Bull. Aust. Math. Soc. **91** (2015), 92–103
doi:10.1017/S0004972714000641

ON *SI*-GROUPS

R. R. ANDRUSZKIEWICZ and M. WORONOWICZ✉

(Received 13 May 2014; accepted 19 July 2014; first published online 12 September 2014)

Abstract

This paper presents new results concerning the structure of *SI*-groups and refines and purifies the results obtained in this field by Shalom Feigelstock ['Additive groups of rings whose subrings are ideals', *Bull. Aust. Math. Soc.* **55** (1997), 477–481]. The structure theorem describing torsion-free *SI*-groups is proved in the associative case. Numerous examples of *SI*-groups are given. Some inconsistencies in Feigelstock's article are noted and corrected.

2010 *Mathematics subject classification*: primary 20K99; secondary 16D25.

Keywords and phrases: *SI*-groups, *H*-rings, ideals, torsion-free groups, associative ring multiplication.

1. Introduction

The inspiration to write this paper was some unfortunate wording in Feigelstock's interesting article entitled *Additive groups of rings whose subrings are ideals* [7]. In the introduction of that paper Feigelstock wrote: 'Rings are assumed to be associative, but need not possess a unity. Most of the results which will be obtained remain true for non-associative rings.' However, the proof of [7, Corollary 11] contradicts the declared assumption of that article. The author, performing an indirect proof, constructed a ring multiplication which is not associative (cf. Remark 4.4). Therefore, that proof shows the validity of [7, Corollary 11] for the class of additive groups of rings which need not be associative. Also, [7, Lemma 13] has been proven only in the general case (cf. Remark 4.6). We show that [7, Corollary 11 and Lemma 13] are false in the class of additive groups of associative rings. Moreover, we present new results on the structure of *SI*-groups in both the cases of associative and general rings. To avoid inconsistencies in the wording of the theorems, we introduce terminology and symbols allowing us to distinguish between the two cases. Where it does not lead to misunderstandings, we retain the notation introduced in [7].

We bring further clarification of [7, Theorem 10], provide numerous examples of associative rings in which every ring is an ideal and prove the structure theorem describing torsion-free *SI*-groups in the associative case. In addition, we give examples of mixed *SI*-groups for both the associative and general cases which are not given in [7].

Many algebraists have conducted research into various aspects of the structure of the additive groups of rings (see, for example, [1–4, 6, 7]).

Symbols $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{P}$ and $\mathbb{N}$ stand for the field of rationals, the ring of integers, the set of all prime numbers and the set of all natural numbers understood as the set of all positive integers, respectively. In this paper, only abelian groups with the traditional additive notation will be considered. By $Z(n)$, we denote the cyclic group of order n. The exponent of the abelian group A is denoted by $\omega(A)$. If $n \in \mathbb{N}$, we will use symbols $\oplus_n$ and $\odot_n$ to denote addition and multiplication in the ring $\mathbb{Z}_n$. The least common multiple of integers k and l is denoted by $\mathrm{LCM}\,(k, l)$. The symbol R^+ stands for the additive group of the ring R. The (two-sided) ideal I of a ring R is denoted by $I \lhd R$. The (two-sided) annihilator of a nonempty subset X of the ring R is denoted by $\mathfrak{a}(X)$. If a is an element of the ring R, then symbols $[a]$, (a), $\langle a\rangle$ and $o(a)$ stand for the ring generated by a, the ideal generated by a in R, the cyclic subgroup of the group R^+ generated by a and the order of a in the group R^+, respectively. If $\{A_i : i \in I\}$, where $I \neq \emptyset$, is a family of abelian groups and $x \in \bigoplus_{i\in I} A_i$, then the support of x is denoted by $\mathrm{supp}\,(x)$. If $i \in I$ and $\mathrm{supp}\,(x) \subseteq \{i\}$, then we will write x_i instead of x.

For background knowledge of divisible groups and the tensor product of abelian groups, we refer the reader to [8, 9].

2. Preliminaries

2.1. Definitions and notation. For an arbitrary abelian group A and a prime number p, we define a p-component A_p of the group A:

$$A_p = \{a \in A : p^n a = 0 \text{ for some } n \in \mathbb{N}\}.$$

Often we will use the designation

$$\mathbb{P}(A) = \{p \in \mathbb{P} : o(a) = p \text{ for some } a \in A\}.$$

The torsion part of A is denoted by $T(A)$. Of course, $T(A) = \bigoplus_{p\in\mathbb{P}(A)} A_p$.

Definition 2.1. Let $(A, +, 0)$ be an abelian group. An operation $*\colon A \times A \to A$ is called a ring multiplication if, for all $a, b, c \in A$,

$$a * (b + c) = a * b + a * c \quad \text{and} \quad (b + c) * a = b * a + c * a.$$

The algebraic system $(A, +, *, 0)$ is called a ring.

Definition 2.2. A ring in which every subring is an ideal (two-sided) is called an *SI*-ring. An abelian group A is called an *SI*-group if every ring R with $R^+ = A$ is an *SI*-ring. The associative *SI*-rings are called hamiltonian rings or H-rings, because these structures are somewhat analogous to the hamiltonian groups.

The H-rings play a significant role in ring theory and they were systematically studied by many authors. The most valuable results were achieved by Rédei and others (see [5, 10, 11]). This work motivates the following definition.

Definition 2.3. An abelian group A is called an SI_H-group if every associative ring R with $R^+ = A$ is an H-ring.

Remark 2.4. Every SI-group is an SI_H-group. Corollaries 11 and 12, Lemma 13 and Theorem 16 of [7] are proven for SI-groups in the sense of Definition 2.2 (cf. Remark 4.4, Corollaries 4.5 and 4.6 and Remark 4.7 in this paper). All other results obtained by Feigelstock in [7] are true both in the class of SI_H-groups and in the class of SI-groups. There exist mixed SI_H-groups which are not SI-groups (cf. Remark 4.3).

Definition 2.5. Let A be an abelian group. If on A there does not exist any nonzero ring multiplication, then A is called a *nil*-group. If on A there does not exist any nonzero associative ring multiplication, then we say that A is a nil_a-group.

Remark 2.6. Every *nil*-group is a nil_a-group. Every torsion nil_a-group is a *nil*-group, by [6, Theorem 3.4] and [9, Theorem 120.3]. By [6, Theorem 4.1], a mixed nil_a-group does not exist. It is easily seen that every ring multiplication on an arbitrary subgroup of the group $\mathbb{Q}^+$ is associative and every abelian torsion-free group of rank one can be embedded in the group $\mathbb{Q}^+$. Thus, the concepts of nil_a-group and *nil*-group are equivalent also in the class of abelian torsion-free groups of rank one. To the best of our knowledge, it is not known whether there exists a torsion-free nil_a-group A of rank more than one such that A is not a *nil*-group.

2.2. Ring multiplication on some specific abelian groups. Every abelian group $(A, +, 0)$ can be provided with a ring structure in a trivial way by defining $a \cdot b = 0$ for all $a, b \in A$; such a ring is called a null ring and it is denoted by A^0.

Remark 2.7. Let p be a prime number. It is a well-known fact that up to isomorphism there exist only two rings of cardinality p: the zero ring $Z(p)^0$ and the field $\mathbb{Z}_p$. So, if $s > 1$ is a square-free number, R is a ring with $R^+ = Z(s)$ and r is the product of all prime divisors p of s for which $R_p \cong \mathbb{Z}_p$, then the ring R satisfies the condition $R^n \cong \mathbb{Z}_r$ for all $n \in \mathbb{N}$ such that $n \geq 2$.

Proposition 2.8. *Let A and H be abelian groups such that $A = T(A)$, $\omega(A_p) < \infty$, $H = pH$ and $H_p = \{0\}$ for all $p \in \mathbb{P}(A)$. Then every ring multiplication $*$ on the group $G = A \oplus H$ satisfies the following conditions:*

(i) $A * H = H * A = \{0\}$;

(ii) $A * A \subseteq A$;

(iii) $H * H \subseteq H$.

Proof. (i) This follows from the p-divisibility of H for every $p \in \mathbb{P}(A)$, and the distributivity of multiplication with respect to addition. In fact, take any $a \in A$ and $h \in H$. Since $o(a) = n$, for some $n \in \mathbb{N}$, and $H = pH$, for every $p \in \mathbb{P}(A)$, we infer that there exists $h' \in H$ such that $h = nh'$. Thus, $a * h = a * (nh') = (na) * h' = 0 * h' = 0$ and analogously $h * a = 0$.

(ii) Take any $a_1, a_2 \in A$. Then $a_1 * a_2 = a_3 + h$ for some $a_3 \in A$, $h \in H$. Let $m = \mathrm{LCM}\,(o(a_1), o(a_2), o(a_3))$. Then $0 = m(a_1 * a_2) = m(a_3 + h) = mh$ and hence $h \in T(H)$.

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Let $p \in \mathbb{P}$. If $p \mid o(h)$, then $p \mid m$; thus, from the definition of m, it follows that $p \in \mathbb{P}(A)$. But $H_p = \{0\}$ for every $p \in \mathbb{P}(A)$ and hence $h = 0$.

(iii) Take any $h_1, h_2 \in H$. Then $h_1 * h_2 = a + h$ for some $a \in A$, $h \in H$. So, there exists a nonempty finite subset P of the set $\mathbb{P}(A)$ such that $a \in \bigoplus_{p\in P} A_p$. Moreover, $n(\bigoplus_{p\in P} A_p) = \{0\}$ for $n = \prod_{p\in P} \omega(A_p)$. But $H = pH$ for every $p \in \mathbb{P}(A)$, so $h_1 = nh_1'$, $h_2 = nh_2'$ and $h = n^2h'$ for some $h_1', h_2', h' \in H$. Therefore, $n^2(h_1' * h_2') = a + n^2h'$ and hence $a = n^2((h_1' * h_2') - h') \in n^2G = \bigoplus_{q\in\mathbb{P}(A)\setminus P}(n^2A_q) \oplus H$. Since $a \in \bigoplus_{p\in P} A_p$, we have $a = 0$. Therefore, $h_1 * h_2 \in H$. □

Proposition 2.9. *Let p be a prime number and n be a positive integer. Let H be a nontrivial nil_a-group with $H_p = \{0\}$. If R is an associative ring such that $R^+ = Z(p^n) \oplus H$, then $R^2 \subseteq Z(p^n)$. In particular, if $n = 1$ and $R^2 \neq \{0\}$, then $R^2 = Z(p)$.*

Proof. As $H_p = \{0\}$, we have $R_p = Z(p^n)$. Hence, $I = Z(p^n)$ is an ideal of the ring R. Suppose, contrary to our claim, that $H^2 \not\subseteq Z(p^n)$. Then $(R/I)^2 \neq \{I\}$. Since $(R/I)^+ \cong H$, there is a nonzero associative ring multiplication on the group H. Therefore, H is not a nil_a-group, which is a contradiction. □

Remark 2.10. If we strengthen the assumptions of the foregoing proposition, assuming that H is a *nil*-group, then the proposition remains true in the general class of rings.

2.3. *H*-rings. Relying on important results obtained by Kruse in [10], we prove a proposition which plays a key role in the description of torsion-free SI_H-groups (cf. Theorem 3.10).

Proposition 2.11. *A torsion-free ring R is an H-ring if and only if either $R^2 = \{0\}$ or $R \cong n\mathbb{Z}$ for some $n \in \mathbb{N}$.*

Proof. Suppose that R is a torsion-free H-ring such that $R^2 \neq \{0\}$. Then the set $\mathcal{N}(R)$ of all nilpotent elements of the ring R is an ideal in R. Take any $a \in \mathcal{N}(R)$. Then $o(a^2) \in \mathbb{N}$, by [10, (1.5)], so $a^2 = 0$. Let $x, y \in \mathcal{N}(R)$ be arbitrary. Then $\langle x\rangle = [x] \lhd R$, so $xy = kx$ for some $k \in \mathbb{Z}$. As $y^2 = 0$, we have $0 = xy^2 = kxy = k^2x$. Thus, $kx = 0$ and $xy = 0$. Therefore, $\mathcal{N}(R)^2 = \{0\}$ and $R \neq \mathcal{N}(R)$. Hence, by [10, (3.3)], we have $\mathcal{N}(R) = \{0\}$, and the ring R is reduced. Thus, by [10, (1.3) and (1.4)], we obtain $R \cong n\mathbb{Z}$ for some $n \in \mathbb{N}$.

The opposite implication is obvious. □

Proposition 2.12. *Let A be an abelian group and let s be a positive integer. If s is square-free, then $R = \mathbb{Z}_s \oplus A^0$ is an H-ring.*

Proof. Take any $\alpha \in R \setminus \{0\}$. Then $\alpha = (k, a)$ for some $k \in \mathbb{Z}_s$ and $a \in A$ such that $k \neq 0$ or $a \neq 0$. If $k = 0$, then $[\alpha] = \{0\} \oplus \langle a\rangle \lhd R$. If $a = 0$, then $[\alpha] = \langle k\rangle \oplus \{0\} \lhd R$, because $\mathbb{Z}_s$ is an H-ring. Now suppose that $k \neq 0$ and $a \neq 0$. Then $\alpha^2 = (k^2, 0) \neq (0, 0)$ and hence, by a simple induction argument, we get $\alpha^n = (k^n, 0) = k^{n-2}\alpha^2 \in \langle\alpha^2\rangle$ for every $n \in \mathbb{N}$ such that $n \geq 2$. Therefore, $[\alpha] = \langle\alpha\rangle + \langle\alpha^2\rangle$. Moreover, $\langle\alpha^2\rangle = \langle k^2(1, 0)\rangle = \langle k(1, 0)\rangle$, because $o((1, 0))$ is square-free. Hence, $[\alpha] = \langle(k, a)\rangle + \langle(k, 0)\rangle = \langle k\rangle \oplus \langle a\rangle \lhd R$. □

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Proposition 2.13. *If A is an H-ring satisfying $A = pA$ for some $p \in \mathbb{P}$, then $R = Z(p)^0 \oplus A$ is an H-ring.*

Proof. Let $\alpha, \beta \in R$. Then $\alpha = (k, a)$, $\beta = (l, b)$ for some $k, l \in Z(p)^0$ and $a, b \in A$. Since $A = pA$, there exists $c \in A$ such that $b = pc$. Moreover, $[a] \lhd A$, so there exist $n \in \mathbb{N}$ and $k_1, k_2, \ldots, k_n \in \mathbb{Z}$ such that $ac = \sum_{i=1}^{n} k_i a^i$. Hence, $ab = a(pc) = p(ac) = p\sum_{i=1}^{n} k_i a^i$. It is clear that $\alpha^m = (0, a^m)$ for every $m \in \mathbb{N}$ such that $m \geq 2$. Notice further that $p(0, a) = (0, pa) = p(k, a) = p\alpha$. Hence, $p(0, a^m) = p\alpha^m$ for all $m \in \mathbb{N}$. Therefore, $\alpha\beta = (0, ab) = (0, p\sum_{i=1}^{n} k_i a^i) = \sum_{i=1}^{n} k_i (p(0, a^i)) = \sum_{i=1}^{n} (pk_i)\alpha^i \in [\alpha]$. Thus, $[\alpha] \lhd R$. □

Lemma 2.14. *Let A be an H-ring satisfying $A = pA$ for some $p \in \mathbb{P}$. If $R = \mathbb{Z}_p \oplus A$, then $(0, a) \in [(1, a)]$ for all $a \in A$.*

Proof. Take any $a \in A$. Then $a = pb$ for some $b \in A$ and $[a] \lhd A$. So, there exist $s \in \mathbb{N}$ and $k_1, k_2, \ldots, k_s \in \mathbb{Z}$ such that $ab = k_1 a + k_2 a^2 + \cdots + k_s a^s$. Hence, $a^2 = a(pb) = p(ab) = (pk_1)a + (pk_2)a^2 + \cdots + (pk_s)a^s$. Therefore, $a = (pk_1 + 1)a + (pk_2 - 1)a^2 + (pk_3)a^3 + (pk_4)a^4 + \cdots + (pk_s)a^s$. Moreover, $p \mid (pk_1 + 1) + (pk_2 - 1) + pk_3 + pk_4 + \cdots + pk_s$ and hence $(0, a) = (pk_1 + 1)(1, a) + (pk_2 - 1)(1, a)^2 + pk_3(1, a)^3 + pk_4(1, a)^4 + \cdots + pk_s(1, a)^s$. Thus, $(0, a) \in [(1, a)]$. □

Proposition 2.15. *If A is an H-ring satisfying $A = pA$ for some $p \in \mathbb{P}$, then $R = \mathbb{Z}_p \oplus A$ is an H-ring.*

Proof. Take any $\alpha \in R$. Then $\alpha = (k, a)$ for some $k \in \mathbb{Z}_p$, $a \in A$. If $k \neq 0$, then there exists $l \in \mathbb{Z}_p$ satisfying $k \odot_p l = 1$. Since $p \nmid l$, there exist $x, y \in \mathbb{Z}$ such that $(0, a) = x(0, pa) + y(0, la)$. It is evident that $(0, pa) = p\alpha$. Moreover, $(0, la) \in [(1, la)]$, by Lemma 2.14, and $(1, la) = (k \odot_p l, la) = l(k, a) = l\alpha$. Thus, $(0, a) \in [\alpha]$. As $(k, 0) = \alpha - (0, a) \in [\alpha]$, we have $(k) \oplus [a] \subseteq [\alpha]$. Moreover, $\alpha \in (k) \oplus [a]$, so $[\alpha] \subseteq (k) \oplus [a]$. Therefore, $[\alpha] = (k) \oplus [a] \lhd R$. □

Proposition 2.16. *Let P be an infinite subset of the set $\mathbb{P}$. Then there does not exist an H-subring S of the ring $\prod_{p \in P} \mathbb{Z}_p$ such that $\bigoplus_{p \in P} \mathbb{Z}_p \subsetneq S$.*

Proof. Suppose that the assertion of the proposition is false. Take any $a \in S \setminus \bigoplus_{p \in P} \mathbb{Z}_p$ and $p \in \operatorname{supp}(a)$. Let $\varepsilon_p = (0, 0, \ldots, 0, \overset{p}{1}, 0, 0, \ldots)$. Then $|\operatorname{supp}(\varepsilon_p)| < \infty$ and hence $\varepsilon_p \in S$. Moreover, $[a] \lhd S$, so $\varepsilon_p a \in [a]$ and $[a]$ is an H-ring. Therefore, $[\varepsilon_p a] \lhd [a]$. Hence, $I = \bigoplus_{p \in \operatorname{supp}(a)} \mathbb{Z}_p \lhd [a]$. Since the ring $[a]$ is finitely generated and commutative, $[a]$ is a noetherian ring. Thus, the ideal I is finitely generated. As $|\operatorname{supp}(a)| = \infty$, we have a contradiction. □

Proposition 2.17. *Let p be a prime number and n be a positive integer. Let A be an abelian group such that $A_p \neq \{0\}$ or $T(A) \neq A$. If $n \geq 2$, then $R = \mathbb{Z}_{p^n} \oplus A^0$ is not an H-ring.*

Proof. Take any $a \in A$. Let $\alpha = (p^{n-1}, a)$. Then $\alpha^2 = 0$, because $n \geq 2$. Therefore, $[\alpha] = \langle \alpha \rangle$. Suppose, contrary to our claim, that $(1, 0)\alpha \in [\alpha]$. Then there exists $k \in \mathbb{Z}$ such that $(p^{n-1}, 0) = k(p^{n-1}, a)$. Hence, $p^{n-1} = kp^{n-1}$ and $0 = ka$. If $o(a) = \infty$, then, from the equality $ka = 0$, it follows that $k = 0$. Thus, $p^{n-1} = 0$ in $\mathbb{Z}_{p^n}$, which is a contradiction. If $o(a) = p$, then $p \mid k$. So, there exists $l \in \mathbb{Z}$ such that $k = lp$. Therefore, $p^{n-1} = kp^{n-1} = lp^n = 0$ in $\mathbb{Z}_{p^n}$, which is a contradiction. Thus, $(1, 0)\alpha \notin [\alpha]$. □

To end these preliminaries, we prove a surprising fact about subgroups of the group $\mathbb{Q}^+$ which we will apply in describing mixed SI_H-groups (see Theorem 3.8 and Example 3.9).

Lemma 2.18. *If A is a subgroup of the group $\mathbb{Q}^+$ satisfying $A \neq pA$ for some $p \in \mathbb{P}$, then $A/pA \cong Z(p)$. In particular, $A = pA + \langle a \rangle$ for all $a \in A \setminus pA$.*

Proof. Suppose, contrary to our claim, that $\dim_{\mathbb{Z}_p} A/pA \geq 2$. Then there exist $a_1, a_2 \in A \setminus pA$ such that $a_1 + pA$, $a_2 + pA$ are linearly independent over the field $\mathbb{Z}_p$. Let $n = \min\{m \in \mathbb{N} : ma_1 \in \langle a_2 \rangle\}$. Then $na_1 = ka_2$ for some $k \in \mathbb{Z}$. Thus, $n(a_1 + pA) - k(a_2 + pA) = pA$ and hence $p \mid n$ and $p \mid k$. Therefore, $n = pn_1$ and $k = pk_1$ for some $n_1 \in \mathbb{N}$, $k_1 \in \mathbb{Z}$. Moreover, A is torsion-free and, consequently, $n_1a_1 = k_1a_2$, which contradicts the minimality of the number n. Therefore, $\dim_{\mathbb{Z}_p} A/pA = 1$ and $A/pA \cong Z(p)$. □

3. SI_H-groups

First observe that [7, Lemma 1] can be somewhat generalised, not assuming the commutativity of the ring R, in the following lemma.

Lemma 3.1. *Let R be an associative ring and let $A = R^+$. Let M be a left-sided R-module satisfying $R \circ M \neq \{0\}$. Then $A \oplus M$ is not an SI_H-group.*

Proof. Let $S = \left(\begin{smallmatrix} R & M \\ 0 & 0 \end{smallmatrix}\right)$. Then S is an associative ring with $S^+ \cong A \oplus M$ and $T = \left(\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a subring of S. As $T \cdot S = \left(\begin{smallmatrix} R^2 & R\circ M \\ 0 & 0 \end{smallmatrix}\right)$ and $R \circ M \neq \{0\}$, we have $T \cdot S \not\subseteq T$. Hence, $T \ntrianglelefteq S$. Therefore, $A \oplus M$ is not an SI_H-group. □

It follows from [7, Theorem 10] that if G is a mixed SI_H-group, then the torsion part $T(G)$ of G satisfies $T(G) = \bigoplus_{p\in\mathbb{P}(G)} Z(p^{n_p})$, where $n_p \in \mathbb{N}$ for all $p \in \mathbb{P}(A)$. Hence, by [7, Lemma 8] and Proposition 2.17, we get a more accurate version of [7, Theorem 10].

Theorem 3.2. *If G is a mixed SI_H-group, then $T(G) = \bigoplus_{p\in\mathbb{P}(G)} Z(p)$.*

Proposition 3.3. *Let A be an SI_H-group satisfying $A = pA$ and $A_p = \{0\}$ for some $p \in \mathbb{P}$. Then $G = Z(p) \oplus A$ is an SI_H-group.*

Proof. Let R be an arbitrary associative ring with $R^+ = G$. Then it follows from Proposition 2.8 that $Z(p)^2 \subseteq Z(p)$, $A^2 \subseteq A$ and $Z(p) \cdot A = A \cdot Z(p) = \{0\}$. Thus, $R = R_1 \oplus R_2$, where R_1 and R_2 are rings such that $R_1^+ = Z(p)$ and $R_2^+ = A$. So, the assertion follows from Remark 2.7 and Propositions 2.13 and 2.15. □

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Theorem 3.4. *Let $\emptyset \neq P \subseteq \mathbb{P}$ and let A be an SI_H-group satisfying $A = pA$ and $A_p = \{0\}$ for all $p \in P$. Then $G = (\bigoplus_{p\in P} Z(p)) \oplus A$ is an SI_H-group.*

Proof. If $|P| < \infty$, then the assertion follows from Lemma 3.3 by a simple induction argument. Now suppose that $|P| = \infty$. Let R be an arbitrary associative ring with $R^+ = G$. Take any $\alpha, \beta \in R$. Then $\alpha = (k, a)$, $\beta = (l, b)$ for some $k, l \in \bigoplus_{p\in P} Z(p)$ and $a, b \in A$. Let $P_{kl} = \operatorname{supp}(k) \cup \operatorname{supp}(l)$. There exists a subgroup H of the group G such that $H \cong \bigoplus_{p\in P_{kl}} Z(p) \oplus A$. Of course, $\alpha, \beta \in H$. Moreover, $|P_{kl}| < \infty$; thus, H is an SI_H-group, by the first part of the proof. It follows from Proposition 2.8 and Remark 2.7 that H is a subring of the ring R. Therefore, $[\alpha] \lhd H$ and $\alpha\beta, \beta\alpha \in [\alpha]$. Hence, by the arbitrary choice of the element β of the ring R, we obtain $[\alpha] \lhd R$. Since α has also been chosen arbitrarily, R is an H-ring. Therefore, G is an SI_H-group. □

Proposition 3.5. *Let H be an abelian group satisfying $H_p = \{0\}$ and $H \neq pH$ for some $p \in \mathbb{P}$. If $\dim_{\mathbb{Z}_p} H/pH \geq 2$, then $G = \mathbb{Z}_p^+ \oplus H$ is not an SI_H-group.*

Proof. Consider the diagram

$$G \xrightarrow{\pi_1} H \xrightarrow{\pi_2} H/pH \xrightarrow{\varphi} \bigoplus_{i\in I} \mathbb{Z}_p^+,$$

where π_1 is a natural projection of the group G on the group H, π_2 is the canonical epimorphism and φ is an isomorphism. Let $f = \varphi \circ \pi_2 \circ \pi_1$. From the assumptions, it follows that there exists $x \in H \setminus pH$ such that $|\operatorname{supp} \varphi(x + pH)| \geq 2$. Take any $i_1, i_2 \in \operatorname{supp} \varphi(x + pH)$ such that $i_1 \neq i_2$. Let $c = (\varphi(x + pH))_{i_1}$. Then $\varphi^{-1}(c) = y + pH$ for some $y \in H \setminus pH$. For $t = 1, 2$, the epimorphisms $\mu_t \colon \bigoplus_{i\in I} \mathbb{Z}_p^+ \to \mathbb{Z}_p^+$ can be defined by

$$\mu_1((k_i)_{i\in I}) = k_{i_1}, \quad \mu_2((k_i)_{i\in I}) = k_{i_2}.$$

Let $*\colon G \times G \to G$ be given by

$$g_1 * g_2 = (\mu_1(f(g_1)) \odot_p \mu_2(f(g_2)), 0).$$

Since the functions used in the definition of the function $*$ are homomorphisms of groups and $\odot_p$ is a ring multiplication, we infer that the operation $*$ is distributive with respect to addition. Moreover, $f(\mathbb{Z}_p^+) = \{0\}$, so $G * \mathbb{Z}_p^+ = \mathbb{Z}_p^+ * G = \{0\}$. Therefore, $R = (G, +, *, 0)$ is the ring satisfying $R^3 = \{0\}$. Next, $[(0, y)] = \langle (0, y) \rangle$ and $(0, y) * (0, x) = (k, 0)$ for some $0 \neq k \in \mathbb{Z}_p^+$. Thus, $(0, y) * (0, x) \notin [(0, y)]$ and hence $[(0, y)] \ntriangleleft R$. Therefore, R is not an H-ring. Hence, G is not an SI_H-group. □

Theorem 3.6. *Let G be a mixed SI_H-group and let $p \in \mathbb{P}(A)$. Then there exists $H \leq G$ such that $G = G_p \oplus H$, where $H = \langle h \rangle + pH$ for some $h \in H$.*

Proof. It follows from [7, Lemma 8] that there exists $H \leq G$ such that $G = G_p \oplus H$. If $H = pH$, then $H = \langle h \rangle + pH$ for all $h \in H$. If $H \neq pH$, then $\dim_{\mathbb{Z}_p} H/pH = 1$, by Proposition 3.5. Hence, $H = \langle h \rangle + pH$ for all $h \in H \setminus pH$. □

Remark 3.7. It follows from Theorem 3.2 that $G_p = Z(p)$ and hence $pG = pH$. If $H \neq pH$, then the subgroup H is not uniquely determined. In fact, let $K = \langle (a, h) \rangle + pH$ for some $0 \neq a \in G_p$, $h \in H \setminus pH$. Then $G = G_p + K$. Suppose that $k(a, h) + (0, ph_1) = l(a, 0)$ for some $k, l \in \mathbb{Z}$, $h_1 \in H$. Then $ka = la$ and $kh + ph_1 = 0$; hence, $k \equiv l \pmod p$ and $kh = -ph_1 \in pH$. As $h \in H \setminus pH$, we have $p \mid k$. Therefore, $G_p \cap K = \{0\}$. Moreover, $(a, h) \notin H$, so $K \neq H$.

Notice that it follows directly from [7, Theorem 6 and Lemma 8] that if G is a nontrivial torsion SI_H-group, then for every $p \in \mathbb{P}(G)$ there exists a p-divisible subgroup H of the group G such that $G = G_p \oplus H$.

Proposition 3.8. *Let H be a nil_a-group with $H_p = \{0\}$ for some $p \in \mathbb{P}$. If there exists $h_0 \in H$ such that $H = \langle h_0 \rangle + pH$, then $G = Z(p) \oplus H$ is an SI_H-group.*

Proof. Let $R = (G, +, *, 0)$ be an arbitrary associative ring. If $R^2 = 0$, then R is an H-ring. Now suppose that $R^2 \neq \{0\}$. Since $H_p = \{0\}$, we obtain $R_p = Z(p)$. Hence, $I = Z(p) \lhd R$. Moreover, $R^2 = I$, by Proposition 2.9. Take any $0 \neq a \in I$. Then $o(a) = p$ and $\langle a \rangle = I$. We have two cases.

(i) $a^2 \neq 0$. Then $I \cong \mathbb{Z}_p$, by Remark 2.7. Thus, the ring I has a unity. So, there exists $J \lhd R$ such that $R = I \oplus J$. Hence, $H \cong (R/I)^+ \cong J^+$. Moreover, H is a nil_a-group, so $J^2 = \{0\}$. Therefore, $R \cong \mathbb{Z}_p \oplus H^0$. Hence, from Proposition 2.12, we infer that R is an H-ring.

(ii) $a^2 = 0$. Then $R^4 = I^2 = \langle a^2 \rangle = \{0\}$. Suppose, contrary to our claim, that $R^3 \neq \{0\}$. Since $R^2 \lhd R$, $R^3 \subseteq R^2$. As $R^3 \leq R$, $R^2 = I$ and $|I| = p$, we have $R^3 = R^2$. Hence, $R^3 = R^2 \cdot R = R^3 \cdot R = R^4 = \{0\}$, which is a contradiction. Therefore, $R^3 = \{0\}$. Take any $h \in H$. Then $a * h \in I$, so there exists $k \in \mathbb{Z}$ such that $a * h = ka$. Thus, $(a * h) * h = (ka) * h = k(a * h) = k(ka) = k^2 a$. As $R^3 = \{0\}$, we have $k^2 a = 0$. Therefore, $p \mid k^2$ and hence $p \mid k$. Thus, $ka = 0$ and, consequently, $a * h = 0$. Hence, $a * H = \{0\}$. Analogously, $H * a = \{0\}$. Moreover, $a^2 = 0$, so $a \in \mathfrak{a}(R)$. Next, $(pH) * R = p(H * R) \subseteq pR^2 = pI = \{0\}$ and hence $(pH) * R = \{0\}$. Similarly, $R * (pH) = \{0\}$. Therefore, $pH \subseteq \mathfrak{a}(R)$. From the assumption, it follows that $H = \langle h_0 \rangle + pH$ for some $h_0 \in H$. Thus, $R = \langle h_0 \rangle + \mathfrak{a}(R)$. Moreover, $R^2 \neq \{0\}$, so $h_0^2 \neq 0$. Hence, by the equality $R^2 = I$, we obtain $o(h_0^2) = p$. Take any $x, y \in R$. Suppose that $x * y \neq 0$. Then $x = \alpha h_0$ and $y = \beta h_0$ for some $\alpha, \beta \in \mathbb{Z}$. Hence, $x * y = \alpha\beta h_0^2 \in \langle \alpha h_0^2 \rangle \cap \langle \beta h_0^2 \rangle$. Since $o(h_0^2)$ is a square-free number, we have $\langle \alpha h_0^2 \rangle = \langle \alpha^2 h_0^2 \rangle$ and $\langle \beta h_0^2 \rangle = \langle \beta^2 h_0^2 \rangle$. Therefore, $x * y \in \langle x^2 \rangle \cap \langle y^2 \rangle$. Moreover, $pR^2 = R^3 = \{0\}$, so the ring R is almost null (cf. [10, Definition (2.1)]). Thus, R is an H-ring. □

Example 3.9. Let H be a nil-subgroup of the group $\mathbb{Q}^+$ such that $H \neq pH$ for some $p \in \mathbb{P}$. Then there exists $h_0 \in H \setminus pH$ and hence $H = \langle h_0 \rangle + pH$, by Lemma 2.18. Moreover, $H_p = \{0\}$, so it follows from Proposition 3.8 that $Z(p) \oplus H$ is an SI_H-group.

Theorem 3.10. *Let A be an abelian torsion-free group. Then A is an SI_H-group if and only if either A is a nil_a-group or $A \cong \mathbb{Z}^+$.*

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Proof. Suppose that A is an SI_H-group which is not a nil_a-group. Then there exists an H-ring R satisfying $R^2 \neq \{0\}$ and $R^+ = A$. It follows from Proposition 2.11 that $R \cong n\mathbb{Z}$ for some $n \in \mathbb{N}$. Therefore, $A \cong \mathbb{Z}^+$. The opposite implication is obvious. □

Corollary 3.11. *The converse theorem to Theorem 3.6 is not true. In fact, take any distinct prime numbers p and q. Let $H = [1/q]^+$. Then $H \neq pH$, by [6, Lemma 2.5], so $H = \langle h \rangle + pH$ for some $h \in H \setminus pH$, by Lemma 2.18. But H is not an SI_H-group, by Theorem 3.10, so it follows from [7, Lemma 3] that $G = Z(p) \oplus H$ is not an SI_H-group.*

4. *SI*-groups

Lemma 4.1. *Let both A and H be abelian groups. If A is not a nil-group and A is a homomorphic image of H, then $A \oplus H$ is not an SI-group.*

Proof. Let $f\colon H \to A$ be an epimorphism. Let $(R, +, \cdot)$ be a ring such that $R^+ = A$ and $R^2 \neq \{0\}$. It is easy to check that the operation $*\colon (A \oplus H) \times (A \oplus H) \to (A \oplus H)$ defined by

$$(a_1, x_1) * (a_2, x_2) = (a_1 \cdot f(x_2) + a_2 \cdot f(x_1), 0) \quad \text{for all } a_1, a_2 \in A,\ x_1, x_2 \in H$$

is a ring multiplication on the group $A \oplus H$. Since $R^2 \neq \{0\}$, there exist $a, b \in A$ such that $a \cdot b \neq 0$. Let $y \in H$ be such that $f(y) = b$. Then $(0, y)^2 = (0, 0)$, so $[(0, y)] = \langle (0, y) \rangle$ and $(a, 0) * (0, y) = (a \cdot f(y), 0) = (a \cdot b, 0) \notin [(0, y)]$. Therefore, $[(0, y)]$ is not an ideal in the ring $(A \oplus H, +, *)$. □

Corollary 4.2. *Let H be an abelian group such that $H \neq pH$ for some $p \in \mathbb{P}$. Then $Z(p) \oplus H$ is not an SI-group.*

Proof. Since $H \neq pH$, it follows that H/pH is a nonzero linear space over the field $\mathbb{Z}_p$. Hence, there exists an epimorphism $f\colon H \to Z(p)$. Therefore, the assertion follows directly from Lemma 4.1. □

Remark 4.3. From the above corollary and Example 3.9, it follows that the class of all *SI*-groups is a proper subclass of the class of all SI_H-groups.

Remark 4.4. The ring multiplication in [7, Corollary 11] is not associative. In fact, $((a, 0)(0, h))(0, h) = (a, 0)(0, h) = (a, 0) \neq (0, 0)$ and $(a, 0)((0, h)(0, h)) = (a, 0)(0, 0) = (0, 0)$. Therefore, Feigelstock's proof provides the truth for this corollary for *SI*-groups referred to in Definition 2.2. Example 3.9 shows that Corollary 11 is false in the class of SI_H-groups. In addition, $G_p = Z(p)$, by Remark 2.4 and Theorem 3.2; hence, $H = pG$.

Corollary 4.5. *Corollary 12 of [7] is true only for SI-groups (in the sense of Definition 2.2).*

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Remark 4.6. Lemma 13 of [7] is true only in the class of *SI*-groups. In fact, let G_1 be a *nil*-subgroup of the group $\mathbb{Q}^+$ satisfying $G_1 \neq pG_1$ for some $p \in \mathbb{P}$, and let $G_2 = Z(p)$. Then $G_1 \otimes G_2 \cong G_1/pG_1$ and hence $\operatorname{Hom}(G_1 \otimes G_2, G_2) \neq \{0\}$. Therefore, the assumptions of [7, Lemma 13] are valid. But $G = G_1 \oplus G_2$ is an SI_H-group, by Lemma 2.18 and Proposition 3.8.

Since the ring multiplication constructed by Feigelstock for the case (1) is associative, [7, Lemma 13] will be true in the class of SI_H-groups if we strengthen the assumptions by requiring that the elements i, j, k are pairwise distinct.

Corollary 4.7. *Theorem 16 of [7] has been proved for SI-groups (in the sense of Definition 2.2).*

Proposition 4.8. *Let s be a positive integer not less than 2 and let H be a torsion-free nil-group satisfying $H = pH$ for every $p \in \mathbb{P}$ such that $p \mid s$. If s is a square-free number, then $A = \mathbb{Z}_s^+ \oplus H$ is an SI-group.*

Proof. Let R be an arbitrary ring with $R^+ = A$. Then it follows from Proposition 2.8 that $R = S \oplus H^0$, where S is some ring with $S^+ = \mathbb{Z}_s^+$. If $R^2 = \{0\}$, then R is an *SI*-ring. If $R^2 \neq \{0\}$, then, by Remark 2.7, we obtain $R^2 \cong \mathbb{Z}_r$, where r is the product of all prime divisors p of s for which $S_p \cong \mathbb{Z}_p$. In particular, it follows that the ring R is commutative and associative. Take any $\alpha \in R$. Then $\alpha = (k, x)$ for some $k \in S$ and $x \in H$. If $k = 0$, then $[\alpha] = \langle(0, x)\rangle$ and $[\alpha] \cdot R = \{0\} \subseteq [\alpha]$, so $[\alpha] \lhd R$. Now suppose that $k \neq 0$. Then $\alpha^2 = (k^2, 0)$ and hence $\alpha^n \in \langle\alpha^2\rangle$ for every $n \in \mathbb{N}$ satisfying $n \geq 2$. Thus, $[\alpha] = \langle\alpha\rangle + \langle\alpha^2\rangle$. Notice that $\alpha^2 = (k, 0)^2 = (k(1, 0))^2 = k^2(1, 0)^2$. Moreover, $o((1, 0)^2)$ is a square-free number, so $\langle\alpha^2\rangle = \langle k(1, 0)^2\rangle$. Therefore, $k(1, 0)^2 = l\alpha^2$ for some $l \in \mathbb{Z}$. Take any $t \in S$. Then $\alpha \cdot (t, 0) = (k, 0) \cdot (t, 0) = kt(1, 0)^2 = t(k(1, 0)^2) = tl\alpha^2 \in \langle\alpha^2\rangle$ and $\alpha^2 \cdot (t, 0) = \alpha \cdot (\alpha \cdot (t, 0)) = tl\alpha^3 \in \langle\alpha^2\rangle$. Thus, $\alpha \cdot R \subseteq \langle\alpha^2\rangle$ and $\alpha^2 \cdot R \subseteq \langle\alpha^2\rangle$. Hence, $[\alpha] \lhd R$. □

Example 4.9. Let $p < p_1 < p_2 < p_3 < \cdots$ be primes, let $A = Z(p)$ and let $H = [1/p]^+ + \langle 1/p_1, 1/p_2, 1/p_3, \ldots\rangle$. It follows from [6, Lemma 2.5] and [6, Example 3.2] that H is a p-divisible *nil*-group. Hence, $G = A \oplus H$ is an *SI*-group, by Proposition 4.8.

Lemma 4.10. *Let A be a mixed SI-group and let $H = \bigcap_{p \in \mathbb{P}(A)} pA$. Then $T(H) = \{0\}$ and $H = pH$ for all $p \in \mathbb{P}(A)$.*

Proof. Suppose, contrary to our claim, that $T(H) \neq \{0\}$. Then there exist $q \in \mathbb{P}(A)$ and $a \in H$ such that $o(a) = q$. From the definition of the group H, it follows that $a \in A_q \cap qA$. But $A_q = Z(q)$, by Theorem 3.2 and Remark 2.4, so $(qA)_q = \{0\}$. Hence, $a = 0$, which is a contradiction.

Take any $p \in \mathbb{P}(A)$ and $h \in H$. Then $h \in pA$. Corollary 11 of [7] implies the existence of a subgroup $H_{(p)}$ of the group A such that $A = A_p \oplus H_{(p)}$ and $H_{(p)} = pH_{(p)}$. Moreover, $A_p = Z(p)$, by Theorem 3.2 and Remark 2.4. Hence, $pA = pH_{(p)} = H_{(p)}$. Therefore, $pA = p^2A$. Thus, $h = px$ for some $x \in pA$. Take any $q \in \mathbb{P}(A) \setminus \{p\}$. Then $x = k(px) + l(qx)$ for some $k, l \in \mathbb{Z}$. Moreover, $px = h \in qA$, by the definition of the group H. Therefore, $x \in qA$. In this way, we have shown that $x \in rA$ for all $r \in \mathbb{P}(A)$. Therefore, $x \in H$. Thus, $H = pH$ for all $p \in \mathbb{P}(A)$. □

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

Remark 4.11. If A is an *SI*-group with $\mathbb{P}(A) = \mathbb{P}$, then A can be embedded in the group $\prod_{p\in\mathbb{P}} Z(p)$. In fact, it follows from Lemma 4.10 that $H = \bigcap_{p\in\mathbb{P}} pA$ is a torsion-free divisible subgroup of the group A. Hence, by [7, Corollary 4], we obtain $H = \{0\}$. It is easy to check that the function $f\colon A \to \prod_{p\in\mathbb{P}}(A/pA)$ defined by

$$f(a) = (a + pA)_{p\in\mathbb{P}} \quad \text{for all } a \in A$$

is a homomorphism. If $a \in \ker f$, then $a \in H$ and hence $a = 0$. Thus, f is a monomorphism. Moreover, [7, Corollary 11] and Remark 4.4 imply that $A = Z(p) \oplus pA$ for all $p \in \mathbb{P}$. Hence, $A/pA \cong Z(p)$ for all $p \in \mathbb{P}$. Thus, there exists a monomorphism $\varphi\colon A \to \prod_{p\in\mathbb{P}} Z(p)$. Let $B = \varphi(A)$. Then B is a mixed *SI*-subgroup of the group $\prod_{p\in\mathbb{P}} Z(p)$ with $\mathbb{P}(B) = \mathbb{P}$. Hence, by Theorem 3.2, we obtain $\bigoplus_{p\in\mathbb{P}} Z(p) \leq B \leq \prod_{p\in\mathbb{P}} Z(p)$.

Proposition 4.12. *Let A be a mixed SI-group with $|\mathbb{P}(A)| < \infty$. Then there exists a subgroup H of the group A such that $T(H) = \{0\}$, $H = pH$ for all $p \in \mathbb{P}(A)$ and $A = T(A) \oplus H$. In particular, H is an SI-group.*

Proof. Let $H = \bigcap_{p\in\mathbb{P}(A)} pA$. Then $T(H) = \{0\}$ and $H = pH$ for all $p \in \mathbb{P}(A)$, by Lemma 4.10. Take any $a \in A$. Let $s = \prod_{p\in\mathbb{P}(A)} p$. Then $sA = \bigcap_{p\in\mathbb{P}(A)} pA = H$ and hence $sa \in H$. Moreover, $H = sH$, by the first part of the proof, so there exists $h \in H$ satisfying $sa = sh$. Therefore, $s(a - h) = 0$ and hence $a - h \in T(A)$. Thus, $a = (a - h) + h \in T(A) + H$. Hence, $A \subseteq T(A) + H$. The inverse inclusion is obvious. Moreover, by the first part of the proof, we have $T(H) = \{0\}$, so $T(A) + H = T(A) \oplus H$. Therefore, $A = T(A) \oplus H$. Hence, from [7, Lemma 3], we infer that H is an *SI*-group. □

Theorem 3.2 together with the following proposition gives a partial description of the structure of mixed *SI*-groups whose torsion part is a direct summand (modulo the description of the structure of torsion-free *SI*-groups).

Proposition 4.13. *Let A be a mixed SI-group. If there exists $H \leq A$ such that $A = T(A) \oplus H$, then $H = \bigcap_{p\in\mathbb{P}(A)} pA$.*

Proof. Take any $p \in \mathbb{P}(A)$. Since $A/T(A) \cong H$, it follows from [7, Corollary 12] that $H = pH$. Moreover, $pH \subseteq pA$, so $H \subseteq pA$. In view of the arbitrary choice of $p \in \mathbb{P}(A)$, we have shown that $H \subseteq \bigcap_{p\in\mathbb{P}(A)} pA$. Hence, by modularity of the lattice of subgroups of an abelian group, we have $H + (T(A) \cap \bigcap_{p\in\mathbb{P}(A)} pA) = (H + T(A)) \cap \bigcap_{p\in\mathbb{P}(A)} pA$. As $T(\bigcap_{p\in\mathbb{P}(A)} pA) = \{0\}$ (cf. Lemma 4.10) and $H + T(A) = A$, we have $H = \bigcap_{p\in\mathbb{P}(A)} pA$. □

Acknowledgement

The authors are grateful to Professor Maciej Mączyński for editorial support.

https://doi.org/10.1017/S0004972714000641 Published online by Cambridge University Press

References

[1] A. M. Aghdam, 'Square subgroup of an Abelian group', *Acta Sci. Math.* **51** (1987), 343–348.

[2] A. M. Aghdam and A. Najafizadeh, 'Square subgroups of rank two Abelian groups', *Colloq. Math.* **117**(1) (2009), 19–28.

[3] A. M. Aghdam and A. Najafizadeh, 'Square submodule of a module', *Mediterr. J. Math.* **7**(2) (2010), 195–207.

[4] A. M. Aghdam, F. Karimi and A. Najafizadeh, 'On the subgroups of torsion-free groups which are subrings in every ring', *Ital. J. Pure Appl. Math.* **31** (2013), 63–76.

[5] V. I. Andrijanov, 'Periodic hamiltonian rings', *Mat. Sb.* **74**(116) (1967), 241–261; (in Russian); English transl. *Mat. Sb.* **74**(116) No. 2 (1967), 225–241.

[6] R. R. Andruszkiewicz and M. Woronowicz, 'On associative ring multiplication on abelian mixed groups', *Comm. Algebra* **42**(9) (2014), 3760–3767.

[7] S. Feigelstock, 'Additive groups of rings whose subrings are ideals', *Bull. Aust. Math. Soc.* **55** (1997), 477–481.

[8] L. Fuchs, *Infinite Abelian Groups*, Vol. 1 (Academic Press, New York, 1970).

[9] L. Fuchs, *Infinite Abelian Groups*, Vol. 2 (Academic Press, New York, 1973).

[10] R. L. Kruse, 'Rings in which all subrings are ideals', *Canad. J. Math.* **20** (1968), 862–871.

[11] L. Rédei, 'Vollidealringe im weiteren Sinn. I', *Acta Math. Acad. Sci. Hungar.* **3** (1952), 243–268.

R. R. ANDRUSZKIEWICZ, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2,
Poland
e-mail: randrusz@math.uwb.edu.pl

M. WORONOWICZ, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2,
Poland
e-mail: mworonowicz@math.uwb.edu.pl