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The algebra of functions with Fourier transforms in a given function space

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Let G be a locally compact abelian group and $\hat{G}$ be its dual group. For $1 \le p < \infty$, let $A_p(G)$ denote the set of all those functions in $L_1(G)$ whose Fourier transforms belong to $L_p(\hat{G})$. Let $M(A_p(G))$ denote the set of all functions φ belonging to $L_\infty(\hat{G})$ such that $\varphi \cdot \hat{f}$ is Fourier transform of an L^1-function on G whenever f belongs to $A_p(G)$. For $1 \le p < q < \infty$, we prove that $A_p(G) \subsetneq A_q(G)$ provided G is nondiscrete. As an application of this result we prove that if G is an infinite compact abelian group and $1 \le p \le 4$ then $l_p(\hat{G}) \subsetneq M(A_p(G))$, and if $p > 4$ then there exists $\psi \in l_p(\hat{G})$ such that ψ does not belong to $M(A_p(G))$.

1. Introduction

Let G be a locally compact abelian group and let $1 \le p < \infty$. $A_p(G)$ is the Banach algebra consisting of all those functions $f \in L_1(G)$ for which $\hat{f} \in L_p(\Gamma)$ where Γ denotes the dual group of G. The multiplication in $A_p(G)$ is the convolution and the norm is given by

$$\|f\|^p = \|f\|_1 + \|\hat{f}\|_p \quad (f \in A_p(G)) .$$

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In [7] and [8] Larsen stated without proof that the $A_p(G)$ are distinct for distinct p unless G is discrete in which case $A_p(G) = L_1(G)$, $\forall\, p$. In a private communication Professor Larsen told us that his proof of this assertion was fallacious and he gave a proof of the fact that $A_p(G) \subsetneqq A_q(G)$, $p < q$ in the case that G is R or T or G is infinite compact and $1 \le p < 2$. In this paper we shall prove this result for nondiscrete G . As an application we shall show that for an infinite compact abelian group G , $l_p(\Gamma) \subsetneqq M(A_p(G))$, if $1 \le p \le 4$, and there exists $\psi \in l_p(\Gamma)$ such that $\psi \notin M(A_p(G))$, if $p > 4$. Here $M(A_p(G))$ denotes the algebra of multipliers of $A_p(G)$. For the results on $A_p(G)$ and its multipliers we refer the reader to [7], where the standard results from harmonic analysis and functional analysis are also given in the appendices.

2. Results on $A_p(G)$

In this section we shall prove that for nondiscrete G and $1 \le p < q < \infty$, $A_p(G) \subsetneqq A_q(G)$. The proof of this fact depends on several intermediary results which are of interest for their own sake.

PROPOSITION 1. *Let G be a nondiscrete locally compact abelian group and let $1 \le p < 2$. Then $A_p(G) \subsetneqq A_q(G)$, provided $p < q < \infty$.*

Proof. Since G is nondiscrete, therefore Γ is non-compact. Let U be a symmetric neighbourhood of 0 in Γ such that $\overline{U}$ is compact. Choose a sequence $\{\gamma_n\}$ in Γ such that (γ_i+U+U) is disjoint from (γ_j+U+U) , unless $i = j$. Let $g = \chi_U$ and

$$h = \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} \chi_{\gamma_k+U+U}$$

(for any set A , χ_A denotes the characteristic function of A). Then g and h both belong to $L_2(\Gamma)$ and hence there exists $f \in L_1(G)$ such that $\hat{f} = g * h$. Moreover, $g \in L_1(\Gamma)$ and $h \in L_q(\Gamma)$; therefore

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$f \in A_q(G)$. But $f \notin A_p(G)$. In fact

$$g * h(\tau) = \frac{1}{k^{1/p}} \rho(U)$$

for each $\tau \in \gamma_k + U$, where ρ is the Haar measure on Γ. Since (γ_i+U) and (γ_j+U) are disjoint for $i \neq j$, it follows that $g * h \notin L_p(\Gamma)$, that is, $f \notin A_p(G)$.

REMARK. The proof of Proposition 1 is a modification of the arguments of Martin and Yap [9], p. 218.

COROLLARY 1. *Let* $G = T$, *the circle group, and* $1 \leq p < q < \infty$. *Then* $A_p(G) \subsetneqq A_q(G)$.

Proof. If $1 \leq p < 2$, then the result follows from Proposition 1. If $p \geq 2$ then $q > 2$ and the conjugate index q' lies between 1 and 2. It is known (Edwards [2], p. 147) that there exists $f \in L_{q'}(G)$ such that $\hat{f}$ does not belong to $L_p(\Gamma)$. Such a function f belongs to $A_q(G)$ but $f \notin A_p(G)$.

PROPOSITION 2. *Let* $G = R$, *the real line, and* $1 \leq p < q < \infty$. *Then* $A_p(G) \subsetneqq A_q(G)$.

Proof. Since $p < q$ it follows that $A_p(G) \subseteq A_q(G)$. Moreover, it can be easily seen that $\|f\|^q \leq 2\|f\|^p$ for every $f \in A_p(G)$. Therefore the assumption that $A_p(G) = A_q(G)$ would lead to the existence of a constant $K > 0$ such that

$$\|f\|_1 + \|\hat{f}\|_p \leq K\left[\|f\|_1 + \|\hat{f}\|_q\right] \tag{1}$$

for every $f \in A_p(G)$. We shall show that (1) leads to a contradiction. For this purpose consider the function

$$\Delta_\alpha(x) = \begin{cases} 1 - \frac{|x|}{\alpha}, & |x| \leq \alpha, \\ 0, & |x| > \alpha, \end{cases}$$

where $\alpha > 0$.

Let $f_\alpha = \check{\Delta}_\alpha$ where $\check{}$ denotes the inverse Fourier transform, that is,

$$f_\alpha(x) = \int_{-\infty}^{\infty} \Delta_\alpha(t)e^{itx}dt .$$

Then $\|f_\alpha\|_1 = 2\pi$; (see [4], pp. 21-22). Moreover $\hat{f}_\alpha = 2\pi\Delta_\alpha$ and

$$(2) \qquad \|\hat{f}_\alpha\|_p = 2\pi\left(\frac{2}{p+1}\right)^{1/p}\alpha^{1/p} \quad \text{for} \quad 1 \le p < \infty .$$

(2) follows from an easy computation. From (1) and (2) it follows that

$$2\pi + 2\pi\left(\frac{2}{p+1}\right)^{1/p}\alpha^{1/p} \le K\left[2\pi+2\pi\left(\frac{2}{q+1}\right)^{1/q}\alpha^{1/q}\right]$$

or

$$(3) \qquad 2\pi\left(\frac{2}{p+1}\right)^{1/p}\alpha^{1/p} \le (K-1)2\pi + 2K\pi\left(\frac{2}{q+1}\right)^{1/q}\alpha^{1/q} .$$

Dividing (3) by $\alpha^{1/q}$ on both sides we get

$$(4) \qquad 2\pi\left(\frac{2}{p+1}\right)^{1/p}\alpha^{1/p-1/q} \le (K-1)2\pi\alpha^{-1/q} + 2K\pi\left(\frac{2}{q+1}\right)^{1/q} .$$

Taking limit as $\alpha \to \infty$ in (4) we see that the right hand side of (4) remains bounded while the left hand side tends to ∞ since $p < q$. This contradiction establishes the proposition.

PROPOSITION 3. *Let G be an infinite compact totally disconnected abelian group and $1 \le p < q < \infty$. Then $A_p(G) \subsetneq A_q(G)$.*

Proof. As in the proof of Proposition 2, the assumption that $A_p(G) = A_q(G)$ would lead to the existence of a constant $K > 0$ such that

$$\|f\|_1 + \|\hat{f}\|_p \le K\left[\|f\|_1+\|\hat{f}\|_q\right]$$

for every $f \in A_p(G)$. Then we shall have

$$(5) \qquad \|\hat{f}\|_p \le K\|f\|_1 + K\|\hat{f}\|_q \quad \text{for every} \quad f \in A_p(G) .$$

We shall show that (5) leads to a contradiction. Since G is compact and totally disconnected, there exists a neighbourhood basis $\{V_\alpha\}_{\alpha \in I}$ of

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0 in G consisting of open and closed subgroups of G ; (see [5], p. 62). Since G is infinite compact, it follows that

(6) $$\lim_{\alpha} \lambda(V_\alpha) = 0 ,$$

where λ denotes the normalized Haar measure on G .

Let $\lambda_\alpha = \lambda(V_\alpha)$ and let X_α denote the annihilator subgroup of V_α . Since V_α is open and closed, it follows that X_α is finite. Let n_α be the number of points in X_α and let $f_\alpha = \chi_{V_\alpha}$. Then $\|f_\alpha\|_1 = \lambda_\alpha$ and $\hat{f}_\alpha = \lambda_\alpha \chi_{X_\alpha}$. Also $\|f_\alpha\|_2^2 = \lambda_\alpha$. Therefore by the Plancherel Theorem we get

$$\lambda_\alpha = \|f_\alpha\|_2^2 = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2 = \lambda_\alpha^2 \cdot n_\alpha .$$

Therefore $n_\alpha = 1/\lambda_\alpha$. Now, for $1 \le p < \infty$, we have $\|\hat{f}_\alpha\|_p = \lambda_\alpha \cdot n_\alpha^{1/p} = \lambda_\alpha^{1/p'}$, where $1/p + 1/p' = 1$.

From (5) it follows that

(7) $$\lambda_\alpha^{1/p'} \le K\lambda_\alpha + \lambda_\alpha^{1/q'} .$$

Dividing both sides of (7) by $\lambda_\alpha^{1/q'}$ we get

(8) $$\lambda_\alpha^{1/p'-1/q'} \le K\lambda_\alpha^{1/q} + K .$$

Taking the limit in (8) we see that the right hand side remains bounded, while the left hand side tends to ∞ , because $1/p' < 1/q'$ and λ_α tends to zero. This contradiction yields the proof.

REMARK. The above proof uses a technique due to Edwards [3], p. 196.

PROPOSITION 4. *Let G_1 and G_2 be locally compact abelian groups and $G = G_1 \times G_2$. Let $1 \le p < q < \infty$. If $A_p(G_i) \ne A_q(G_i)$ for some i , then $A_p(G) \ne A_q(G)$.*

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Proof. We may assume that $A_p(G_1) \neq A_q(G_1)$ (the proof in the other case is exactly similar). Choose $f \in A_q(G_1)$ such that $f \notin A_p(G_1)$. Let g be any non-zero function belonging to $A_p(G_2)$. Define

$$h(x, y) = f(x) \cdot g(y) \quad \text{for} \quad (x, y) \in G_1 \times G_2 .$$

Then $h \in A_q(G)$ but $h \notin A_p(G)$, because $\hat{h}(r, \eta) = \hat{f}(r) \cdot \hat{g}(\eta)$ and $\hat{h} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma_1)$ and $\hat{g} \in L_p(\Gamma_2)$, where Γ, Γ_1 and Γ_2 are the dual groups of G, G_1 and G_2 respectively.

Let G be a locally compact abelian group and let H be a closed subgroup. For a continuous function f on G which has compact support, define

$$\pi_H(f)(x) = \int_H f(xy)dy ,$$

where dy denotes Haar integral on H. It is well known (see [10], Chapter 3) that $\pi_H(f)$ is constant on cosets modulo H and that $\pi_H(f)$ defines a continuous function on G/H which has compact support. This gives a mapping π_H from $C_c(G)$ into $C_c(G/H)$, where $C_c(G)$ denotes the space of continuous functions with compact support. This mapping π_H extends to $L_1(G)$ and it maps $L_1(G)$ *onto* $L_1(G/H)$. Reiter [11] has shown that if $S(G)$ is a Segal algebra on G, then $\pi_H(S(G))$ becomes a Segal algebra on G/H. The following proposition is interesting because it shows that $\pi_H(A_p(G)) = A_p(G/H)$ under the hypothesis that H is compact. We shall use this proposition to prove Theorem 1 of this paper.

PROPOSITION 5. *Let G be a locally compact abelian group and let H be a compact subgroup of G. Then $\pi_H(A_p(G)) = A_p(G/H)$.*

Proof. Let Λ be the annihilator of H. Since H is compact, it follows that Λ is open. Also for $f \in L_1(G)$, $\pi_H(f)^\wedge = \hat{f}|_\Lambda$. Therefore $f \in A_p(G)$ implies that $\pi_H(f) \in A_p(G/H)$ and hence $\pi_H(A_p(G)) \subseteq A_p(G/H)$. To prove the other inclusion take $f' \in A_p(G/H)$. Then there exists

$f \in L_1(G)$ such that $\pi_H(f) = f'$. Consider the function $g = m_H * f$ where m_H is the normalized Haar measure on H and m_H is considered as a bounded measure on the whole group G in the usual manner. Since $\hat{m}_H = \chi_\Lambda$ it follows that $\pi_H(g) = f'$, because $\pi_H(g)$ and f' have the same Fourier transform. Also since Λ is open and $\hat{g} = \hat{f}'$ on Λ and $\hat{g} = 0$ outside Λ , it follows that $g \in A_p(G)$. Therefore $\pi_H\left(A_p(G)\right) = A_p(G/H)$.

THEOREM 1. *Let G be an infinite compact abelian group and let $1 \le p < q < \infty$. Then $A_p(G) \subsetneq A_q\ G$.*

Proof. We have already proved the theorem for totally disconnected G in Proposition 3. Let us now suppose that G is not totally disconnected. Then the dual group Γ has an element of infinite order; (see [12], p. 47). Therefore Γ contains Z (the group of integers) as a subgroup. Let H be the annihilator of this subgroup. Then the dual of G/H is isomorphic to Z and hence G/H is isomorphic to T (the circle group). By Corollary 1 it follows that $A_p(G/H) \subsetneq A_q(G/H)$. The theorem now follows from Proposition 5.

COROLLARY 2. *Let G be a nondiscrete locally compact, compactly generated abelian group and $1 \le p < q < \infty$. Then $A_p(G) \subsetneq A_q(G)$.*

Proof. From Theorem 9.8 of [5] it follows that $G = R^a \times Z^b \times F$, where a and b are nonnegative integers and F is a compact abelian group. If $a > 0$ then the result follows from Propositions 2 and 4. If $a = 0$ then since G is nondiscrete it follows that F is an infinite compact abelian group. The result then follows from Theorem 1 and Proposition 4.

PROPOSITION 6. *Let G be a locally compact abelian group and let H be an open subgroup of G . Let $f \in L_1(H)$. Define g on G by setting $g = f$ on H and $g = 0$ outside H . Then $g \in L_1(G)$ and $\hat{g} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma/\Lambda)$, where $1 \le p < \infty$ and Λ is the annihilator of H .*

Proof. It is obvious that $g \in L_1(G)$. Let F be the set of all those elements $\varphi \in L_1(\Gamma)$ which are almost everywhere constant on each coset of Λ . Let $\eta : \Gamma \to \Gamma/\Lambda$ be the quotient map. Then it follows from Theorem 28.55 of [6] that $h \to h \circ \eta$ is a Banach algebra isomorphism of $L_1(\Gamma/\Lambda)$ onto $F^{\cdot}$. Since g is zero outside H , $\hat{g}$ is constant on cosets of Λ . Moreover $\hat{g} = \hat{f} \circ \eta$ and $|\hat{g}|^p = |\hat{f}|^p \circ \eta$. Hence it follows that $\hat{g} \in L_p(\Gamma)$ if and only if $\hat{f} \in L_p(\Gamma/\Lambda)$.

COROLLARY 3. *Let G be a locally compact abelian group and $1 \le p < q < \infty$. Let H be an open subgroup of G such that $A_p(H) \subsetneq A_q(H)$. Then $A_p(G) \subsetneq A_q(G)$.*

Proof. Let $f \in A_q(H)$ such that $f \notin A_p(H)$. Define g as in Proposition 6. Then Proposition 6 implies that $g \in A_q(G)$ but $g \notin A_p(G)$.

THEOREM 2. *Let G be a nondiscrete locally-compact abelian group and $1 \le p < q < \infty$. Then $A_p(G) \subsetneq A_q(G)$.*

Proof. Theorem 2.4.1 of [12] implies that there exists an open subgroup H of G such that $H = R^n \times F$ where n is a nonnegative integer and F is a compact abelian group. If $n > 0$ then $A_p(H) \subsetneq A_q(H)$ by Proposition 4. If $n = 0$ then since G is nondiscrete and H is open it follows that F is an infinite compact abelian group. Therefore $A_p(H) \subsetneq A_q(H)$ by Theorem 1. Thus, in any case, $A_p(H) \subsetneq A_q(H)$. The proof now follows from Corollary 3.

3. Multipliers of $A_p(G)$

In this section G will denote an infinite compact abelian group and Γ its dual group. We shall prove:

THEOREM 3. *$l_p(\Gamma) \subsetneq M(A_p(G))$ for $1 \le p \le 4$. If $4 < p < \infty$ then there exists $\psi \in l_p(\Gamma)$ such that $\psi \notin M(A_p(G))$.*

Proof. It is known that every bounded function on Γ defines a

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multiplier on $A_p(G)$, provided $1 \le p \le 2$ (see [1], p. 207). Therefore we may assume that $p > 2$. Suppose $p \le 4$ and $\varphi \in l_p(\Gamma)$ and $f \in A_p(G)$. By Hölder's inequality we have

$$\sum_{\gamma} |\varphi\hat{f}(\gamma)|^2 \le \left\{\sum_{\gamma} |\varphi(\gamma)|^{2p/p-2}\right\}^{1-2/p} \left\{\sum_{\gamma} |\hat{f}(\gamma)|^p\right\}^{2/p} < \infty ,$$

because $\varphi \in l_p(\Gamma)$ and $\frac{2p}{p-2} \ge p$ for $2 < p \le 4$. Therefore there exists $g \in L_2(G)$ such that $\hat{g} = \varphi \cdot \hat{f}$. Clearly $g \in A_p(G)$. Therefore $\varphi \in M(A_p(G))$. Since the constant functions on Γ define multipliers on $A_p(G)$ it follows that $l_p(\Gamma) \subsetneq M(A_p(G))$.

Let us now consider the case when $4 < p < \infty$. By Theorem 1, there exists a function $f \in L_1(G)$ such that $\sum_{\gamma} |\hat{f}(\gamma)|^4 = \infty$ and $\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)|^p < \infty$. Let $\varphi = \hat{f}$. Then $\varphi \in l_p(\Gamma)$. However $\varphi^2 \notin l_2(\Gamma)$. Therefore, from Theorem 1.1 of [1] it follows that there exists a function ε on Γ such that $\varepsilon(\gamma) = \pm 1$ and for no integrable function g on G we have $\hat{g}(\gamma) = \varepsilon(\gamma)\varphi(\gamma)\cdot\varphi(\gamma)$. Now the function $\psi(\gamma) = \varepsilon(\gamma)\varphi(\gamma)$ is a function belonging to $l_p(\Gamma)$, but $\psi\cdot\hat{f}$ is not Fourier transform of any integrable function. Therefore $\psi \notin M(A_p(G))$ even though $\psi \in l_p(\Gamma)$.

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