). \tag{A.7}$$

Recall from (A.3) that $R_1 \in \mathcal{P}_q^\bullet$. It follows by Theorem A.4 that the left-hand side (LHS) of each of the equations (A.4), (A.5), (A.6) and (A.7) belongs to $\mathcal{P}_{\mathrm{wt}(\mathfrak{s})}^\bullet$. Thus, in any case of the four equations above, we have that

$$0 = \mathscr{L}^\bullet(\mathrm{LHS}) = \mathscr{L}^\bullet(\mathrm{RHS}) = \mathscr{L}^\bullet(\mathfrak{s}) - \mathscr{L}^\bullet(\mathscr{U}^\bullet(\mathfrak{s})),$$

whence we obtain

$$\mathscr{L}^\bullet(\mathfrak{s}) = \mathscr{L}^\bullet(\mathscr{U}^\bullet(\mathfrak{s})).$$

□

We set

$$\mathscr{U}_1^\bullet(\mathfrak{s}) := \begin{cases} -[\mathfrak{s}^{\mathrm{T}}, q^{\{m+1\}}, \mathfrak{s}''] + L_1^{m+1}[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m+1}(\mathfrak{s}'')] & (\mathfrak{s}' \neq \emptyset)\\ L_1^m[\mathfrak{s}^{\mathrm{T}}, \alpha_q^{\bullet,m}(\emptyset)] & (\mathfrak{s}' = \emptyset)\end{cases},$$

$$\mathscr{U}_2^\bullet(\mathfrak{s}) := \begin{cases} L_1^{m+1}(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m+1}(\mathfrak{s}'')) & (\mathfrak{s}' \neq \emptyset)\\ L_1^m(\mathfrak{s}^{\mathrm{T}} \boxplus \alpha_q^{\bullet,m}(\emptyset)) & (\mathfrak{s}' = \emptyset)\end{cases},$$

$$\mathscr{U}_3^\bullet(\mathfrak{s}) := \begin{cases} -[\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, D_{q,\mathfrak{s}''}^\bullet] & (\mathfrak{s}' \neq \emptyset)\\ 0 & (\mathfrak{s}' = \emptyset)\end{cases}.$$

It is clear that $\mathscr{U}_i^\bullet(\mathfrak{s}) \in \mathcal{H}_{\mathrm{wt}(\mathfrak{s})}$ for $1 \leq i \leq 3$ and $\mathscr{U}^\bullet(\mathfrak{s}) = \mathscr{U}_1^\bullet(\mathfrak{s}) + \mathscr{U}_2^\bullet(\mathfrak{s}) + \mathscr{U}_3^\bullet(\mathfrak{s})$.

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press

Lemma A.6. *Let $\mathfrak{s} \in \mathcal{I} \setminus \mathcal{I}^{\mathrm{T}}$. Then the following hold:*

(1) *If* $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_1^{\bullet}(\mathfrak{s}))$, *then* $\mathrm{dep}(\mathfrak{n}) > \mathrm{dep}(\mathfrak{s})$.
(2) *If* $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_2^{\bullet}(\mathfrak{s}))$, *then* $\mathrm{dep}(\mathfrak{n}) \geq \mathrm{dep}(\mathfrak{s})$ *and* $\mathrm{Init}(\mathfrak{n}) > \mathrm{Init}(\mathfrak{s})$ *(using lexicographical order).*
(3) *If* $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_3^{\bullet}(\mathfrak{s}))$, *then* $\mathrm{dep}(\mathfrak{n}) \geq \mathrm{dep}(\mathfrak{s}) > \ell_1 := \mathrm{dep}(\mathrm{Init}(\mathfrak{s}))$, $\mathrm{Init}(\mathfrak{n}) \geq \mathrm{Init}(\mathfrak{s})$ *and* $1 \leq n_{\ell_1+1} < s_{\ell_1+1}$.

Proof. First, we note that since $\mathfrak{s} \notin \mathcal{I}^{\mathrm{T}}$, if $\mathfrak{s}' = \emptyset$, then $m \geq 1$. By Corollary A.2, we have $\mathrm{dep}(\mathfrak{n}) \geq \mathrm{dep}(\mathfrak{s})$ for all $1 \leq i \leq 3$ and $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_i^{\bullet}(\mathfrak{s}))$ and $\mathrm{dep}(\mathfrak{n}) > \mathrm{dep}(\mathfrak{s})$ for all $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_1^{\bullet}(\mathfrak{s}))$.

By the definition of $\alpha_q^{\bullet}$, when $\mathfrak{n} \in \mathrm{Supp}(\mathcal{U}_2^{\bullet}(\mathfrak{s}))$, we can write $\mathfrak{n} = \mathfrak{s}^{\mathrm{T}} \boxplus [1, \mathfrak{n}_0]$ for some $\mathfrak{n}_0 \in \mathcal{I}$. We may assume $\mathfrak{s}^{\mathrm{T}} \neq \emptyset$ since if $\mathfrak{s}^{\mathrm{T}} = \emptyset$, we have $\mathcal{U}_2^{\bullet}(\mathfrak{s}) = 0$ by the definition of $\boxplus$. Let $j := \mathrm{dep}\,\mathfrak{s}^{\mathrm{T}}$ and so $s_j < q$. Since $s_j + 1 \leq q$, we have $\mathrm{Init}(\mathfrak{n}) = (s_1, \ldots, s_{j-1}, s_j + 1, \ldots) > (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}) = \mathrm{Init}(\mathfrak{s})$.

Let $\mathfrak{n} = (n_1, \ldots) \in \mathrm{Supp}(\mathcal{U}_3^{\bullet}(\mathfrak{s}))$. We may assume that $\mathfrak{s}' \neq \emptyset$ and $\bullet = \zeta$. In particular, $\mathrm{dep}(\mathfrak{s}) > \ell_1$. Note that any $\mathfrak{n}^{\dagger} \in \mathrm{Supp}(D_{q,\mathfrak{s}''}^{\zeta})$ can be written as $\mathfrak{n}^{\dagger} = (s_{\ell_1+1} - i, \mathfrak{n}_-^{\dagger})$ for some $1 \leq i < s_{\ell_1+1}$ and $\mathfrak{n}_-^{\dagger} \in \mathcal{I}$. Then by the definition of $\mathcal{U}_3^{\bullet}(\mathfrak{s})$, we have $\mathfrak{n} = (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, \mathfrak{n}^{\dagger}) = (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, s_{\ell_1+1} - i, \mathfrak{n}_-^{\dagger})$. Thus, $\mathrm{Init}(\mathfrak{n}) = (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}, \ldots) \geq (\mathfrak{s}^{\mathrm{T}}, q^{\{m\}}) = \mathrm{Init}(\mathfrak{s})$ and $n_{\ell_1+1} = s_{\ell_1+1} - i < s_{\ell_1+1}$. □

Theorem A.7 (Theorem 3.6 (2)). *For each $P \in \mathcal{H}$, there exists $e \geq 0$ such that* $\mathrm{Supp}((\mathcal{U}^{\bullet})^e(P)) \subset \mathcal{I}^{\mathrm{T}}$, *where $(\mathcal{U}^{\bullet})^0$ is defined to be the identity map and for $e \in \mathbb{Z}_{>0}$, $(\mathcal{U}^{\bullet})^e$ is defined to be the e-th iteration of $\mathcal{U}^{\bullet}$.*

Proof. We may assume that $P = \mathfrak{s} = (s_1, \ldots) \in \mathcal{I}_w$ for some $w \geq 0$. By Lemma A.6, for each index $\mathfrak{n} = (n_1, \ldots) \in \mathrm{Supp}(\mathcal{U}^{\bullet}(\mathfrak{s}))$, one of the following conditions holds:

(1) $\mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}}$ (and hence $\mathfrak{n} = \mathfrak{s}$);
(2) $\mathfrak{s} \notin \mathcal{I}_w^{\mathrm{T}}$, $\mathrm{dep}(\mathfrak{n}) > \mathrm{dep}(\mathfrak{s})$;
(3) $\mathfrak{s} \notin \mathcal{I}_w^{\mathrm{T}}$, $\mathrm{dep}(\mathfrak{n}) = \mathrm{dep}(\mathfrak{s})$, $\mathrm{Init}(\mathfrak{n}) > \mathrm{Init}(\mathfrak{s})$ (using lexicographical order);
(4) $\mathfrak{s} \notin \mathcal{I}_w^{\mathrm{T}}$, $\mathrm{dep}(\mathfrak{n}) = \mathrm{dep}(\mathfrak{s})$, $\mathrm{Init}(\mathfrak{n}) = \mathrm{Init}(\mathfrak{s})$, $n_{\mathrm{dep}(\mathrm{Init}(\mathfrak{n}))+1} < s_{\mathrm{dep}(\mathrm{Init}(\mathfrak{s}))+1}$.

This means that $\mathfrak{n} = \mathfrak{s} \in \mathcal{I}_w^{\mathrm{T}}$ or

$$(\mathrm{dep}(\mathfrak{n}); \mathrm{Init}(\mathfrak{n}); -n_{\mathrm{dep}(\mathrm{Init}(\mathfrak{n}))+1}) > (\mathrm{dep}(\mathfrak{s}); \mathrm{Init}(\mathfrak{s}); -s_{\mathrm{dep}(\mathrm{Init}(\mathfrak{s}))+1})$$

in $\{0, 1, \ldots, w\} \times \mathrm{Init}(\mathcal{I}_w) \times \{-w, \ldots, -2, -1\}$ with the lexicographical order. Here, when $\mathrm{Init}(\mathfrak{s}) = \mathfrak{s}$ (resp. $\mathrm{Init}(\mathfrak{n}) = \mathfrak{n}$), temporarily we put $s_{\mathrm{dep}(\mathrm{Init}(\mathfrak{s}))+1} := 1$ (resp. $n_{\mathrm{dep}(\mathrm{Init}(\mathfrak{n}))+1} := 1$). Since this totally ordered set is finite, this procedure will stop after a finite number of steps. □

Acknowledgments. We are grateful to D. Brownawell, M. Papanikolas and J. Yu for many helpful discussions and their encouragement when writing up the manuscript. We thank the referee for careful reading and providing some helpful comments, which improve the presentation of this paper. We further thank NCTS and Kyushu University for their hospitality and partial financial support regarding our collaboration.

Competing interest. The authors have no competing interest to declare.

Funding statement. Chieh-Yu Chang was supported by NSTC Grant Number 111-2628-M-007-002-, and MOST Grant Number MOST Grant 107-2628-M-007-002-MY4. Yen-Tsung Chen was partially supported by MOST Grant Number 107-2628-M-007-002-MY4. Yoshinori Mishiba was supported by JSPS KAKENHI Grant Numbers JP18K13398 and JP23K03073.

References

[1] G. W. Anderson, 't-motives', *Duke Math. J.* **53**(2) (1986), 457–502.
[2] G. W. Anderson, W. D. Brownawell and M. A. Papanikolas, 'Determination of the algebraic relations among special Γ-values in positive characteristic', *Ann. of Math. (2)* **160**(1) (2004), 237–313.
[3] G. W. Anderson and D. S. Thakur, 'Tensor powers of the Carlitz module and zeta values', *Ann. of Math. (2)* **132**(1) (1990), 159–191.
[4] G. W. Anderson and D. S. Thakur, 'Multizeta values for F_q [t], their period interpretation, and relations between them', *Int. Math. Res. Not. IMRN* (**11**) (2009), 2038–2055.

[5] Y. André, *Une Introduction aux Motifs (Motifs Purs, Motifs Mixtes, Périodes), Panoramas et Synthèses* vol. 17 (Société Mathématique de France, Paris, 2004).

[6] F. Brown, 'Mixed Tate motives over Z', *Ann. of Math. (2)* **175**(2) (2012), 949–976.

[7] J. I. Burgos Gil and J. Fresán, 'Multiple zeta values: from numbers to motives', to appear in Clay Mathematics Proceedings.

[8] L. Carlitz, 'On certain functions connected with polynomials in a Galois field', *Duke Math. J.* **1**(2) (1935), 137–168.

[9] C.-Y. Chang, 'Linear independence of monomials of multizeta values in positive characteristic', *Compositio Math.* **150** (2014), 1789–1808.

[10] C.-Y. Chang, 'Linear relations among double zeta values in positive characteristic', *Camb. J. Math.* **4**(3) (2016), 89–331.

[11] C.-Y. Chang, Y.-T. Chen and Y. Mishiba, 'Algebra structure on multiple zeta values in positive characteristic', *Camb. J. Math.* **10**(4) (2022), 743–783.

[12] Y.-T. Chen and R. Harada, 'On lower bounds of the dimension of multizeta values in positive characteristic', *Doc. Math.* **26** (2021), 537–559.

[13] C.-Y. Chang and Y. Mishiba, 'On a conjecture of Furusho over function fields', *Invent. Math.* **223** (2021), 49–102.

[14] C.-Y. Chang, M. A. Papanikolas and J. Yu, 'An effective criterion for Eulerian multizeta values in positive characteristic', *J. Eur. Math. Soc. (JEMS)* **21**(2) (2019), 405–440.

[15] H.-J. Chen, 'On shuffle of double zeta values over $\mathrm{F}_q[\mathrm{t}]$', *J. Number Theory* **148** (2015), 153–163.

[16] P. Deligne and A. B. Goncharov, 'Groupes fondamentaux motiviques de Tate mixte', *Ann. Sci. École Norm. Sup. (4)* **38**(1) (2005), 1–56.

[17] H. Gangl, M. Kazeko and D. Zagier, 'Double zeta values and modular forms', in *Automorphic Forms and Zeta Functions* (World Sci. Publ., Hackensack, NJ, 2006), 71–106.

[18] A. B. Goncharov, 'Multiple polylogarithms and mixed Tate motives', https://arxiv.org/abs/math/0103059.

[19] D. Goss, 'L-adic zeta functions, v-series, and measures for function fields. With an addendum', *Invent. Math.* **55**(2) (1979), 107–119.

[20] D. Goss, *Basic Structures of Function Field Arithmetic* (Springer-Verlag, Berlin, 1996).

[21] M. Hoffman, 'The algebra of multiple harmonic series', *J. Algebra* **194** (1997), 477–495.

[22] B. H. Im, H. Kim, K. N. Le, T. Ngo Dac and L. H. Pham, 'Zagier-Hoffman's conjectures in positive characteristic', https://arxiv.org/abs/2205.07165.

[23] K. Ihara, M. Kaneko and D. Zagier, 'Derivation and double shuffle relations for multiple zeta values', *Compositio Math.* **142** (2006), 307–338.

[24] Y.-L. Kuan and Y.-H. Lin, 'Criterion for deciding zeta-like multizeta values in positive characteristic', *Exp. Math.* **25**(3) (2016), 246–256.

[25] J. A. Lara Rodríguez and D. S. Thakur, 'Zeta-like multizeta values for $\mathrm{F}_q[\mathrm{t}]$', *Indian J. Pure Appl. Math.* **45**(5) (2014), 785–798.

[26] T. Ngo Dac, 'On Zagier-Hoffman's conjectures in positive characteristic', *Ann. of Math. (2)* **194**(1) (2021), 361–392.

[27] M. A. Papanikolas, 'Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms', *Invent. Math.* **171**(1) (2008), 123–174.

[28] G. Racinet, 'Doubles mélanges des polylogarithmes multiples aux racines de l'unité', *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 185–231.

[29] T. Terasoma, 'Mixed Tate motives and multiple zeta values', *Invent. Math.* **149**(2) (2002), 339–369.

[30] G. Todd, 'A conjectural characterization for $\mathrm{F}_q(\mathrm{t})$-linear relations between multizeta values', *J. Number Theory* **187**, 264–287 (2018).

[31] D. S. Thakur, *Function Field Arithmetic* (World Scientific Publishing, River Edge NJ, 2004).

[32] D. S. Thakur, 'Power sums with applications to multizeta and zeta zero distribution for $\mathrm{F}_q[\mathrm{t}]$', *Finite Fields Appl.* **15**(4) (2009), 534–552.

[33] D. S. Thakur, 'Relations between multizeta values for $\mathrm{F}_q[\mathrm{t}]$', *Int. Math. Res. Not. IMRN* (**12**) (2009), 2318–2346.

[34] D. S. Thakur, 'Shuffle relations for function field multizeta values', *Int. Math. Res. Not. IMRN* (**11**) (2010), 1973–1980.

[35] D. S. Thakur, 'Multizeta values for function fields: a survey', *J. Théor. Nombres Bordeaux* **29**(3) (2017), 997–1023.

[36] D. Zagier, 'Values of zeta functions and their applications', in *ECM Volume (Progress in Math.)* vol. 120 (1994), 497–512.

[37] J. Zhao, *Multiple Zeta Functions, Multiple Polylogarithms and Their Special Values* (Series on Number Theory and Its Applications, 12) (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016).

https://doi.org/10.1017/fmp.2023.26 Published online by Cambridge University Press