BULL. AUSTRAL. MATH. SOC
VOL. 35 (1987) 49-57.

65F35, 15A48
15A60, 15A18
65F15

LOWER BOUNDS FOR TAU COEFFICIENTS AND OPERATOR NORMS USING COMPOSITE MATRIX NORMS

CHOON PENG TAN

Lower bounds for the tau coefficients and operator norms are derived by using composite matrix norms. For a special class of matrices B, our bounds on $||B||_p$ (the operator norm of B induced by the ℓ_p norm) improve upon a general class of Maitre (1967) bounds for $p \geq 2$.

1. Introduction

The tau coefficient of a nonnegative matrix A, denoted by $\tau(A)$, can be used as an upper bound on the maximum modulus of the subdominant eigenvalues of A [5]. Explicit functional forms for the tau coefficients in terms of the entries of A are available for the coefficients which are defined with respect to the ℓ_p norms for $p = 1$ and ∞, denoted by $\tau_1(A)$ and $\tau_\infty(A)$ respectively. Rothblum and Tan [5] have shown that for an $n \times n$ nonnegative matrix A, $\tau_1(A)$ and $\tau_\infty(A)$ can be computed by using $O(n^3)$ and $O(n^2)$ algorithms respectively. With regard to the other tau coefficients, their functional forms are not

Received 24 February 1986.

known and hence they cannot be computed explicitly. The coefficient $\tau_2(A)$, which is defined with respect to the Euclidean norm, can be evaluated as $\lambda_0^{1/2}$ where λ_0 is the maximal root of a polynomial equation of degree $n - 1$ [6] . For the tau coefficients which cannot be computed explicitly, it is desirable that bounds on them be available. Computable upper bounds are available in [7, p.310] for the tau coefficients which are defined with respect to the ℓ_p norms for $1 \le p \le \infty$. By virtue of their definitions, lower bounds for the tau coefficients and operator norms are readily available. However, our aim is to study lower bounds of the type obtained by using composite matrix norms. We shall show that in some cases, our bounds on the operator norms improve upon some well-known bounds [2], [3].

2. Some Basic Definitions

We assume throughout the paper that A is an $n \times n$ nonnegative, irreducible matrix with spectral radius $\rho(A) = r$. We denote the positive right eigenvector of A corresponding to r , which is known as the Perron vector, by $\underset{\sim}{\omega}$ and $\underset{\sim}{\omega} = (\omega_i)$ is unique up to a positive, multiplicative scalar. Vector norms will be denoted by ϕ and ψ; $||\cdot||_p$ will denote an ℓ_p norm for $1 \le p \le \infty$, that is $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ for any vector $x = (x_i)$.

DEFINITION 1. The tau coefficient of A with respect to a given vector norm ϕ is defined as;

$$\tau_\phi(A) = \max\,\{\phi(x'A) : x'.\underset{\sim}{\omega} = 0,\ \phi(x) = 1,\ x \in \mathbb{R}^n\}$$

where $\underset{\sim}{\omega}$ is assumed to be known.

If ϕ is an ℓ_p norm for $1 \le p \le \infty$, then we write $\tau_\phi(\cdot)$ as $\tau_p(\cdot)$. $\tau_\phi(A^k)$ where A^k is reducible is defined in a similar manner where $\underset{\sim}{\omega}$ is the Perron vector of A . Definition 1 may also be extended to any real matrix A and any real vector $\underset{\sim}{\omega}$ which is not necessarily the Perron vector of A. In this case, $\tau_\phi(A)$ as defined may not bound the subdominant eigenvalues of A.

DEFINITION 2. The operator norm of a matrix B induced by the vector norm ϕ is defined as :

$$||B||_\phi = \max \{\phi(Bx) : \phi(x) = 1, \; x \in \mathcal{C}^n\}$$

A vector norm ψ on $\mathcal{C}^n$ is said to be absolute if $\psi(x) = \psi(|x|)$ for any $x \in \mathcal{C}^n$, where $|x|$ denotes the vector $(|x_i|)$. If $x_i \le y_i$ for $i = 1,2,\ldots,n$, then we write $x \le y$. The vector norm ψ is said to be monotonic if, whenever $|x| \le |y|$, then $\psi(x) \le \psi(y)$ for any two vectors x and y. It is well known that a vector norm is absolute if and only if it is monotonic [1].

A composite matrix norm is derived from the composition of 2 vector norms where one of them is monotonic [4]. Specifically, we denote such a norm by $[\;]_{\phi,\psi}$ where

$$(1) \qquad [B]_{\phi,\psi} = \psi[\phi(B_{.1}), \phi(B_{.2}), \ldots, \phi(B_{.n})],$$

and $B_{.j}$ denotes the j^{th} column of B for $j = 1,2,\ldots,n$ and ψ is a monotonic norm. Note that $[B]_{\phi,\psi}$ can be evaluated explicitly in terms of the entries of B if the norms ϕ and ψ are defined explicitly in terms of the components of any given vector. We use the notation $[B]_{p,q}$ and $||B||_p$ for $[B]_{\phi,\psi}$ and $||B||_\phi$ respectively, when ϕ is an ℓ_p norm and ψ is an ℓ_q norm for $1 \le p \le \infty$ and $1 \le q \le \infty$.

3. Main Results

THEOREM 1.

$$\tau_\phi(A) = \max \{[(VA)']_{\phi,\psi} / [V']_{\phi,\psi} : V\underline{w} = \underline{0}, \; V \in \mathbb{R}_{n\times n}\}$$

where ψ is any monotonic norm.

https://doi.org/10.1017/S0004972700013034 Published online by Cambridge University Press

Proof. Let V be any $n \times n$ real matrix such that $V\underset{\sim}{\omega} = \underset{\sim}{0}$. We have

$$\phi((VA)_{i.}) \le \tau_\phi(A)\phi(V_{i.})$$

for $i = 1,2,\ldots,n$ where $(VA)_{i.}$ and $V_{i.}$ denote the i^{th} rows of VA and V respectively. Exploiting the fact that ψ is a monotonic norm, it follows that

$$[(VA)']_{\phi,\psi} \le \tau_\phi(A)[V']_{\phi,\psi} .$$

We now show that equality in the above inequality holds for some particular V. First, note that there exists a real vector x such that $\tau_\phi(A) = \phi(x'A)$ where $x'.\underset{\sim}{\omega} = 0$ and $\phi(x) = 1$. Let V be a matrix with identical rows where the common row is x'. Hence,

$$[(VA)']_{\phi,\psi} = \tau_\phi(A)\psi(\underset{\sim}{1}) = \tau_\phi(A)[V']_{\phi,\psi}$$

where $\underset{\sim}{1} = (1,1,\ldots,1)$ and the proof is complete. □

For particular matrices V, we have the following corollaries:

COROLLARY 1.

$$[B']_{\phi,\psi} \le |\underset{\sim}{v}'.\underset{\sim}{\omega}|^{-1}[W']_{\phi,\psi}\tau_\phi(A) \le |\underset{\sim}{v}'.\underset{\sim}{\omega}|^{-1}[W']_{\phi,\psi}||B'||_\phi$$

where $\underset{\sim}{v}$ *is any real vector,* $B = (I - (\underset{\sim}{v}'.\underset{\sim}{\omega})^{-1}\underset{\sim}{\omega}\underset{\sim}{v}')A,$

$W = (\underset{\sim}{v}'.\underset{\sim}{\omega})I - \underset{\sim}{\omega}\,\underset{\sim}{v}'$ *and* ψ *is any monotonic norm.*

COROLLARY 2.

$$[B']_{\phi,\psi} \le ||\underset{\sim}{\omega}||_2^{-2}[W]_{\phi,\psi}\tau_\phi(A) \le ||\underset{\sim}{\omega}||_2^{-2}[W]_{\phi,\psi}||B'||_\phi$$

where $B = [A - \underset{\sim}{\omega}(\underset{\sim}{\omega}'A/||\underset{\sim}{\omega}||_2^2)]$, $W = ||\underset{\sim}{\omega}||_2^2\, I - \underset{\sim}{\omega}\,\underset{\sim}{\omega}'$ *(a symmetric matrix) and* ψ *is any monotonic norm.*

COROLLARY 3.

$$[A'(I - \underset{\sim}{\omega}\,\underset{\sim}{\omega}'/||\underset{\sim}{\omega}||_2^2)]_{\phi,\psi} \le ||\underset{\sim}{\omega}||_2^{-2}[W]_{\phi,\psi}||A'||_\phi$$

where $W = ||\underset{\sim}{\omega}||_2^2 I - \underset{\sim}{\omega}\,\underset{\sim}{\omega}'$ *and* ψ *is any monotonic norm.*

https://doi.org/10.1017/S0004972700013034 Published online by Cambridge University Press

Remarks.

(i) For $V = I - (\underset{\sim}{\nu}'.\underset{\sim}{\omega})^{-1}\underset{\sim}{\omega}\,\underset{\sim}{\nu}'$ and $B = VA$, we have $\tau_\phi(A) \le ||B'||_\phi$ and hence Corollary 1 follows. By letting $\underset{\sim}{\nu} = \underset{\sim}{\omega}$, we obtain Corollary 2. Corollary 3 follows from Corollary 2 by noting that $\tau_\phi(A) \le ||A'||_\phi$.

(ii) In the proof of Theorem 1, we do not make use of the property that $\underset{\sim}{\omega}$ is an eigenvector nor assume that A is nonnegative. Hence Theorem 1 and Corollaries 1 - 3 are true for any <u>real</u> matrix A and any <u>real</u> vector $\underset{\sim}{\omega}$ by defining an appropriate $\tau_\phi(A)$. If the norms ϕ and ψ are defined explicitly in terms of the components of any given vector, then the lower bounds for $\tau_\phi(A)$, $||B'||_\phi$ and $||A'||_\phi$ given in Corollaries 1-3 are computable since we can choose any 2 known vectors $\underset{\sim}{\omega}$ and $\underset{\sim}{\nu}$ (in the case of $\tau_\phi(A)$, there is only freedom of choice in $\underset{\sim}{\nu}$).

If A is an $n \times n$ stochastic matrix, we may choose $\underset{\sim}{\omega} = \underset{\sim}{1} = (1, 1, \ldots, 1)$ and hence by Corollary 2, a lower bound for $\tau_p(A)$ is given by:

$$(2)\qquad \left\{\frac{n^{1-1/q}}{[(n-1)+(n-1)^p]^{1/p}}\right\} [(A - \tfrac{1}{n}\underset{\sim}{1}\,(\underset{\sim}{1}'A))']_{p,q} \le \tau_p(A)$$

where $1 \le p \le \infty$ and $1 \le q \le \infty$ since $[W]_{p,q} = [(n-1)+(n-1)^p]^{1/p} n^{1/q}$ and $||\underset{\sim}{\omega}||_2^2 = n$.

Theorem 3.3 in [5] states that for an irreducible and aperiodic matrix A,

$$(3)\qquad \lim_{k\to\infty} \tau_\phi(A^k)^{1/k} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A \text{ and } \lambda \ne \rho(A)\}.$$

As an application of Theorem 1 (in particular, Corollary 2), we show that this theorem can be proved in an alternative way. Let $B = A - \underset{\sim}{\omega}(\underset{\sim}{\omega}'A/||\underset{\sim}{\omega}||_2^2)$ and it follows that $B^k = A^k - \underset{\sim}{\omega}(\underset{\sim}{\omega}'A^k/||\underset{\sim}{\omega}||_2^2)$ for $k = 1, 2, 3, \ldots$. By Corollary 2,

$$(4)\qquad [(B^k)']_{\phi,\psi} \le ||\underset{\sim}{\omega}||_2^{-2}[W]_{\phi,\psi}\tau_\phi(A^k) \le ||\underset{\sim}{\omega}||_2^{-2}[W]_{\phi,\psi}||(B^k)'||_\phi$$

https://doi.org/10.1017/S0004972700013034 Published online by Cambridge University Press

for $k=1, 2, 3, \ldots$ where $W = \|\underset{\sim}{\psi}\|_2^2 I - \underset{\sim}{\psi}\,\underset{\sim}{\psi}'$. It is well known that for any two given matrix norms $\|\cdot\|_\phi$ and $[\ ]_{\phi,\psi}$, there exists a constant $c(\phi,\psi) > 0$ which depends only on ϕ and ψ such that

$$c(\phi,\psi)\|(B^k)'\|_\phi \le [(B^k)']_{\phi,\psi} \tag{5}$$

(see for example [1, p. 199]). From (4) and (5), we obtain

$$c(\phi,\psi)\|\underset{\sim}{\psi}\|_2^2\|(B^k)'\|_\phi[W]_{\phi,\psi}^{-1} \le \tau_\phi(A^k) \le \|(B^k)'\|_\phi \tag{6}$$

for $k = 1, 2, 3, \ldots$. Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \tau_\phi(A^k)^{1/k} = \lim_{k\to\infty} \|(B^k)'\|_\phi^{1/k} = \rho(B') = \rho(B)$$

and hence (3) is proved.

Merikoski [3] has shown that for any $n \times n$ (real or complex) matrix B,

$$[B]_{2,q} \le n^{1/q}\|B\|_2 \tag{7}$$

where $1 \le q \le \infty$ and the bound is attainable for some particular B. A more general form of the inequality (7) is given in [2, eqn. 10]:

$$[B]_{\phi,\psi} \le [I]_{\phi,\psi}\|B\|_\phi \tag{8}$$

where ϕ and ψ (monotonic) are any two vector norms. If ϕ and ψ are ℓ_p and ℓ_q norms respectively, then (8) reduces to:

$$[B]_{p,q} \le n^{1/q}\|B\|_p \tag{9}$$

where $1 \le p \le \infty$ and $1 \le q \le \infty$. We proceed to show that our bounds on $\|B\|_p$ improve upon the Merikoski bound (7) and in general, the bounds given in (9) when B is confined to a special class of matrices and $p \ge 2$.

THEOREM 2. *Let B be any $n \times n$ matrix of the form $B = F(I-\underset{\sim}{\chi}\,\underset{\sim}{\chi}'/\|\underset{\sim}{\chi}\|_2^2)$ where F and $\underset{\sim}{\chi} = (\nu_i)$ are real. Then for $2 \le p \le \infty$ and $1 \le q \le \infty$,*

$$[B]_{p,q} \le \|\underset{\sim}{\chi}\|_2^{-2}[W]_{p,q}\|B\|_p < n^{1/q}\|B\|_p .$$

where $W = ||\underset{\sim}{\chi}||_2^2 I - \underset{\sim}{\chi}\,\underset{\sim}{\chi}'$.

Proof. The left inequality follows from Corollary 2 and hence it suffices to show the right inequality, namely

$$||\underset{\sim}{\chi}||_2^{-2}[W]_{p,q} < n^{1/q} \quad \text{for} \quad p \geq 2 \,.$$

Note that the i^{th} column ξ_i of the symmetric matrix W is given by

$$\xi_i = -(\nu_1\nu_i, \nu_2\nu_i, \ldots, \nu_{i-1}\nu_i, -\sum_{\substack{j=1\\ j\neq 1}}^{n} \nu_j^2, \nu_{i+1}\nu_i, \ldots, \nu_n\nu_i)$$

and therefore,

$$||\xi_i||_p = [|\nu_i|^p (\sum_{\substack{j=1\\ j\neq i}}^{n}|\nu_j|^p) + (\sum_{\substack{j=1\\ j\neq i}}^{n} \nu_j^2)^p]^{1/p}$$

for $i = 1,2,\ldots,n$. It is well known that the ℓ_p norm of a given vector is a monotonic non-increasing function of p [4] and thus $||\xi_i||_p \leq ||\xi_i||_2$ for $p \geq 2$. We have

$$||\xi_i||_2 = [(\sum_{\substack{j=1\\ j\neq i}}^{n}\nu_j^2)(\sum_{j=1}^{n} \nu_j^2)]^{1/2} \leq ||\underset{\sim}{\chi}||_2^2$$

and hence $||\xi_i||_p \leq ||\underset{\sim}{\chi}||_2^2$ for $i = 1,2,\ldots,n$. Since $\underset{\sim}{\chi} \neq \underset{\sim}{0}$, there exists a k such that $\nu_k \neq 0$ and for this k, $||\xi_k||_p \leq ||\xi_k||_2 < ||\underset{\sim}{\chi}||_2^2$. Using the fact that the ℓ_q norm is monotonic, it follows that

$$[W]_{p,q} < ||\underset{\sim}{\chi}||_2^2\, n^{1/q}$$

□

Example. We consider the matrix A in Example 6.1 of [5] :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 5 & 4 \\ 0 & 3 & 0 \end{bmatrix}$$

Let $\underset{\sim}{\chi} = \underset{\sim}{\omega} = (1, 7, 3)$, the Perron vector of A corresponding to the eigenvalue 7 . Consider

$$B = [A - \underset{\sim}{\omega}(\underset{\sim}{\omega}'A/||\underset{\sim}{\omega}||_2^2)]' = A'(I - \underset{\sim}{\omega}\,\underset{\sim}{\omega}'/||\underset{\sim}{\omega}||_2^2)$$

https://doi.org/10.1017/S0004972700013034 Published online by Cambridge University Press

which can be evaluated as:

$$B = 59^{-1} \begin{bmatrix} -14 & 20 & -42 \\ 14 & -20 & 42 \\ -28 & 40 & -84 \end{bmatrix}$$

The matrix $W = ||\underset{\sim}{\omega}||_2^2 I - \underset{\sim}{\omega}\,\underset{\sim}{\omega}'$ is given by:

$$W = \begin{bmatrix} 58 & -7 & -3 \\ -7 & 10 & -21 \\ -3 & -21 & 50 \end{bmatrix}$$

Thus,

$$(10) \qquad ||\underset{\sim}{\omega}||_2^2[B]_{p,q}/[W]_{p,q} \le ||B||_p$$

and the values of these bounds on $||B||_p$ are compared with those given by (9) for $p = 1,2, \infty$ and $q = 1,2, \infty$ in the following table.

Values of the bounds $||\underset{\sim}{\omega}||_2^2[B]_{p,q}/[W]_{p,q}$ and $n^{-1/q}[B]_{p,q}$ (in brackets) for $p = 1,2, \infty$ and $q = 1,2, \infty$ in the given Example

p \ q	1	2	∞
1	1.6889 (1.7175)	1.8085 (1.9016)	2.2703 (2.8475)
2	1.3578 (1.0518)	1.4262 (1.1645)	1.7587 (1.7437)
∞	1.1783 (0.8588)	1.2236 (0.9508)	1.4482 (1.4237)

We observe from the table that the bounds given by (10) are better than those given by (9) for $p = 2, \infty$ and $q = 1,2, \infty$. The actual values of $||B||_p$ are 2.8475, 2.0169 and 2.5763 for $p = 1,2, \infty$ respectively.

References

[1] P. Lancaster, *Theory of Matrices*, (Academic Press, New York, 1969).

[2] J. -F. Maitre, "Norme composée et norme associée généralisée d'une matrice", *Numer. Math.* 10 (1967) 132-141.

[3] J.K. Merikoski, "Lower Bounds for $||A||_2$ by Hölder Norms", *Linear and Multilinear Algebra* 9 (1981) 341-344.

[4] A. Ostrowski, "Über Normen von Matrizen", *Math. Z.* 63 (1955), 2-18.

[5] U.G. Rothblum and C.P. Tan, "Upper Bounds on the Maximum Modulus of Subdominant Eigenvalues of Nonnegative Matrices", *Linear Algebra Appl.* 66 (1985), 45-86.

[6] E. Seneta and C.P. Tan, "The Euclidean and Frobenius Ergodicity Coefficients and Spectrum Localization", *Bull. Malaysian Math. Soc. (2)* 7 (1984), 1-7.

[7] C.P. Tan, "A Bound Problem in the Modeling of Computer Systems and Queueing Networks", *Mathematical Computer Performance and Reliability*, (G. Iazeolla, P.J. Courtois and A. Hordijk, Eds.) North-Holland, Amsterdam, (1984) 303-311.

Department of Mathematics,
University of Malaya,
Kuala Lumpur 59100,
Malaysia.