Appendix B

The groups of the Standard Model

The Standard Model is constructed by insisting that the equations of the model retain the same form after certain transformations. For instance, we require that the equations take the same form in every inertial frame of reference, so that they are covariant under a Lorentz transformation; this may be a rotation of axes or a boost, or a combination of rotation and boost. The Lagrangian density that describes the Standard Model takes the same form in the new coordinate system, and the Lorentz transformation is said to be a *symmetry transformation*. In the Standard Model, as well as symmetries under coordinate transformations, there are ‘internal’ symmetries of the particle fields. The corresponding symmetry transformations are conveniently represented by matrices.

It is characteristic of symmetry transformations that they satisfy the mathematical axioms of a *group*, which we set out below. In this appendix we consider some properties of the groups that play a special role in the Standard Model.

B.1 Definition of a group

A group G is a set of elements $a, b, c, \ldots$, together with a rule that combines any two elements a,b of G to form an element ab, which also belongs to G, satisfying the following conditions.

(i) The rule is *associative*: $a(bc) = (ab)c$.
(ii) G contains a unique *identity element* I such that, for every element a of G,

$$aI = Ia = a.$$

(iii) For every element a of G there exists a unique *inverse* element a^{-1} such that

$$aa^{-1} = a^{-1}a = I.$$

If also $ab = ba$ for all a, b the group is said to be *commutative* or *Abelian*.

It is usually easy to determine whether or not a given set of elements and their combination law satisfy these axioms. For example, the set of all integers forms an Abelian group under addition, with 0 the identity element. The set of all non-singular $n \times n$ matrices ($n > 1$) forms a non-Abelian group under matrix multiplication. The permutations of the numbers $1, 2, \ldots, n$ form a group which has $n!$ elements; this is an example of a *finite group*. The group of rotations of the coordinate axes is a three-parameter *continuous group*: an element is specified by three parameters that take on a continuous range of values. We shall be concerned principally with groups of this type.

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B.2 Rotations of the coordinate axes, and the group *SO(3)*

Consider a rotation of the coordinate axes about the origin. If the coordinates of a point P are (x^1, x^2, x^3) in a frame of reference K, and (x'^1, x'^2, x'^3) in a frame K', rotated relative to K, the x'^i are related to the x^i by a real linear transformation of the form

$$x'^i = R^i_j x^j. \tag{B.1}$$

$\mathbf{R} = (R^i_j)$ is the *rotation matrix*. For example, a rotation of the axes through an angle θ about the 03 axis in a right-handed sense is given by

$$\begin{aligned} x'^1 &= x^1 \cos\theta + x^2 \sin\theta, \\ x'^2 &= -x^1 \sin\theta + x^2 \cos\theta, \\ x'^3 &= x^3, \end{aligned}$$

and corresponds to the matrix

$$\mathbf{R}_{03}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.2}$$

We may regard the x'^i and x^i as 3×1 (column) matrices $\mathbf{x}'$ and $\mathbf{x}$, and write the transformation (B.1) as

$$\mathbf{x}' = \mathbf{R}\mathbf{x}.$$

The transpose $\mathbf{x}^{\mathrm{T}}$ of $\mathbf{x}$ is a 1×3 (row) matrix, and the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$x' y' = \mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{y}^{\mathrm{T}}\mathbf{x}.$$

In particular, the length OP is given by $\surd(\mathbf{x}^{\mathrm{T}}\mathbf{x})$. Since a rotation of axes preserves scalar products,

$$\mathbf{x}'^{\mathrm{T}}\mathbf{y}' = \mathbf{x}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}.$$

This holds for *all* pairs **x, y**. Hence

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \tag{B.3}$$

where $\mathbf{I}$ is the identity matrix: hence the inverse of $\mathbf{R}$ is the transpose $\mathbf{R}^{\mathrm{T}}$ of $\mathbf{R}$ and $\mathbf{R}$ is said to be an *orthogonal matrix*.

$$\text{Since } \det \mathbf{R}^{\mathrm{T}} \det \mathbf{R} = \det(\mathbf{R}^{\mathrm{T}}\mathbf{R}) = \det \mathbf{I} = 1 \text{ and } \det \mathbf{R}^{\mathrm{T}} = \det \mathbf{R}, \tag{B.4}$$

$$(\det \mathbf{R})^2 = 1, \quad \det \mathbf{R} = \pm 1.$$

Matrices corresponding to pure or ‘proper’ rotations have $\det \mathbf{R} = +1$. We can see this by noting that the identity rotation is a proper rotation, and $\det \mathbf{I} = 1$. Any proper rotation can be constructed as a sequence of infinitesimal rotations starting from $\mathbf{I}$ and hence by continuity also has determinant $+1$.

The product of two orthogonal matrices is an orthogonal matrix, since

$$(\mathbf{R}_1\mathbf{R}_2)^{\mathrm{T}} = \mathbf{R}_2{}^{\mathrm{T}}\mathbf{R}_1{}^{\mathrm{T}} = \mathbf{R}_2{}^{-1}\mathbf{R}_1{}^{-1} = (\mathbf{R}_1\mathbf{R}_2)^{-1},$$

and if $\det \mathbf{R}_1 = 1$ and $\det \mathbf{R}_2 = 1$,

$$\det(\mathbf{R}_1\mathbf{R}_2) = \det \mathbf{R}_1 \det \mathbf{R}_2 = 1.$$

Hence real orthogonal 3×3 matrices with $\det \mathbf{R} = 1$ form a group under matrix multiplication. This group is called *the special orthogonal group* and is denoted by $SO(3)$.

Orthogonal matrices with $\det \mathbf{R} = -1$ also preserve scalar products. It is easy to see that inversion of the coordinate axes in the origin, $x'^i = -x^i$, corresponds to an

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orthogonal matrix with determinant -1; a general 'improper' rotation corresponds to inversion in the origin together with a proper rotation. Improper rotation matrices do not form a group, since the product of two improper rotations is a proper rotation.

A general proper rotation may be built up as a sequence of rotations about three different axes. For example, consider

$$\mathbf{R}(\psi,\theta,\phi) = \mathbf{R}_{03''}(\psi)\mathbf{R}_{02'}(\theta)\mathbf{R}_{03}(\phi), \tag{B.5}$$

in an obvious notation. The direction of $03''$ is defined by θ and ϕ, and then ψ defines the final orientation of $01''2''$ in the plane perpendicular to $03''$. Thus each element of $SO(3)$ is specified by just three parameters. (ψ, θ, ϕ are known as the Euler angles.)

We can also interpret the transformation (B.1) in an *active* sense. Consider a system described by a wave function $\mathbf{\Phi}(\mathbf{x})$ in the frame K. The system is described by $\mathbf{\Phi}'(\mathbf{x}') = \mathbf{\Phi}(\mathbf{R}^{-1}\mathbf{x}')$ in the frame K'. This is the *passive* interpretation. We might, alternatively, drop the primes on the coordinates and give this equation an active interpretation, supposing that the axes have been held fixed and the system given the inverse rotation $\mathbf{R}^{-1}$. The wave function of the rotated system is $\mathbf{\Phi}'(\mathbf{x}) = \mathbf{\Phi}(\mathbf{R}^{-1}\mathbf{x})$.

B.3 The group *SU(2)*

An $n \times n$ matrix $\mathbf{U}$ is *unitary* if $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$. The product of two unitary matrices is unitary. Hence $n \times n$ unitary matrices form a group under matrix multiplication, denoted by $U(n)$.

Since

$$\det(\mathbf{U}\mathbf{U}^\dagger) = \det\mathbf{U}\det\mathbf{U}^* = \det\mathbf{U}(\det(\mathbf{U}))^* = \det\mathbf{I} = 1,$$

we may write $\det\mathbf{U} = \mathrm{e}^{\mathrm{i}n\alpha}$, where α is real.

The *special unitary group SU*(2) is the group of all 2×2 unitary matrices with determinant equal to 1. These form a group, since if $\det\mathbf{U}_1 = 1$ and $\det\mathbf{U}_2 = 1$ then $\det(\mathbf{U}_1\mathbf{U}_2) = \det\mathbf{U}_1\det\mathbf{U}_2 = 1$. $SU(2)$ is a *sub-group* of $U(2)$. Every element of $U(2)$ is the product of a phase factor $\mathrm{e}^{\mathrm{i}\alpha}$, which is an element of $U(1)$, and an element of $SU(2)$.

The group $SU(2)$ is related in a remarkable way to the rotation group $SO(3)$ described in Section B.2. It is central to the electroweak sector of the Standard Model.

Any element of $U(2)$ can be put in the form

$$\mathbf{U} = \exp(\mathbf{iH})$$

where $\mathbf{H}$ is a Hermitian matrix (Appendix A). A general 2×2 Hermitian matrix may be taken as

$$\mathbf{H} = \begin{pmatrix} \alpha^0 + \alpha^3 & a^1 - \mathrm{i}\alpha^2 \\ \alpha^1 + \mathrm{i}\alpha^2 & \alpha^0 - \alpha^3 \end{pmatrix}$$

where the $\alpha^\mu (\mu = 0, 1, 2, 3)$ are four real parameters. This choice enables us to write

$$\mathbf{H} = \alpha^0\mathbf{I} + \alpha^{\mathrm{k}}\sigma^{\mathrm{k}}, \tag{B.6}$$

where the index k runs from 1 to 3, and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The σ^k are the same as the Pauli spin matrices, and hence they satisfy

$$\begin{aligned} &(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = \mathbf{I};\ \sigma^j\sigma^k + \sigma^k\sigma^j = 0,\ j \neq k; \\ &[\sigma^1, \sigma^2] = \sigma^1\sigma^2 - \sigma^2\sigma^1 = 2\mathrm{i}\sigma_3,\ \text{etc.} \end{aligned} \tag{B.7}$$

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Since the unit matrix $\mathbf{I}$ commutes with all matrices, a general member of $U(2)$ can be written as

$$\mathbf{U} = \exp\ \mathrm{i}(\alpha^0\mathbf{I} + \alpha^k\sigma^k) = \exp(\mathrm{i}\alpha^0)\exp(\mathrm{i}\alpha^k\sigma^k).$$

The phase factor $\exp(\mathrm{i}\alpha^0)$ belongs to the group $U(1)$. Hence elements of $SU(2)$ are of the form

$$\mathbf{U}_s = \exp(\mathrm{i}\alpha^k\sigma^k). \tag{B.8}$$

An element may be specified by the three parameters α^k; the matrices σ^k are the corresponding *generators* of the group. Each has zero trace (see Problem B.1).

The algebra of the σ^k matrices enables us to write these elements in closed form. Let us formally consider the α^k to make up a vector $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{\alpha}}$, where $\hat{\boldsymbol{\alpha}}$ is the corresponding unit vector, and write the 'scalar product' $\alpha^k\sigma^k$ as $\alpha\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}$. It is easy to see that

$$(\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma})^2 = \hat{\alpha}^j\sigma^j\hat{\alpha}^k\sigma^k = \hat{\alpha}^j\hat{\alpha}^j\mathbf{I} = \mathbf{I},$$

since $\sigma^j\sigma^k + \sigma^k\sigma^j = 0$ and $(\sigma^1)^2 = \mathbf{I}$, etc. Then the power series expansion of (B.8) gives

$$\begin{aligned}\mathbf{U}_s &= \mathbf{I} + \mathrm{i}\alpha(\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}) + \frac{(\mathrm{i}\alpha)^2}{2!}\mathbf{I} + \cdots \\ &= \cos\alpha\mathbf{I} + \mathrm{i}\ \sin\alpha(\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}).\end{aligned} \tag{B.9}$$

To establish the connection between the groups $SU(2)$ and $SO(3)$, we associate with each point $\mathbf{x}$ the Hermitian matrix

$$\mathbf{X}(\mathbf{x}) = \begin{pmatrix} x^3 & x^1 - \mathrm{i}x^2 \\ \mathrm{x}^1 + \mathrm{i}x^2 & -x^3 \end{pmatrix}. \tag{B.10}$$

This matrix has $\mathrm{Tr}\,\mathbf{X} = 0$ and $\det\,\mathbf{X} = -x^kx^k$.

Consider now an element $\mathbf{U}$ of $SU(2)$ and the matrix

$$\mathbf{X}' = \mathbf{U}\mathbf{X}\mathbf{U}^\dagger. \tag{B.11}$$

(We are now dropping the suffix s on $\mathbf{U}$.)

$\mathbf{X}'$ is also Hermitian, and $\mathrm{Tr}\,\mathbf{X}' = \mathrm{Tr}(\mathbf{U}\mathbf{X}\mathbf{U}^\dagger) = \mathrm{Tr}(\mathbf{U}^\dagger\mathbf{U}\mathbf{X}) = \mathrm{Tr}\,\mathbf{X} = 0$. Hence $\mathbf{X}'$ is of the form

$$\mathbf{X}' = \begin{pmatrix} x'^3 & x'^1 - \mathrm{i}x'^2 \\ x'^1 + \mathrm{i}x'^2 & -x'^3 \end{pmatrix}$$

where the x'^k are related to the x^k by a real linear transformation.

Also $\det\,\mathbf{X}' = \det\,\mathbf{U}\ \det\,\mathbf{X}\ \det\,\mathbf{U}^\dagger = \det(\mathbf{U}\mathbf{U}^\dagger)\det\,\mathbf{X} = \det\,\mathbf{X}$, so that $x'^kx'^k = x^kx^k$. Since the length of $\mathbf{x}$ is preserved and the transformation may be continuously generated from the identity matrix (see Problem B.3), the transformation must correspond to a proper rotation of the coordinate axes and hence to a rotation matrix $\mathbf{R}(\mathbf{U})$.

As an example, the $SU(2)$ matrix

$$\mathbf{U} = \exp[\mathrm{i}(\theta/2)\sigma^3] = \cos(\theta/2)\mathbf{I} + \mathrm{i}\ \sin(\theta/2)\sigma^3 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2} \end{pmatrix}, \tag{B.12}$$

where we have used (B.9), corresponds to the rotation matrix $\mathbf{R}_{03}(\theta)$ of equation (B.2). This may be verified by direct matrix multiplication.

The matrices $\mathbf{U}$ and $-\mathbf{U}$ give the same transformation (B.11), and hence correspond to the same rotation matrix: to every element of $SO(3)$ there correspond two elements of $SU(2)$, differing by a factor of -1. In the example (B.12) above, rotations of θ and $\theta + 2\pi$ about the 03 axis correspond to the same rotation matrix, but give matrices $\mathbf{U}$ and $-\mathbf{U}$, respectively in $SU(2)$.

B.4 The group $SL(2,\mathrm{C})$ and the proper Lorentz group

The set of all 2×2 matrices with complex elements and with determinant equal to 1 evidently forms a group under matrix multiplication. This group is denoted by $SL(2,\mathrm{C})$. It is related to the group of proper Lorentz transformations in much the same way as the group $SU(2)$ is related to the group of proper rotations.

We now associate with each point $x = (x^0, \mathbf{x})$ in space-time the general Hermitian matrix

$$\mathbf{X}(x) = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} \tag{B.13}$$

which has

$$\det \mathbf{X} = (x^0)^2 - x^k x^k.$$

Consider an element $\mathbf{M}$ of $SL(2,\mathrm{C})$ and the matrix $\mathbf{X}'$ given by

$$\mathbf{M}^\dagger \mathbf{X}' \mathbf{M} = \mathbf{X} \text{ or } \mathbf{X}' = (\mathbf{M}^{-1})^\dagger \mathbf{X} \mathbf{M}^{-1}. \tag{B.14}$$

Then $\mathbf{X}'$ is also Hermitian and hence we can write

$$\mathbf{X}' = \begin{pmatrix} x'^0 + x'^3 & x'^1 - \mathrm{i}x'^2 \\ x'^1 + \mathrm{i}x'^2 & x'^0 - x'^3 \end{pmatrix},$$

where the x'^μ are related to the $x^\mu (\mu = 0, 1, 2, 3)$ by a real linear transformation. Also

$$\det \mathbf{M}^\dagger \mathbf{X}' \mathbf{M} = \det \mathbf{M}^\dagger \det \mathbf{X}' \det \mathbf{M} = \det \mathbf{X}' = \det \mathbf{X}$$

so that

$$(x'^0)^2 - x'^k x'^k = (x^0)^2 - x^k x^k.$$

Hence the matrix $\mathbf{M}$ corresponds to a Lorentz transformation matrix $\mathbf{L(M)}$. The matrices $\mathbf{L(M)}$ form a group that includes the identity transformation $\mathbf{L(I)} = \mathbf{I}$, and hence by continuity correspond to proper Lorentz transformations.

A general proper Lorentz transformation between frames K and K' is specified by six parameters: three parameters to give the velocity $\mathbf{v}$ of K' relative to K and three parameters to give the orientation of K' relative to K. A general 2×2 complex matrix is defined by eight real parameters. The condition $\det \mathbf{M} = 1$ reduces this number to six. Hence a matrix $\mathbf{M}$ can be found corresponding to every proper Lorentz transformation. The matrices $\mathbf{M}$ and $-\mathbf{M}$ give the same transformation (B.14): two elements of $SL(2,\mathrm{C})$ correspond to each element of the proper Lorentz group.

The matrix

$$\mathbf{P} = \exp[(\theta/2)\sigma^3] = \cosh(\theta/2)\mathbf{I} + \sinh(\theta/2)\sigma^3 = \begin{pmatrix} \mathrm{e}^{\theta/2} & 0 \\ 0 & \mathrm{e}^{-\theta/2} \end{pmatrix} \tag{B.15}$$

corresponds to the Lorentz boost (2.3) of Chapter 2, as may be verified by direct matrix multiplication.

More generally, a Lorentz boost from a frame K to a frame K' moving with velocity $v = \tanh \theta$ in the direction of the unit vector $\hat{\mathbf{v}}$ is given by

$$\mathbf{P} = \exp[(\theta/2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}] = \cosh(\theta/2)\mathbf{I} + \sinh(\theta/2)\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}$$

where $\boldsymbol{\sigma} = \left(\sigma^1, \sigma^2, \sigma^3\right)$.

Note that, since the matrices σ^k are Hermitian, so also is any matrix $\mathbf{P}$ corresponding to a Lorentz boost.

B.5 Transformations of the Pauli matrices

In discussing Lorentz transformations, it is convenient to write $\mathbf{I} = \sigma^0$ and introduce the notation

$$\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3), \qquad \tilde{\sigma}^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3). \tag{B.16}$$

Then from (B.13)

$$\mathbf{X}(x) = x^0\sigma^0 + x^k\sigma^k = x_\mu\tilde{\sigma}^\mu, \qquad \mathbf{X}'(x') = x'_\mu\tilde{\sigma}^\mu.$$

The relation

$$\mathbf{M}^\dagger\mathbf{X}'\mathbf{M} = \mathbf{X}$$

gives

$$x'_\mu\mathbf{M}^\dagger\tilde{\sigma}^\mu\mathbf{M} = x_\nu\tilde{\sigma}^\nu = L^\mu{}_\nu\tilde{\sigma}^\nu x'_\mu$$

(see Problem 2.2). Since the x'_μ are arbitrary, we can deduce

$$\mathbf{M}^\dagger\tilde{\sigma}^\mu\mathbf{M} = L^\mu{}_\nu\tilde{\sigma}^\nu. \tag{B.17}$$

Also (Problem B.6)

$$L^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\nu M^\dagger\tilde{\sigma}^\mu M).$$

Similarly, by considering the matrix

$$\mathbf{X}_1(x) = x^0\sigma^0 - x^k\sigma^k = x_\nu\sigma^\nu,$$

which also has $\det \mathbf{X}_1 = (x^0)^2 - x^kx^k$, we can show that there exists a matrix $\mathbf{N}$ belonging to $SL(2,\mathrm{C})$ such that

$$\mathbf{N}^\dagger\sigma^\mu\mathbf{N} = L^\mu{}_\nu\sigma^\nu. \tag{B.18}$$

The matrices $\mathbf{M}$ and $\mathbf{N}$ are evidently related. The reader may verify directly that when $\mathbf{M} = \mathbf{P}$, where $\mathbf{P}$ is given by (B.15) and corresponds to a Lorentz boost, we can take $\mathbf{N} = \mathbf{P}^{-1}$, and this will be true for a Lorentz boost in any direction. For a pure rotation of axes, we take $\mathbf{M} = \mathbf{N} = \mathbf{U}$, where $\mathbf{U}$ is a unitary matrix. A general $\mathbf{M}$ can be constructed as a product of a rotation followed by a boost: $\mathbf{M} = \mathbf{PU}$. The corresponding $\mathbf{N}$ is given by $\mathbf{N} = \mathbf{P}^{-1}\mathbf{U}$.

Now $\mathbf{U}$ satisfies $\mathbf{UU}^\dagger = \mathbf{I}$, and we noted that $\mathbf{P}$ is Hermitian, $\mathbf{P} = \mathbf{P}^\dagger$. Hence

$$\mathbf{NM}^\dagger = (\mathbf{P}^{-1}\mathbf{U})(\mathbf{U}^\dagger\mathbf{P}) = \mathbf{I}, \tag{B.19}$$

so that $\mathbf{N}$ is the inverse of $\mathbf{M}^\dagger$.

The results (B.17) and (B.18), together with (B.19), are useful in constructing Lorentz scalars, vectors and higher order tensors.

B.6 Spinors

We define a *left-handed spinor*

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

as a complex two-component entity that transforms under a Lorentz transformation with matrix $\mathbf{L}(\mathbf{M})$ by the rule

$$\mathbf{l}' = \mathbf{Ml} \tag{B.20}$$

i.e. $l'_a = M_{ab}l_b$, where a and b take on the values 1, 2.

We similarly define a right-handed spinor

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \tag{B.21}$$

as a two-component entity that transforms by

$$\mathbf{r}' = \mathbf{N}\mathbf{r}.$$

Electrons, and all other fermions in the Standard Model, are described by spinor fields. The nomenclature of 'left-handed' and 'right-handed' is elucidated in Section 6.3.

Spinors have the remarkable property that they can be combined in pairs to make Lorentz scalars, Lorentz four-vectors and higher order Lorentz tensors. For example, $\mathbf{l}^{\dagger}\mathrm{r} = l^*{}_a r_a$ is a (complex) Lorentz scalar, since

$$\mathbf{l}'^{\dagger}\mathbf{r}' = (\mathbf{M}\mathbf{l})^{\dagger}\mathbf{N}\mathbf{r} = \mathbf{l}^{\dagger}\mathbf{M}^{\dagger}\mathbf{N}\mathbf{r} = \mathbf{l}^{\dagger}\mathbf{r}, \tag{B.22}$$

where we have used (B.19).

The quantities

$$\begin{aligned} \mathbf{l}^{\dagger}\tilde{\sigma}\mathbf{l} &= \mathbf{l}^{\dagger}(\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)\mathbf{l}, \\ \mathbf{r}^{\dagger}\sigma\mathbf{r} &= \mathbf{r}^{\dagger}(\sigma^0, \sigma^1\sigma^2, \sigma^3)\mathbf{r}, \end{aligned}$$

transform like (real) contravariant four-vectors, since

$$\mathbf{l}'^{\dagger}\tilde{\sigma}^{\mu}\mathbf{l}' = \mathbf{l}^{\dagger}\mathbf{M}^{\dagger}\tilde{\sigma}^{\mu}\mathbf{M}\mathbf{l} = L^{\mu}{}_{\nu}(\mathbf{l}^{\dagger}\tilde{\sigma}^{\nu}\mathbf{l}), \tag{B.23}$$

using (B.17), and

$$\mathbf{r}'^{\dagger}\sigma^{\mu}\mathbf{r}' = \mathbf{r}^{\dagger}\mathbf{N}^{\dagger}\sigma^{\mu}\mathbf{N}\mathbf{r} = L^{\mu}{}_{\nu}(\mathbf{r}^{\dagger}\sigma^{\nu}\mathbf{r}), \tag{B.24}$$

using (B.18).

B.7 The group *SU(3)*

The special unitary group $SU(3)$ is the group of all 3×3 unitary matrices with determinant equal to 1. Our discussion will parallel our discussion of the group $SU(2)$ in Section B.3. An element of $SU(3)$ can be expressed as

$$\mathbf{U} = \exp(\mathrm{i}\mathbf{H})$$

where $\mathbf{H}$ is a 3×3 Hermitian matrix. A general 3×3 Hermitian matrix is specified by $3^2 = 9$ real parameters (Appendix A). The condition det $\mathbf{U} = 1$, or equivalently Tr$\mathbf{H} = 0$ (Problem B.1), reduces this number to 8. In place of the σ^k matrices used in Section B.3, we have the eight traceless Hermitian matrices introduced by Gell-Mann:

$$\begin{aligned} &\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ &\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = (1/\sqrt{3}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{B.25}$$

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A general traceless Hermitian matrix is of the form

$$\mathbf{H} = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_8\lambda_8$$
$$= \begin{pmatrix} \alpha_3 + \alpha_8/\sqrt{3} & \alpha_1 - \mathrm{i}\alpha_2 & \alpha_4 - \mathrm{i}\alpha_5 \\ \alpha_1 + \mathrm{i}\alpha_2 & -\alpha_3 + \alpha_8/\sqrt{3} & \alpha_6 - \mathrm{i}\alpha_7 \\ \alpha_1 + \mathrm{i}\alpha_5 & \alpha_6 + \mathrm{i}\alpha_7 & -2\alpha_8/\sqrt{3} \end{pmatrix} \tag{B.26}$$

The matrices λ_a satisfy the commutation relations

$$[\lambda_a,\ \lambda_b] = 2\mathrm{i}\sum_{c=1}^{8} f_{abc}\lambda_c \tag{B.27}$$

where the f_{abc} are the structure constants (cf. equations (B.7)). The f_{abc} are odd in the interchange of any pair of indices, and the non-vanishing f_{abc} are given by the permutations of $f_{123} = 1,\ f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = 1/2,\ f_{458} = f_{678} = \sqrt{3}/2.$

The matrices also have the property

$$\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab,} \tag{B.28}$$

where δ_{ab} is the Kronecker δ.

These results may be verified by direct calculation.

Problems

B.1 Show that if $\mathbf{U} = \exp(\mathrm{i}\mathbf{H})$ and $\mathrm{Tr}\,\mathbf{H} = 0$, then $\det \mathbf{U} = 1$. (Make $\mathbf{H}$ diagonal with a unitary transformation. $\mathbf{U}$ is then also diagonal.)

B.2 Verify that the $SU(2)$ matrices $\exp[\mathrm{i}(\theta/2)\sigma^1]$ and $\exp\,[\mathrm{i}(\theta/2)\sigma^2]$ correspond to rotations $R_{01}(\theta)$ and $R_{02}(\theta)$, respectively.

B.3 Show that the $SU(2)$ matrix corresponding to the rotation $R(\psi, \theta, \phi)$ (equation (B.5)) is

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2}\cos(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} & \mathrm{e}^{\mathrm{i}\psi/2}\sin(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ -\mathrm{e}^{-\mathrm{i}\psi/2}\sin(\theta/2)\mathrm{e}^{i\phi/2} & \mathrm{e}^{-\mathrm{i}\psi/2}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}.$$

B.4 Show that $\mathbf{l}^\dagger\tilde{\sigma}^\mu\sigma^\nu\mathbf{r}$ transforms as a tensor and $\mathbf{l}^\dagger(\tilde{\sigma}^\mu\sigma^\nu + \tilde{\sigma}^\nu\sigma^\mu)\mathbf{r} = \mathbf{2}g^{\mu\nu}\mathbf{l}^\dagger\mathbf{r}$.

B.5 Show that the rotation matrix R^i_j of equation (B.1) is related to the $SU(2)$ matrix $\mathbf{U}$ of (B.11) by

$$R^i_j = \frac{1}{2}\mathrm{Tr}(\mathbf{U}\sigma^i\mathbf{U}^\dagger\sigma^{\mathrm{j}}).$$

B.6 Show from (B.17) that

$$\mathrm{L}^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\nu M^\dagger\tilde{\sigma}^\mu M).$$

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