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HILBERT RINGS ARISING AS PULLBACKS

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ABSTRACT. Let R be the pullback $A \times_C B$, where $B \longrightarrow C$ is a surjective homomorphism of commutative rings and A is a subring of C. It is shown that R and C are Hilbert rings if and only if A and B are Hilbert rings. Applications are given to the $D + XE[X]$, $D+M$, and $D+(X_1, \ldots, X_n)D_S[X_1, \ldots, X_n]$ constructions. For these constructions, new examples are given of Hilbert domains R which are unruly, in the sense that R is non-Noetherian and each of its maximal ideals is finitely generated. Related examples are also given.

1. **Introduction.** All rings considered in this paper are commutative with unit; all subrings and homomorphisms are unital. Consider a surjective ring-homomorphism $u: B \longrightarrow C$ and a subring A of C. Our interest here is to determine when the pullback $R = u^{-1}(A) = A \times_C B$ is an unruly Hilbert domain. (By definition, a ring D is a *Hilbert ring* if each prime ideal of D is an intersection of maximal ideals of D; Hilbert rings are also known as Jacobson rings [2]. Following [11], we say that a Hilbert domain D is *unruly* if D is non-Noetherian and each maximal ideal of D is finitely generated.) An unruly Hilbert domain was first constructed by Gilmer-Heinzer [9] in response to a question of Geramita. Recently, Mott-Zafrullah [11, Corollaries 6 and 7] used a particular pullback, the $D + XE[X]$ construction with E a field containing D, to provide a family of unruly Hilbert domains. Our main result, Theorem 3, asserts (using the above notation) that R and C are Hilbert rings if and only if A and B are Hilbert rings. As applications, we characterize the Hilbert property for the $D + XE[X]$ construction in Corollary 4 (which generalizes [11, Theorem 5]); for the $D+(X_1, \ldots, X_n)D_S[X_1, \ldots, X_n]$ construction (*cf.* [4], [6]) in Corollary 5 (which generalizes [1, Theorem 4.1]); and for the generalized $D + M$ construction (*cf.* [3]) in Corollary 6. Some unruly Hilbert domains arising from these constructions are identified in Corollaries 12 and 15. The complexity of characterizing the unruly property for pullbacks is addressed in Example 10.

If D is a ring, the set of prime (resp., maximal) ideals of D is denoted by Spec(D) (resp., Max(D)). For the proof of Theorem 3 and the alternate proof sketched in Remark 8, familiarity with gluing techniques, as in [5], [6], is assumed. Any unexplained material is standard, as in [2], [7].

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2. **Results.** Let us *fix notation* as follows: R denotes the pullback $R = u^{-1}(A) = A \times_C B$, where $u: B \longrightarrow C$ is a surjective ring-homomorphism and A is a subring of C. As usual, we may identify $C = B/I$, where $I = \ker(u)$, and so $A = R/I$. We first isolate two useful lemmas. Lemma 1 can also be proved directly, using [5, Theorem 1.4(c) and Corollary 1.5(3)].

LEMMA 1. *(a) If $P \in \mathrm{Spec}(R)$, then either $P = Q \cap R$ for some (unique) $Q \in \mathrm{Spec}(B)$ or $P = u^{-1}(Q)$ for some (unique) $Q \in \mathrm{Spec}(A)$.*

(b) $\mathrm{Max}(R) = \{u^{-1}(P) : P \in \mathrm{Max}(A)\} \cup \{M \cap R : M \in \mathrm{Max}(B), I \not\subset M\}$.

PROOF. According to [5, Theorem 1.4], $\mathrm{Spec}(R)$ is identified, up to homeomorphism, as a quotient space of the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. (In detail, if $I \subset Q \in \mathrm{Spec}(B)$, then Q is identified with an $A \cap (Q/I)$.) This homeomorphism entails a similar identification of $\mathrm{Spec}(R)$ as a partially ordered set (under inclusion), two consequences of which are the desired assertions. (*cf.* also [6, Proposition 1.2(c)].) ■

LEMMA 2. *Let $P \in \mathrm{Spec}(B)$ such that $I \not\subset P$. Suppose that there exist families $\{M_\alpha\} \subset \mathrm{Max}(B)$, $\{P_\beta\} \subset \mathrm{Max}(A)$ such that $I \not\subset M_\alpha$ for each α and $P \cap R = [\bigcap(M_\alpha \cap R)] \cap \left[\bigcap\left(u^{-1}(P_\beta)\right)\right]$. Then $P = \bigcap M_\alpha$.*

PROOF. First note that $\{M_\alpha\}$ is nonempty since $I \not\subset P$. To show that $P \subset \bigcap M_\alpha$, consider $x \in P$ and for each α, choose $t_\alpha \in I \setminus M_\alpha$. Then, for each α, $xt_\alpha \in P \cap I \subset P \cap R \subset M_\alpha \cap R \subset M_\alpha$. Since M_α is prime, $x \in M_\alpha$, as desired.

For the reverse inclusion, consider $y \in \bigcap M_\alpha$ and choose $t \in I \setminus P$. Then, for each α, we have $yt \in M_\alpha \cap I \subset M_\alpha \cap R$; and for each β, we have $yt \in I \subset u^{-1}(P_\beta)$. Thus, $yt \in P \cap R \subset P$ and so, since P is prime, we have $y \in P$. ■

We proceed to characterize the rings of the title. In the proof of Theorem 3 (and several places later), we appeal to the easy fact that any homomorphic image of a Hilbert ring is itself a Hilbert ring: *cf.* [2, Example (3), page 351].

THEOREM 3. *R and C are Hilbert rings if and only if A and B are Hilbert rings.*

PROOF. Suppose that R and C are Hilbert rings. Since it is a homomorphic image of R, A is also a Hilbert ring. As for B, consider $P \in \mathrm{Spec}(B)$. If $I \subset P$ then, since $C = B/I$ is a Hilbert ring, P/I is an intersection of maximal ideals of C, whence P is an intersection of maximal ideals of B. Thus, without loss of generality, $I \not\subset P$. Since R is a Hilbert ring, Lemma 1 provides families $\{M_\alpha\} \subset \mathrm{Max}(B)$, $\{P_\beta\} \subset \mathrm{Max}(A)$ such that $I \not\subset M_\alpha$ for each α and $P \cap R = [\bigcap(M_\alpha \cap R)] \cap \left[\bigcap\left(u^{-1}(P_\beta)\right)\right]$. Then, by Lemma 2, $P = \bigcap M_\alpha$, and so B is a Hilbert ring.

Conversely, suppose that A and B are Hilbert rings. Then so is $B/I = C$. As for R, consider $P \in \mathrm{Spec}(R)$. If $I \subset P$ then, since $R/I = A$ is a Hilbert ring, P/I is an intersection of maximal ideals of A, and so P is an intersection of maximal ideals of R. Thus, without loss of generality, $I \not\subset P$. By Lemma 1(a), $P = Q \cap R$ for some (unique) $Q \in \mathrm{Spec}(B)$. Since B is a Hilbert ring, $Q = (\bigcap N_i) \cap (\bigcap W_j)$, where the maximal ideals being intersected have been labelled so that $I \not\subset N_i$ for each i and $I \subset W_j$ for each j.

Intersecting with R leads to $P = \left(\bigcap(N_i \cap R)\right) \cap \left(\bigcap(W_j \cap R)\right)$. As noted in Lemma 1, $N_i \cap R \in \mathrm{Max}(R)$ for each i, and so it suffices to show that each $V_j = W_j \cap R$ is an intersection of maximal ideals of R. However, this follows since A is a Hilbert ring; indeed, $V_j/I = \bigcap(V_{jk}/I)$, with $\{V_{jk}\}$ a family of maximal ideals containing I, entails $V_j = \bigcap V_{jk}$. ∎

We proceed to three applications of Theorem 3. In the first two of these, we consider polynomial rings in finitely many algebraically independent indeterminates. The restriction to finitely many variables is required by the following facts. A ring D is a Hilbert ring if and only if $D[X]$ is a Hilbert ring (and, hence, if and only if $D[X_1, \ldots, X_n]$ is a Hilbert ring for each positive integer n): *cf.* [7, Theorem 31.8]. However, if the set of variables $\{X_i\}$ is denumerable, then $\mathbb{Q}[\{X_i\}]$ is not a Hilbert ring (although $\mathbb{Q}$ is trivially a Hilbert ring): *cf.* [2, Exercise 10(d), page 373], [7, Exercise 14, page 389], [8], [10].

We begin the applications of Theorem 3 with a generalization of [11, Theorem 5]. The latter result established the "if" assertion of Corollary 4 for the special case in which $n = 1$ and E is a field.

COROLLARY 4. *Let $X_1, \ldots, X_n$ be finitely many algebraically independent indeterminates over a ring E, and let D be a subring of E. Then $R = D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ is a Hilbert ring if and only if D and E are Hilbert rings.*

PROOF. $R = u^{-1}(A) = A \times_C B$, where $B = E[X_1, \ldots, X_n]$, $C = E$, $A = D$, and $u\colon B \longrightarrow C$ is the surjective E-algebra homomorphism sending each X_i to 0. By Theorem 3, R and E are Hilbert rings if and only if D and $E[X_1, \ldots, X_n]$ are Hilbert rings; that is, by the above remarks, if and only if D and E are Hilbert rings. Therefore, it suffices to show that if R is a Hilbert ring, then so is E. This, in turn follows from the fact that E is a homomorphic image of R, by virtue of the map $R \longrightarrow E, f \longmapsto f(1, \ldots, 1)$. ∎

[1, Theorem 4.1] established the special case of Corollary 5 in which $n = 1$, D is an integral domain, and $S \cap P \neq \emptyset$ entails $S \cap Q \neq \emptyset$ for nonzero prime ideals $Q \subset P$ of D. The case in which $n = 1$ and D is an integral domain was conjectured in [1, page 119].

COROLLARY 5. *Let $X_1, \ldots, X_n$ be finitely many algebraically independent indeterminates over a ring D, and let S be a multiplicatively closed subset consisting of (some) non-zerodivisors of D. Then $D + (X_1, \ldots, X_n)D_S[X_1, \ldots, X_n]$ is a Hilbert ring if and only if D and D_S are Hilbert rings.*

PROOF. Put $E = D_S$ in Corollary 4. ∎

COROLLARY 6. *Let $B = K + M$ be a ring, where K is a field and $M \in$* Max(B)*; let D be a subring of K. Then $R = D + M$ is a Hilbert ring if and only if D and B are Hilbert rings.*

PROOF. $R = u^{-1}(A) = A \times_C B$, where $C = K$, $A = D$, and $u\colon B \longrightarrow C$ is the surjective ring-homomorphism, $k + m \longmapsto k$, for $k \in K$, $m \in M$. The assertion now follows from Theorem 3 since $C = K$, being a field, is trivially a Hilbert ring. ∎

It is perhaps somewhat surprising that, in the general context of Theorem 3, R being a Hilbert ring is not equivalent to A and B being Hilbert rings. (By Theorem 3, if C is a Hilbert ring, for instance, a field, then R is a Hilbert ring if and only if A and B are Hilbert rings.) The next example illustrates this by emphasizing the role of C.

EXAMPLE 7. *In the general pullback context of Theorem 3, R being a Hilbert ring does not imply that B is a Hilbert ring.*

PROOF. Let X be an indeterminate over $\mathbb{Q}$, put $B = \mathbb{Z}_{2\mathbb{Z}} + X\mathbb{Q}[X]$, $C = \mathbb{Z}_{2\mathbb{Z}}$, $A = \mathbb{Z}$, and $u: B \longrightarrow C$ the C-algebra map sending X to 0. Then $R = u^{-1}(A) = A \times_C B = \mathbb{Z} + X\mathbb{Q}[X]$ is a Hilbert ring, by Corollary 6. (This result applies since $\mathbb{Z}$ and $\mathbb{Q}[X]$, being PIDs with infinitely many prime ideals, are Hilbert rings.) However, B is not a Hilbert ring since its homomorphic image C is (quasilocal and hence) not a Hilbert ring. ∎

We pause next to sketch a proof of Theorem 3 that avoids the use of Lemma 2.

REMARK 8. By extensive use of [5, Theorem 1.4], it is possible to avoid appealing to Lemma 2 in the proof of Theorem 3. The issue is to show that B is a Hilbert ring, given that R and C are Hilbert rings. As in the earlier proof, we consider $P \in \operatorname{Spec}(B)$ such that, without loss of generality, $I \not\subset P$. Put $p = P \cap R$. Since R is a Hilbert ring, $p = (\bigcap n_i) \cap (\bigcap w_j)$, where the maximal ideals being intersected have been labelled so that $I \not\subset n_i$ for each i and $I \subset w_j$ for each j. As noted in Lemma 1, for each i, there exists a unique $N_i \in \operatorname{Spec}(B)$ such that $N_i \cap R = n_i$ (and, hence, $I \not\subset N_i$). Moreover, by virtue of the identifications leading to the quotient space structure in [5, Theorem 1.4], we find that for each j, there exists $V_j \in \operatorname{Spec}(B)$ such that $p \subset V_j \cap R \subset w_j$ and $P \subset V_j$. (*cf.* [6, Proposition 1.2(c)].) In addition, $p \subset n_i$ entails $P \subset N_i$. Thus, $P \subset (\bigcap N_i) \cap (\bigcap V_j)$. As each V_j/I is an intersection of maximal ideals in the Hilbert ring $B/I = C$, V_j is an intersection of maximal ideals of B. Hence, it suffices to show that $(\bigcap N_i) \cap (\bigcap V_j) \subset P$. To this end, consider $x \in (\bigcap N_i) \cap (\bigcap V_j)$, and choose $y \in I \setminus P$. Since

$$xy \in \Big(\big(\bigcap N_i\big) \cap \big(\bigcap V_j\big)\Big) \cap R \subset \big(\bigcap n_i\big) \cap \big(\bigcap w_j\big) = p \subset P,$$

the primeness of P yields $x \in P$, as desired. ∎

The remainder of the paper is concerned with finding pullbacks which are unruly Hilbert domains, in the sense of Mott-Zafrullah [11]. In developing this material, we shall find it convenient to defer discussion of the "non-Noetherian" aspect of "unruly" to the corollaries. As in [9], [11], the unruly domains that are most accessible have the stronger property that each of their maximal ideals is principal. Proposition 9 studies the impact of assuming that R has this stronger property.

In several proofs below, we need the (easy) fact that the ring-theoretic property of having all maximal ideals being finitely generated (resp., principal) is preserved by homomorphic images.

PROPOSITION 9. *Suppose that each maximal ideal of R is finitely generated (resp., principal). Then each maximal ideal of A is finitely generated (resp., principal) and, for each* $N \in \mathrm{Max}(B)$ *such that* $I \not\subset N$, *N is finitely generated (resp., principal).*

PROOF. Since $A = R/I$ is a homomorphic image of R, the first assertion is immediate. Now, suppose that $N \in \mathrm{Max}(B)$ satisfies $I \not\subset N$. By Lemma 1, $M = N \cap R \in \mathrm{Max}(R)$. By hypothesis, $M = \Sigma R z_i$ for some finite (resp., singleton) set $\{z_i\}$. It suffices to show that $N = \Sigma B z_i$. One inclusion is clear. For the reverse inclusion, consider $x \in N$, and choose $y \in I \setminus N$. Since N is maximal, $N + By = B$, and so $n + by = 1$, for some $n \in N$, $b \in B$. It follows that $n \in N \cap (1 + I) \subset N \cap R = M$, whence $n = \Sigma r_i z_i$ for some $r_i \in R$. Then

$$x = x(n + by) = \Sigma x r_i z_i + xby \in \Sigma B z_i$$

since $xby \in N \cap I \subset M \subset \Sigma B z_i$. ∎

We next address some of the subtleties of characterizing the unruly property, by showing that the converses of the "finitely generated" and "principal" assertions of Proposition 9 both fail, even for Hilbert domains.

EXAMPLE 10. (a) Put $B = \mathbb{Q}[X]$, $I = (X^2 + 1)B$, $C = B/I$, $u: B \longrightarrow C$ the canonical surjection, $A = \mathbb{Z}$, and $R = u^{-1}(A) = A \times_C B$. Then $C = \mathbb{Q}(i)$ and $R = A + I$. Both A and B are Hilbert PIDs and, by Theorem 3, R is a Hilbert domain. However, Lemma 1(b) shows that $M = XB \cap R$ is in $\mathrm{Max}(R)$, and it can be shown by a degree argument (exploiting the fact that $X^2, X(X^2 + 1) \in M$) that M is not a principal ideal of R. Thus, the converse of the "principal" assertion of Proposition 9 is invalid.

(b) To show that the converse of the "finitely generated" assertion in Proposition 9 is invalid, we return to the $D + M$ context of Corollary 6. Put $B = \mathbb{Q}[X_1, X_2]$, $I = (X_1, X_2)B$, $C = B/I \cong \mathbb{Q}$, $u: B \longrightarrow C$ the canonical surjection, $A = \mathbb{Z}$, and $R = u^{-1}(A) = A \times_C B = \mathbb{Z} + I$. Both A and B are Noetherian Hilbert domains and, by Theorem 3, R is a Hilbert domain. By Lemma 1(b), $M = (X_1, X_2 + 1)B \cap R \in \mathrm{Max}(R)$; also, $\mathbb{Q}X_1 \subset M$. Then, essentially because $\mathbb{Q}$ is not a finitely generated abelian group, an easy degree argument shows that M is not a finitely generated ideal of R. ∎

Despite Example 10(b), we show that in the $D + M$ context of Corollary 6, we have the converse of the "principal" assertion of Proposition 9, except when D is a field.

THEOREM 11. *Let* $B = K + M$ *be an integral domain, where K is a field and* $M \in \mathrm{Max}(B)$. *Let D be a subring of K which is not a field, and put* $R = D + M$. *Then the following conditions are equivalent:*

(1) Each maximal ideal of D is principal and, for each $N \in \mathrm{Max}(B)$ *such that* $N \neq M$, *N is a principal ideal of B;*

(2) Each maximal ideal of R is principal.

PROOF. View R as a pullback, as in the proof of Corollary 6; in particular, $I = M$ and $A = D$. Then Proposition 9 immediately yields that (2) ⇒ (1). For the converse, assume (1), and consider $Q \in \mathrm{Max}(R)$. According to Lemma 1, there are two cases.

In the first case, $Q = u^{-1}(P) = P + M$ for some $P \in \mathrm{Max}(D)$. By (1), $P = Dy$ for some $y \in D$, and so $Q = P + M = Dy + My = Ry$, a principal ideal of R. (Note that $My = M$ since $y \neq 0$: this is the only place in the proof that we need the hypothesis that D is not a field.)

In the remaining case, $Q = N \cap R$ for some (uniquely determined) maximal ideal N of B such that $N \neq M$. By (1), N is principal. Since K is a field, we can write $N = B(1 + m)$ for some $m \in M$. It suffices to show that $Q = R(1 + m)$. One inclusion is clear. For the reverse inclusion, consider $x \in Q$. As $x \in N$, we have $x = b(1 + m)$ for some $b \in B$. Without loss of generality, b is not in M, and so $b = k + m_1$, with $0 \neq k \in K$ and $m_1 \in M$. Then $x = k + m_2$ for some $m_2 \in M$. As $x \in R = D + M$, directness of the sum yields $k \in D$, whence $b \in R$ and $x \in R(1 + m)$, as desired. ∎

According to [3, Theorem 4], the hypothesis that D is not a field in Theorem 11 ensures that $R = D + M$ is not Noetherian. We are thus ready to produce some new examples of unruly Hilbert domains.

COROLLARY 12. *Let $B = K + M$ be a Hilbert domain, where K is a field and $M \in$ $\mathrm{Max}(B)$. Let D be a Hilbert subring of K which is not a field. Assume that each maximal ideal of D is principal and, for each $N \in \mathrm{Max}(B)$ such that $N \neq M$, N is a principal ideal of B. Then $R = D + M$ is an unruly Hilbert domain; moreover, each maximal ideal of R is principal.*

PROOF. R is a Hilbert domain by Theorem 3, each maximal ideal of R is principal by Theorem 11, and R is non-Noetherian by the above remarks. Accordingly, R is unruly. ∎

EXAMPLE 13. *(a) In view of Proposition 9, Theorem 11 and Corollary 12, it is of interest to find a ring $K + M$ where each maximal ideal $N \neq M$ is principal, but M is not principal. One can show that $B = \mathbb{R} + X\mathbb{C}[X]$ is such an example. (To see this, note that if $0 \neq \alpha \in \mathbb{C}$, then $(X - \alpha)\mathbb{C}[X] \cap B = (\alpha^*\alpha - \alpha^* X)B$, where α^* is the complex conjugate of α.)*

(b) A trivial example of the phenomenon noted in (a) is provided by the ring $R =$ $F + XL[[X]]$, where $F \subset L$ are distinct fields. Indeed, R is quasilocal and its maximal ideal $XL[[X]]$ is nonprincipal. This example also shows that the hypothesis that D is not a field is needed for Corollary 12 and also needed for the implication (1) $\Rightarrow$ (2) in Theorem 11.

We close, in Corollary 15, by generalizing the families of unruly Hilbert domains found in [11, Corollaries 6 and 7]. First, we give a result which is set in the context of Corollary 4.

THEOREM 14. *Let $X_1, \ldots, X_n$ be finitely many ($n \geq 1$) algebraically independent indeterminates over an integral domain E; and let D be a subring of E which is not a field. Put $R = D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$. Then the following are equivalent:*

(1) Each maximal ideal of D is principal, E is a field, and $n = 1$;

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(2) Each maximal ideal of R is principal.

PROOF. Given (1), write the principal ideal domain $B = E[X_1] = E + M$, where $M = X_1B$. An application of Theorem 11 yields (2).

Conversely, assume (2). View R as a pullback as in the proof of Corollary 4; in particular, $B = E[X_1, X_2, \ldots, X_n]$ and $I = (X_1, X_2, \ldots, X_n)B$. As D is a homomorphic image of R, each maximal ideal of D is principal. Next, if E is not a field, choose nonzero $M \in \mathrm{Max}(E)$, and consider $N = (M, X_1 + 1, X_2, \ldots, X_n) \in \mathrm{Max}(B)$. As $I \not\subseteq N$, Lemma 1 gives $N \cap R \in \mathrm{Max}(R)$. By (2), $N \cap R$ is then principal. However, since $0 \neq m \in M$ leads to $mX_1, X_1 + 1 \in N \cap R$, a degree argument shows that $N \cap R$ cannot be a principal ideal, the desired contradiction. Therefore, E is a field. Another straightforward degree argument (focusing on X_2X_1, $X_1 + 1$) leads similarly to $n = 1$. ■

COROLLARY 15. *Let D be a subring of a field E. Put $R = D + XE[X]$. Then the following two conditions are equivalent:*

(1) D is a Hilbert domain, D is not a field, and each maximal ideal of D is finitely generated;

(2) R is an unruly Hilbert domain.

Moreover, if these (equivalent) conditions hold, then each maximal ideal of D is principal if and only if each maximal ideal of R is principal.

PROOF. (2) ⇒ (1): Assume (2). Since D is a homomorphic image of R, D is a Hilbert domain and each maximal ideal of D is finitely generated. It remains to prove that D is not a field. Suppose that D is a field. Then either $[E : D] < \infty$ or $[E : D] = \infty$. In the former case, R is Noetherian by [3, Theorem 4]. In the latter case, $XE[X]$ is a non-finitely generated maximal ideal of R. Thus, in either case, we have a contradiction to the "unruly" hypothesis, and so D is not a field.

(1) ⇒ (2): Assume (1). Then R is a Hilbert domain by Corollary 4, and R is non-Noetherian by [3, Theorem 4]. It remains only to show that each $Q \in \mathrm{Max}(R)$ is finitely generated. According to Lemma 1(b), there are two cases as to the form of Q. These are handled, *mutatis mutandis*, as in the final two paragraphs of the proof of Theorem 11.

Since D is a homomorphic image of R, the final assertion follows directly from either Theorem 11 or Theorem 14. ■

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