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RESEARCH ARTICLE

Almost Everywhere Behavior of Functions According to Partition Measures

William Chan[1], Stephen Jackson[2] and Nam Trang[3]

[1]Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstraße 8-10/104, Vienna, 1040, Austria; E-mail: William.Chan@tuwien.ac.at.
[2]Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, Texas, 76203, USA; E-mail: Stephen.Jackon@unt.edu.
[3]Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, Texas, 76203, USA; E-mail: Nam.Trang@unt.edu.

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Abstract
This paper will study almost everywhere behaviors of functions on partition spaces of cardinals possessing suitable partition properties. Almost everywhere continuity and monotonicity properties for functions on partition spaces will be established. These results will be applied to distinguish the cardinality of certain subsets of the power set of partition cardinals.
The following summarizes the main results proved under suitable partition hypotheses.

- If κ is a cardinal, $\epsilon < \kappa$, $\mathrm{cof}(\epsilon) = \omega$, $\kappa \to_* (\kappa)^{\epsilon\cdot\epsilon}_2$ and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, then Φ satisfies the almost everywhere short length continuity property: There is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.
- If κ is a cardinal, ϵ is countable, $\kappa \to_* (\kappa)^{\epsilon\cdot\epsilon}_2$ holds and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, then Φ satisfies the strong almost everywhere short length continuity property: There is a club $C \subseteq \kappa$ and finitely many ordinals $\delta_0, ..., \delta_k \leq \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if for all $0 \leq i \leq k$, $\sup(f \restriction \delta_i) = \sup(g \restriction \delta_i)$, then $\Phi(f) = \Phi(g)$.
- If κ satisfies $\kappa \to_* (\kappa)^\kappa_2$, $\epsilon \leq \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, then Φ satisfies the almost everywhere monotonicity property: There is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.
- Suppose dependent choice (DC), $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and the almost everywhere short length club uniformization principle for ω_1 hold. Then every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ satisfies a finite continuity property with respect to closure points: Let $\mathfrak{C}_f$ be the club of $\alpha < \omega_1$ so that $\sup(f \restriction \alpha) = \alpha$. There is a club $C \subseteq \omega_1$ and finitely many functions $\Upsilon_0, ..., \Upsilon_{n-1} : [C]^{\omega_1}_* \to \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, for all $g \in [C]^{\omega_1}_*$, if $\mathfrak{C}_g = \mathfrak{C}_f$ and for all $i < n$, $\sup(g \restriction \Upsilon_i(f)) = \sup(f \restriction \Upsilon_i(f))$, then $\Phi(g) = \Phi(f)$.
- Suppose κ satisfies $\kappa \to_* (\kappa)^\epsilon_2$ for all $\epsilon < \kappa$. For all $\chi < \kappa$, $[\kappa]^{<\kappa}$ does not inject into ${}^\chi\mathrm{ON}$, the class of χ-length sequences of ordinals, and therefore, $|[\kappa]^\chi| < |[\kappa]^{<\kappa}|$. As a consequence, under the axiom of determinacy (AD), these two cardinality results hold when κ is one of the following weak or strong partition cardinals of determinacy: ω_1, ω_2, $\boldsymbol{\delta}^1_n$ (for all $1 \leq n < \omega$) and $\boldsymbol{\delta}^2_1$ (assuming in addition $\mathsf{DC}_\mathbb{R}$).

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1. Introduction

Partition relations appear frequently in combinatorics. Ramsey showed that the set of natural numbers, ω, satisfies the finite partition relations $\omega \to (\omega)^k_2$ for each $k < \omega$. The infinite exponent partition relation $\omega \to (\omega)^\omega_2$ (also called the Ramsey property for all partitions) is a natural generalization which is not compatible with the axiom of choice. However, simply definable partitions such as Borel or analytic partitions always satisfy the Ramsey property by results of Galvin and Prikry [7] and Silver [17]. Mathias [15] produced many important results concerning the Ramsey property including the technique of Mathias forcing which is used to verify $\omega \to (\omega)^\omega_2$ in the Solovay model and Woodin's extension AD^+ of the axiom of determinacy, AD. Mathias also studied the Ramsey almost everywhere behavior of functions on the Ramsey space $[\omega]^\omega$ such as when every function $\Phi : [\omega]^\omega \to \mathbb{R}$ is Ramsey almost everywhere continuous or every relation $R \subseteq [\omega]^\omega \times \mathbb{R}$ has a Ramsey almost everywhere uniformization. Recently, these two properties have been used by Schritteser and Törnquist [16] to show that $\omega \to (\omega)^\omega_2$ implies there are no maximal almost disjoint families on ω. Finite exponent partition relations on uncountable cardinals are important in set theory and motivate large cardinal axioms such as the weakly compact and Ramsey cardinals. Martin, Kunen [18], Jackson [8], Kechris, Kleinberg, Moschovakis and Woodin [12] showed that the axiom of determinacy is a natural theory in which ω_1 and many other cardinals κ possess even the strong partition relation: $\kappa \to (\kappa)^\kappa_2$. Kleinberg [14], Martin and Paris studied functions on the finite partition spaces of ω_1 and produced ultrapower representations for ω_n, showed ω_2 has weak partition property and established combinatorial properties such as Jónssonness for ω_n, for all $n \in \omega$. Under the axiom of determinacy, the authors ([4], [2], [6] and [5]) studied variations of almost everywhere continuity properties for functions on the partition spaces of ω_1 and ω_2 according to suitable partition measures and applied these results to distinguish the cardinalities below $\mathscr{P}(\omega_1)$ and $\mathscr{P}(\omega_2)$. There, AD provided useful motivation and elegant arguments, but the techniques have severe limitations. Here, the authors will prove stronger almost everywhere behaviors for functions on partition spaces (such as continuity and monotonicity) from pure combinatorial principles, and these results will be applied to distinguish important cardinalities below the power set of partition cardinals. This will lead to new results about the most important weak and strong partition cardinals of determinacy.

A basic question of infinitary combinatorics is the computation of the size of infinite sets. Cantor formalized the notion of size and the comparison of sizes. Let X and Y be two sets. One says X and Y have the same cardinality (denoted $|X| = |Y|$) if and only if there is a bijection $\Phi : X \to Y$. The cardinality of X is the (proper) class of sets Y which are in bijection with X. The cardinality of X is less than or equal the cardinality of Y (denoted $|X| \leq |Y|$) if and only if there is an injection $\Phi : X \to Y$. The cardinality of X is strictly smaller than the cardinality of Y (denoted $|X| < |Y|$) if and only if $|X| \leq |Y|$ but $\neg(|Y| \leq |X|)$.

The axiom of choice, AC, implies every set is wellorderable. Thus, the class of cardinalities forms a wellordered class under the injection relation. Each cardinality class has a canonical wellordered member (an ordinal) called the cardinal of the class. Wellorderings of sets (even $\mathbb{R}$) are incompatible with certain definability perspectives. This is usually the consequence of definable sets possessing combinatorial regularity properties.

Let ω denote the set of natural numbers or the first infinite cardinal. Cantor showed that ω does not surject onto $\mathscr{P}(\omega)$. Thus, $\omega < |\mathscr{P}(\omega)|$. Let ω_1 denote the first uncountable cardinal. With the axiom of choice, $\omega_1 \leq |\mathscr{P}(\omega)|$ using a wellordering of $\mathscr{P}(\omega)$ or $\mathbb{R}$. However, if the axiom of choice is omitted and instead $\mathbb{R}$ is assumed to satisfy the perfect set property and the property of Baire, then a classical argument involving the Kuratowski–Ulam theorem would show that there is no injection of ω_1 into $\mathbb{R}$ or $\mathscr{P}(\omega)$. Thus, ω_1 and $|\mathscr{P}(\omega)| = |\mathbb{R}|$ are incompatible cardinalities. Moreover, the perfect set property

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completely characterizes the structure of the cardinalities below $|\mathscr{P}(\omega)|$ in a manner which satisfies a choiceless continuum hypothesis: The only uncountable cardinality below $|\mathscr{P}(\omega)|$ is $|\mathscr{P}(\omega)|$.

With the perfect set property and the Baire property, the structure of the cardinalities below $\mathscr{P}(\omega_1)$ is nonlinear since ω_1 and $|\mathbb{R}| = |\mathscr{P}(\omega)|$ are two incompatible cardinalities below $|\mathscr{P}(\omega_1)|$. For each $\epsilon \leq \omega_1$, let $[\omega_1]^\epsilon$ be the increasing sequence space consisting of increasing functions $f : \epsilon \to \omega_1$. $\mathscr{P}(\omega_1)$ and $[\omega_1]^{\omega_1}$ are in bijection. Therefore, sequence spaces represent natural combinatorial cardinalities below $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$. Another important example is $[\omega_1]^{<\omega_1} = \bigcup_{\epsilon<\omega_1}[\omega_1]^\epsilon_*$, which is the set of countable length increasing sequences of countable ordinals. A natural question is to distinguish $|[\omega_1]^\omega|$, $|[\omega_1]^{<\omega_1}|$ and $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}|$ under suitable regularity properties. A helpful combinatorial property possessed by ω_1 (in some natural theories) is the strong partition property, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$.

Partition properties will be discussed in detail in Section 2. Let κ be a cardinal, $\epsilon \leq \kappa$ and $A \subseteq \kappa$. Let $[A]^\epsilon_*$ be the collection of increasing functions $f : \epsilon \to A$ of the correct type (i.e., discontinuous everywhere and has uniform cofinality ω). The partition relation $\kappa \to_* (\kappa)^\epsilon_2$ is the assertion that for all $P : [\kappa]^\epsilon_* \to 2$, there is a closed and unbounded (club) $C \subseteq \omega_1$ and $i \in 2$ so that for all $f \in [C]^\epsilon_*$, $P(f) = i$. If for all $\epsilon < \kappa$, $\kappa \to_* (\kappa)^\epsilon_2$ holds, then κ is called a weak partition cardinal. If $\kappa \to_* (\kappa)^\kappa_2$, then κ is called a strong partition cardinal. If $\epsilon \leq \kappa$ and $\kappa \to_* (\kappa)^\epsilon_2$ holds, then the partition filter μ^κ_ϵ on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\kappa_\epsilon$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$ is an ultrafilter.

If κ satisfies suitable partition relations, then the partition spaces $[\kappa]^\epsilon_*$ for $\epsilon < \kappa$, $[\kappa]^{<\kappa}_*$ and $[\kappa]^\kappa_*$ represent important cardinalities below $\mathscr{P}(\kappa)$. Distinguishing the cardinality of these partition spaces involve understanding the possible injections that exist between these partition spaces. To answer such questions, this paper will use partition properties to obtain very deep understandings of the behavior of functions $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ on measure one sets according to the relevant partition measure, μ^κ_ϵ. The following will summarize and motivate the main results of the paper concerning these almost everywhere behaviors of functions.

In [2], it is shown that if $\kappa \to_* (\kappa)^\kappa_2$, then every function $\Lambda : [\kappa]^\kappa_* \to \mathrm{ON}$, there is an ordinal α so that $\Lambda^{-1}[\{\alpha\}]| = |[\kappa]^\kappa_*|$. This asserts that $|[\kappa]^\kappa_*| = |\mathscr{P}(\kappa)|$ satisfies a regularity property with respect to wellordered decompositions. The set $[\kappa]^{<\kappa}_*$ does not satisfy such regularity. This is used in [2] to show that $|[\kappa]^{<\kappa}_*| < |[\kappa]^\kappa_*| = |\mathscr{P}(\kappa)|$. This paper is motivated by the question of distinguishing the cardinality of $[\kappa]^\epsilon$ for $\epsilon < \kappa$ and $[\kappa]^{<\kappa}_*$. For these computations, it will be important to understand functions $\Phi : [\kappa]^\epsilon_* \to \kappa$ through continuity properties.

To motivate continuity, suppose $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$. Given $f \in [\kappa]^\epsilon_*$, Φ can be considered as an abstract procedure which uses information about f to assign an ordinal value. Examples of such information include specific values of $f(\alpha)$ for $\alpha < \epsilon$, initial segments $f \upharpoonright \alpha$ for $\alpha < \epsilon$ or possibly the entirety of f or the values of f on some unbounded subsets of ϵ. An almost everywhere continuity property intuitively asserts that for μ^κ_ϵ-almost all f, Φ can assign an ordinal to f using only information from f which comes from a well-defined bounded subset of ϵ.

One appealing continuity property for a function $\Phi : [\kappa]^\epsilon_* \to \kappa$ (with $\epsilon < \kappa$) would be that for μ^κ_ϵ-almost all f, there exists a $\delta < \epsilon$ so that $\Phi(f)$ only depends on $f \upharpoonright \delta$. However, such a property is impossible by the following illustrative example. If κ satisfies $\kappa \to_* (\kappa)^2_2$, then κ is a regular cardinal. Thus, the function $\Psi : [\kappa]^\epsilon_* \to \kappa$ defined by $\Psi(f) = \sup(f)$ is well defined and it depends on more than any initial segment. This suggests that perhaps a general function $\Phi : [\kappa]^\epsilon \to \kappa$ might have a fixed $\delta < \epsilon$ so that for μ^κ_ϵ-almost all f, $\Phi(f)$ depends only on the initial segment $f \upharpoonright \delta$ and $\sup(f)$. Under suitable partition properties, such a continuity will be true more generally for functions $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ with $\mathrm{cof}(\epsilon) = \omega$ (and this cofinality assumption is generally necessary).

Fix $\epsilon < \kappa$ a limit ordinal with $\mathrm{cof}(\epsilon) = \omega$. Define an equivalence relation E_0 on $[\kappa]^\epsilon$ by $f \; E_0 \; g$ if and only if there exists an $\alpha < \epsilon$ so that for all β with $\alpha < \beta < \epsilon$, $f(\beta) = g(\beta)$. A function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ is E_0-invariant if and only if whenever $f \; E_0 \; g$, $\Phi(f) = \Phi(g)$. The first step is the following independently interesting result that functions which are E_0-invariant μ^κ_ϵ-almost everywhere depend only on the supremum μ^κ_ϵ-almost everywhere under suitable partition relations.

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Theorem 3.6. Suppose κ is a cardinal, $\epsilon < \kappa$ is a limit ordinal with $\mathrm{cof}(\epsilon) = \omega$ and $\kappa \to_* (\kappa)_2^{\epsilon \cdot \epsilon}$ holds. Let $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$ be a function which is E_0-invariant μ_ϵ^κ-almost everywhere. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]_*^\epsilon$, if $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

Using this theorem, the desired almost everywhere short length continuity result is established.

Theorem 3.7. Suppose κ is a cardinal, $\epsilon < \kappa$ is a limit ordinal with $\mathrm{cof}(\epsilon) = \omega$ and $\kappa \to_* (\kappa)_2^{\epsilon \cdot \epsilon}$ holds. For any function $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]_*^\epsilon$, if $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

The almost everywhere short length continuity of Theorem 3.7 is used to show that if κ is a weak partition cardinal, then for any $\chi < \kappa$, ${}^{<\kappa}\kappa$ does not inject into ${}^{\chi}\kappa$ or even ${}^{\chi}\delta$ for any ordinal δ by providing a sufficiently complete analysis of potential injections.

Theorem 4.4. Suppose κ is a cardinal so that $\kappa \to_* (\kappa)_2^{<\kappa}$. Then for all $\chi < \kappa$, there is no injection of ${}^{<\kappa}\kappa$ into ${}^{\chi}\mathrm{ON}$, the class of χ-length sequences of ordinals. In particular, for all $\chi < \kappa$, $|{}^{\chi}\kappa| < |{}^{<\kappa}\kappa|$.

A stronger continuity notion would assert that a function $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$ (with $\epsilon < \kappa$) has finitely many locations in ϵ depending solely on Φ so that $\Phi(f)$ depends only on the behavior of f at these finitely many locations. (By the previous example, one of these locations must be allowed to be the supremum of f.) The next result states that if ϵ is countable and κ satisfies a suitable partition relation, then $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$ will satisfy a strong almost everywhere short length continuity.

Theorem 3.9. Suppose κ is a cardinal, $\epsilon < \omega_1$ and $\kappa \to_* (\kappa)_2^{\epsilon \cdot \epsilon}$ holds. Let $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$. Then there is a club $C \subseteq \kappa$ and finitely many ordinals $\delta_0, ..., \delta_k \leq \epsilon$ so that for all $f, g \in [C]_*^\epsilon$, if for all $0 \leq i \leq k$, $\sup(f \restriction \delta_i) = \sup(g \restriction \delta_i)$, then $\Phi(f) = \Phi(g)$.

Suppose $\epsilon \leq \kappa$ and $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$. A natural question is that, if one increases the information stored in f by increasing the values of f, could the value of Φ possibly decrease? An almost everywhere monotonicity property for Φ would assert that for μ_ϵ^κ almost all $f, g \in [\kappa]_*^\epsilon$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$. By Fact 5.1, for all functions of the form $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$ to satisfy this almost everywhere monotonicity property, one must at least have the partition relation $\kappa \to_* (\kappa)_2^\epsilon$. If ϵ is countable, then the strong almost everywhere short length continuity of Theorem 3.9 implies the following almost everywhere monotonicity result.

Theorem 4.8. Suppose κ is a cardinal, $\epsilon < \omega_1$, $\kappa \to_* (\kappa)_2^{\epsilon \cdot \epsilon}$ holds and $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]_*^\epsilon$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.

When $\mathrm{cof}(\epsilon) = \omega$, one only has the weaker almost everywhere short length continuity property of Theorem 3.7. Moreover, there are functions on partition spaces of high dimension which do not satisfy a recognizable continuity property. Regardless, almost everywhere monotonicity still holds for functions on partition spaces assuming the appropriate partition relation.

Theorem 5.3. Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)_2^\kappa$. For any function $\Phi : [\kappa]_*^\kappa \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]_*^\kappa$, if for all $\alpha < \kappa$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.

Adapting this argument, one can also show monotonicity for $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$ when $\epsilon < \kappa$.

Theorem 5.7. Suppose κ is a weak partition cardinal. For any $\epsilon < \kappa$ and function $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [\kappa]_*^\epsilon$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.

The last section will establish the strongest known continuity result for functions of the form $\Phi : [\omega_1]_*^{\omega_1} \to \omega_1$ from the strong partition relation on ω_1 and a certain club selection principle. A certain club uniformization principle will be an important tool. Let club_{ω_1} denote the set of club subset of ω_1. The almost everywhere short length club uniformization principle at ω_1 is the assertion that for all $R \subseteq [\omega_1]_*^{<\omega_1} \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed (in the sense that for all $\ell \in [\omega_1]_*^{<\omega_1}$, for all clubs $C \subseteq D$, if $R(\ell, D)$ holds, then $R(\sigma, C)$ holds), there is a club $C \subseteq \omega_1$ and a function $\Lambda : [C]_*^{<\omega_1} \cap \mathrm{dom}(R) \to \mathsf{club}_{\omega_1}$ so that for all $\ell \in [C]_*^{<\omega_1} \cap \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.

Consider a function $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. Asking that there exists a $\delta < \omega_1$ so that $\Phi(f)$ only depending on $f \restriction \delta$ for $\mu_{\omega_1}^{\omega_1}$-almost $f \in [\omega_1]_*^{\omega_1}$ is impossible in general. (For instance, consider $\Phi(f) = f(f(0))$.

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See Example 6.1.) Using the almost everywhere short length club uniformization at ω_1, [4] showed that functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ do satisfy $\mu^{\omega_1}_{\omega_1}$-almost everywhere continuity where $[\omega_1]^{\omega_1}_*$ is endowed with the topology generated by $\{N_\ell : \ell \in [\omega_1]^{\omega_1}_*\}$ as a basis, where $N_\ell = \{f \in [\omega_1]^{\omega_1}_* : \ell \subseteq f\}$ for each $\ell \in [\omega_1]^{<\omega_1}_*$ and ω_1 is given the discrete topology. Explicitly, there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, there exists an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) = \Phi(g)$. [2] showed that the almost everywhere short length club uniformization at ω_1 can be used to get an even finer continuity result which asserts that there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$ and all $\alpha < \omega_1$, if $\Phi(f) < f(\alpha)$, then $f \restriction \alpha$ is a continuity point for Φ relative to C. (For these results, the condition that Φ maps into ω_1 is generally necessary.)

A natural question is whether $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ satisfies any form of continuity in which $\Phi(f)$ depends only on the behavior of f at finitely many locations on ω_1. By the function from Example 6.1, it is impossible to have finitely many ordinals $\delta_0, ..., \delta_{n-1} < \omega_1$ which are independent of any input f so that $\Phi(f)$ depends only on the behavior of f at these finitely many points. One can conjecture if there are finitely many continuity locations for Φ which do depend on f. That is, are there finitely many functions $\Upsilon_0, ..., \Upsilon_{n-1}$ so that there is a club $C \subseteq \omega_1$ with the property that for all $f \in [C]^{\omega_1}$, for all $g \in [C]^{\omega_1}_*$, if for all $i < n$, $\sup(g \restriction \Upsilon_i(f)) = \sup(f \restriction \Upsilon_i(f))$, then $\Phi(f) = \Phi(g)$? This is also not possible. For each $f \in [\omega_1]^{\omega_1}_*$, call an ordinal α a closure point of f if and only if for all $\beta < \alpha$, $f(\beta) < \alpha$ or equivalently $\sup(f \restriction \alpha) = \alpha$. Let $\mathfrak{C}_f$ denote the club set of closure points of f. Let $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$ be defined by $\Psi(f) = \min(\mathfrak{C}_f)$, that is, the smallest closure point of f. Example 6.3 shows that there is no collection of finite functions $\Upsilon_0, ..., \Upsilon_{n-1}$ which satisfies the proposed continuity property with respect to Ψ. Closure points necessarily contain infinite information concerning f. The next result shows that closure points are the only obstruction to a $\mu^{\omega_1}_{\omega_1}$-almost everywhere continuity property asserting finite dependence:

Theorem 6.18. Assume DC, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and that the almost everywhere short length club uniformization principle holds at ω_1. Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. There is a club $C \subseteq \omega_1$ and finitely many functions $\Upsilon_0, ..., \Upsilon_{n-1}$ so that for all $f \in [C]^{\omega_1}_*$, for all $g \in [C]^{\omega_1}_*$, if $\mathfrak{C}_g = \mathfrak{C}_f$ and for all $i < n$, $\sup(g \restriction \Upsilon_i(f)) = \sup(f \restriction \Upsilon_i(f))$, then $\Phi(f) = \Phi(g)$.

To put these results in context and discuss examples, one needs to consider the natural theories which possess combinatorially regular properties. Let $A \subseteq {}^\omega\omega$. Consider a game G_A where two players take turns picking natural numbers to jointly produce an infinite sequence f. Player 1 is said to win G_A if and only if $f \in A$. The axiom of determinacy, denoted AD, asserts that, for all $A \subseteq {}^\omega\omega$, one of the two players has a winning strategy for G_A. Under AD, the perfect set property and the Baire property hold for all sets of reals, and ω_1 and many other cardinals possess partition properties. Many weak versions of the continuity results mentioned here have been previously established for ω_1 and ω_2 under AD. This paper evolved from attempts to establish continuity properties and cardinality computations at the most important weak and strong partition cardinals of determinacy.

See Section 2 for a summary of partition properties under AD. Martin showed under AD that ω_1 is a strong partition cardinal and ω_2 is a weak partition cardinal which is not a strong partition cardinal. Jackson [8] showed under AD that for all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal and $\boldsymbol{\delta}^1_{2n+2}$ is a weak partition cardinal which is not a strong partition cardinal. The next strong partition cardinal after ω_1 is $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. Kechris, Kleinberg, Moschovakis and Woodin [12] showed that $\boldsymbol{\delta}^2_1$ and the Σ_1-stable ordinals $\boldsymbol{\delta}_A$ of $L(A, \mathbb{R})$ for any $A \subseteq \mathbb{R}$ are strong partition cardinals under AD.

Previously known continuity results at ω_1 and ω_2 heavily used determinacy methods. For instance, Kunen trees and Kunen functions ([9] and [3]) are very important for many combinatorial questions at ω_1 and for the description analysis below ω_ω which leads to the strong partition property for $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$. [6] and [5] Fact 2.5 used these Kunen functions to provide a very simple argument that every function $\Phi : [\omega_1]^\epsilon \to \omega_1$ with $\epsilon < \omega_1$ satisfies the almost everywhere short length continuity expressed in Theorem 3.7 and even the stronger version expressed in Theorem 3.9 (but only when the range of the function goes into ω_1). [6] used this result to show that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$ under AD. Using Martin's ultrapower representation of $\omega_{n+1} = \prod_{[\omega_1]^n} / \mu^{\omega_1}_n$ for each $1 \leq n < \omega$, [5] showed that $[\omega_1]^{<\omega_1}$ does not inject into ${}^\omega(\omega_\omega)$. Using a variety of determinacy specific techniques (the full wellordered additivity

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of the meager ideal, generic coding arguments, Banach–Mazur games, Wadge theory and Steel's Suslin bounding), [5] showed that $[\omega_1]^{<\omega_1}$ does not inject into ${}^{\omega}\mathrm{ON}$ under AD and $\mathsf{DC}_{\mathbb{R}}$. (Note that Theorem 4.4 improved this result to just the hypothesis AD without $\mathsf{DC}_{\mathbb{R}}$.) Extending these methods to studying the next strong partition cardinal $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ seems difficult. Although $\omega_{\omega+1}$ has analogs of Kunen functions and generic coding functions ([13]) using supercompactness measures, there is no analog of the full wellordered additivity of the meager ideal which can be a major obstacle to generalizing results to $\omega_{\omega+1}$ as observed by Becker at the end of [1]. Moreover, $\boldsymbol{\delta}^2_1$ and the Σ_1-stable ordinals $\boldsymbol{\delta}_A$ ($A \subseteq \mathbb{R}$) are strong partition cardinals which are limit cardinals and cannot possess analogs of the desired Kunen functions. The methods for ω_1 are much less applicable here. Although, $\boldsymbol{\delta}^1_3$, $\boldsymbol{\delta}^2_1$ and $\boldsymbol{\delta}_A$ are important cardinals of determinacy possessing numerous scales and reflection properties, unlike ω_1, these properties do not seem to facilitate the analysis of cardinality. The pure combinatorial methods of Theorem 3.7, 4.4, 3.9, 5.3 and 5.7 are the only known method for establishing these properties for these important strong partition cardinals of determinacy.

Corollary 3.10. Assume AD. Suppose κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). If $\epsilon < \omega_1$ and $\Phi : [\kappa]^{\epsilon}_* \to \mathrm{ON}$, then there is a club $C \subseteq \kappa$ and finitely many ordinals $\beta_0 < \beta_1 < \ldots < \beta_{p-1} \leq \epsilon$ (where $p \in \omega$) so that for all $f, g \in [C]^{\epsilon}_*$, if for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$, then $\Phi(f) = \Phi(g)$.

Assume AD. Suppose κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$, where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). If $\epsilon < \kappa$ with $\mathrm{cof}(\epsilon) = \omega$ and $\Phi : [\kappa]^{\epsilon}_* \to \mathrm{ON}$, then there is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^{\epsilon}_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Phi(f) = \Phi(g)$.

Corollary 4.6. Assume AD. Suppose κ is $\omega_1, \omega_2, \boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$, where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). Then for any $\chi < \kappa$, $|{}^{\chi}\kappa| < |{}^{<\kappa}\kappa|$ and ${}^{<\kappa}\kappa$ does not inject into ${}^{\chi}\mathrm{ON}$.

Corollary 5.5. Assume AD. Suppose κ is ω_1, $\boldsymbol{\delta}^1_{2n+1}$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). For any $\epsilon \leq \kappa$ and any function $\Phi : [\kappa]^{\epsilon}_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^{\epsilon}_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.

Corollary 5.8. Assume AD. Suppose κ is $\omega_1, \omega_2, \boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). For any $\epsilon < \kappa$ and $\Phi : [\kappa]^{\epsilon}_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^{\epsilon}_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.

Determinacy provides examples to show that the hypothesis in Theorem 3.7 and Theorem 3.9 are generally necessary. Let $\Psi : [\omega_2]^{\omega_1} \to \omega_3$ be defined by $\Phi(f) = [f]_{\mu_1^{\omega_1}}$, that is, the ordinal represented by f in the ultrapower $\prod_{\omega_1} \omega_2/\mu_1^{\omega_1}$ of ω_2 by the club measure on ω_1. Ψ will not satisfy the weak or strong version of the almost everywhere short length continuity. (See Example 3.13.) Letting $\Upsilon : [\omega_2]^{\omega_1+\omega} \to \omega_3$ defined by $\Upsilon(f) = \Psi(f \restriction \omega_1)$ is an example of a function satisfying the weak short length continuity of Theorem 3.7 (note $\mathrm{cof}(\omega_1 + \omega) = \omega$) and does not satisfy the strong short length continuity of Theorem 3.9 (note that $\omega_1 < \omega_1 + \omega$). In the two examples above, the range goes into ω_3. Curiously, it is shown in [6] that every function $\Phi : [\omega_2]^{\omega_1}_* \to \omega_2$ satisfies even the strong almost everywhere short length continuity property (despite $\mathrm{cof}(\omega_1) > \omega$). This remarkable property is unique only to ω_2 and is made possible by Martin's ultrapower representation of ω_2 under AD.

[4] shows the almost everywhere short length club uniformization holds for ω_1 under AD. (By a more general argument, [2] shows that nearly all known strong partition cardinals of AD also satisfies this club uniformization principle.) By absorbing functions $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ into the inner model $L(\mathbb{R})$ which satisfies AD and DC, Theorem 6.18 implies the following holds in AD.

Theorem 6.22. Assume AD. Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. There is a club $C \subseteq \omega_1$ and finitely many function $\Gamma_0, \ldots, \Gamma_{n-1}$ so that for all $f \in [C]^{\omega_1}_*$, for all $g \in [C]^{\omega_1}_*$, if $\mathfrak{C}_g = \mathfrak{C}_f$ and for all $i < n$, $\sup(g \restriction \Gamma_i(f)) = \sup(f \restriction \Gamma_i(f))$, then $\Phi(f) = \Phi(g)$.

In this result, it is necessary that the range goes into ω_1. For example under AD, the function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_2$ defined by $\Psi(f) = [f]_{\mu_1^{\omega_1}}$ (f is mapped to the ordinal below ω_2 represented by f in the ultrapower of ω_1 by the club measure on ω_1) does not satisfy any recognizable continuity property.

2. Partition properties

ON will denote the class of ordinals.

Definition 2.1. Suppose $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$ is a function. The function f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \restriction \alpha) = \sup\{f(\beta) : \beta < \alpha\} < f(\alpha)$.

The function f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ with the following properties:

1. For all $\alpha < \epsilon$, for all $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$.
2. For all $\alpha < \epsilon$, $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.

The function f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

Definition 2.2. If A and B are two sets, then ${}^A B$ denote the set of functions $f : A \to B$.

Let $\epsilon \in \mathrm{ON}$ and X be a class of ordinals. Let $[X]^\epsilon$ be the class of increasing functions $f : \epsilon \to X$. Let $[X]^\epsilon_*$ be the class of increasing functions $f : \epsilon \to X$ of the correct type.

Definition 2.3. (Ordinary partition relation) Suppose κ is a cardinal and $\epsilon \leq \kappa$, then let $\kappa \to (\kappa)^\epsilon_2$ state that for all $P : [\kappa]^\epsilon \to 2$, there is an $A \subseteq \kappa$ with $|A| = \kappa$ and an $i \in 2$ so that for all $f \in [A]^\epsilon$, $P(f) = i$.

Definition 2.4. (Correct type partition relations) Suppose κ is a cardinal, $\epsilon \leq \kappa$ and $\gamma < \kappa$, let $\kappa \to_* (\kappa)^\epsilon_\gamma$ assert that for all $\Phi : [\kappa]^\epsilon_* \to \gamma$, there is a club $C \subseteq \kappa$ and an $\eta < \gamma$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) = \eta$.

κ is a strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_2$. κ is a very strong partition cardinal if and only if $\kappa \to_* (\kappa)^\kappa_{<\kappa}$. κ is a weak partition cardinal if and only if $\kappa \to (\kappa)^{<\kappa}_2$.

The correct type partition relations will be used in this paper. Under the axiom of determinacy, partition relations are often established by proving the correct type partition relation and many applications directly involve the correct type partition relation. The ordinary and correct type partition relations are nearly equivalent by the following result.

Fact 2.5. ([3] Fact 2.6) Suppose κ is a cardinal and $\epsilon \leq \kappa$. $\kappa \to_* (\kappa)^\epsilon_2$ implies $\kappa \to (\kappa)^\epsilon_2$. $\kappa \to (\kappa)^{\omega \cdot \epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_2$.

It is not known if $\kappa \to_* (\kappa)^\kappa_2$ implies $\kappa \to (\kappa)^\kappa_{<\kappa}$, that is, whether a strong partition cardinal is a very strong partition cardinal. (Although all known strong partition cardinals are very strong partition cardinals.) However, one does have the following related results for weak partition cardinals. The first follows from an induction argument.

Fact 2.6. Suppose κ is a cardinal and $\epsilon \leq \kappa$. $\kappa \to_* (\kappa)^\epsilon_2$ implies $\kappa \to_* (\kappa)^\epsilon_n$ for all $n \in \omega$.

Fact 2.7. ([3] Fact 2.13) Suppose κ is a cardinal and $\epsilon < \kappa$. Then $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \to (\kappa)^\epsilon_{<\kappa}$.

Thus, $\kappa \to_* (\kappa)^{<\kappa}_2$ implies $\kappa \to_* (\kappa)^{<\kappa}_{<\kappa}$.

Definition 2.8. If κ is a cardinal and $\epsilon \leq \kappa$, then let μ^κ_ϵ be the filter on $[\kappa]^\epsilon_*$ defined by $X \in \mu^\kappa_\epsilon$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq X$.

If $\kappa \to_* (\kappa)^\epsilon_2$ holds, then μ^κ_ϵ is an ultrafilter and is called the ϵ-partition measure on κ. If $\kappa \to_* (\kappa)^\kappa_2$ holds, then μ^κ_κ is called the strong partition measure on κ.

Note that μ^κ_ϵ is κ-complete if and only if $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$ holds. Thus, if $\epsilon < \kappa$ and $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ holds, then μ^κ_ϵ is κ-complete by Fact 2.7.

Definition 2.9. Suppose A is a set of ordinals. Let $\xi = \mathrm{ot}(A)$. Let $\mathsf{enum}_A : \xi \to A$ denote the increasing enumeration of A.

Suppose κ be a regular cardinal. Let $X \subseteq \kappa$ be an unbounded subset of κ. Let $\mathsf{next}_X : \kappa \to X$ be defined by $\mathsf{next}_X(\alpha)$ is the least element of X greater than α. Let $\mathsf{next}^0_X : \kappa \to \kappa$ be the identity function. For each $0 < \gamma < \kappa$, let $\mathsf{next}^\gamma_X : \kappa \to X$ be defined by $\mathsf{next}^\gamma_X(\alpha)$ is the γ^{th}-element of X strictly greater than α. (Note that $\mathsf{next}_X(\alpha) = \mathsf{next}^1_X(\alpha)$.)

Suppose κ is a cardinal, $\epsilon \leq \kappa$ and $f : \epsilon \to \kappa$. Let $\mathcal{C}_f$ denote the closure of $f[\epsilon]$ in κ.

If $\epsilon \in \mathrm{ON}$, $f : \epsilon \to \mathrm{ON}$ and $\alpha < \epsilon$, then let $\mathsf{drop}(f, \alpha) : (\epsilon - \alpha) \to \mathrm{ON}$ be defined by $\mathsf{drop}(f, \alpha)(\beta) = f(\alpha + \beta)$.

Fact 2.10. Suppose κ is a cardinal, $\epsilon \leq \kappa$ and $\kappa \to_* (\kappa)^{\epsilon}_{<\kappa}$ holds. (By Fact 2.7, if $\epsilon < \kappa$, then $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ is enough to ensure this condition.) Let $\Phi : [\kappa]^{\epsilon}_* \to \kappa$ have the property that for μ^{κ}_{ϵ}-almost all f, $\Phi(f) < f(0)$. Then there is a club $C \subseteq \kappa$ and a $\zeta < \kappa$ so that for all $f \in [C]^{\epsilon}_*$, $\Phi(f) = \zeta$.

Proof. Let $C_0 \subseteq \kappa$ be such that for all $f \in [C_0]^{\epsilon}_*$, $\Phi(f) < f(0)$. Define $P : [\kappa]^{1+\epsilon}_* \to 2$ by $P(g) = 0$ if and only if $\Phi(\mathsf{drop}(g, 1)) < g(0)$. By $\kappa \to_* (\kappa)^{\epsilon}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Let $C_2 \subseteq C_1$ be the club of limit points of C_1. Take any $f \in [C_2]^{\epsilon}_*$. By the stated property of C_0, $\Phi(f) < f(0)$. Since $f(0) \in C_2$, there is a $\gamma \in C_1$ so that $\Phi(f) < \gamma < f(0)$. Let $g \in [C_1]^{1+\epsilon}_*$ be defined by $g(0) = \gamma$ and $\mathsf{drop}(g, 1) = f$. Then $\Phi(\mathsf{drop}(g, 1)) = \Phi(f) < \gamma = g(0)$ and hence $P(g) = 0$. This shows that C_1 must be homogeneous for P taking value 0. Let $f \in [C_2]^{\epsilon}_*$. Since $f(0) \in C_2$, $\min(C_1) < f(0)$. Let $g \in [C_1]^{1+\epsilon}_*$ be defined so that $\mathsf{drop}(g, 1) = f$ and $g(0) = \min(C_1)$. Then $P(g) = 0$ implies that $\Phi(f) = \Phi(\mathsf{drop}(g, 1)) < g(0) = \min(C_1)$. Since f was arbitrary, it has been shown that for all $f \in [C_2]^{\epsilon}_*$, $\Phi(f) < \min(C_1)$. Since $\kappa \to_* (\kappa)^{\epsilon}_{<\kappa}$ implies that μ^{κ}_{ϵ} is κ-complete, there is a $C_3 \subseteq C_2$ and a $\zeta < \kappa$ so that for all $f \in [C_3]^{\epsilon}_*$, $\Phi(f) = \zeta$. □

Fact 2.11. Assume $\kappa \to_* (\kappa)^2_2$. Then μ^{κ}_1 (i.e., the ω-club filter on κ) is a normal κ-complete ultrafilter.

Proof. Let $\Phi : \kappa \to \kappa$ be a μ^{κ}_1-almost everywhere regressive function. Then there is a club $C \subseteq \kappa$ so that for all $\alpha \in [C]^1_*$, $\Phi(\alpha) < \alpha$. By Fact 2.10, there is a club $D \subseteq \kappa$ and a $\zeta < \kappa$ so that for all $\alpha \in [D]^1_*$, $\Phi(\alpha) = \zeta$. So Φ is μ^{κ}_1-almost everywhere constant. □

An ordinal γ is additively indecomposable if and only if for all $\alpha < \gamma$ and $\beta < \gamma$, $\alpha + \beta < \gamma$. An ordinal γ is multiplicatively indecomposable if and only if for all $\alpha < \gamma$ and $\beta < \gamma$, $\alpha \cdot \beta < \gamma$. An ordinal is indecomposable if and only if it is additively and multiplicatively indecomposable. In all discussions, 0 and 1 will be excluded and hence additively indecomposable ordinals will always be limit ordinals. For every limit ordinal ϵ, there exists $\epsilon_0 < \epsilon$ and $\epsilon_1 \leq \epsilon$ so that $\epsilon = \epsilon_0 + \epsilon_1$ and ϵ_1 is additively indecomposable. Because of this decomposition, it will be useful to establish results for sequences whose lengths are additively indecomposable (but possibly not multiplicatively indecomposable) before deducing the general result. One will frequently assume club subsets consists entirely of (additively and multiplicatively) indecomposable ordinals.

Fact 2.12. Let $C_0 \subseteq \kappa$ be a club subset of κ consisting entirely of indecomposable ordinals. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Then $C_1 \subseteq C_0$ is a club subset of κ consisting entirely of indecomposable ordinals. For any $\gamma < \kappa$, $\alpha < \gamma$ and $\beta < \gamma$ with $\gamma \in C_1$, $\mathsf{next}^{\beta}_{C_0}(\alpha) < \gamma$ and in particular, $\mathsf{next}^{\omega\cdot(\beta+1)}_{C_0}(\alpha) < \gamma$.

Proof. Fix $\gamma \in C_1$ and $\alpha < \gamma$. Let $\zeta = \sup\{\eta < \kappa : \mathsf{enum}_{C_0}(\eta) \leq \alpha\}$. Note that $\mathsf{next}^0_{C_0}(\alpha) = \alpha$ and if $0 < \beta < \gamma$, $\mathsf{next}^{\beta}_{C_0}(\alpha) = \mathsf{enum}_{C_0}(\zeta + \beta)$. Since $\alpha < \gamma$ and $\gamma \in C_1$, $\zeta < \mathsf{enum}^{-1}_{C_0}(\gamma) = \gamma$. Because γ is additively indecomposable, $\zeta + \beta < \gamma$. Thus, $\mathsf{next}^{\beta}_{C_0}(\alpha) = \mathsf{enum}_{C_0}(\zeta + \beta) < \mathsf{enum}_{C_0}(\gamma) = \gamma$. The last statement follows from the first statement and the fact that since γ is additively and multiplicatively indecomposable and for all $\beta < \gamma$, $\omega \cdot (\beta + 1) < \gamma$. □

Fact 2.13. (Almost everywhere fixed length measure witness uniformization) Let κ be a cardinal, $1 \leq \delta < \kappa$ and $1 \leq \epsilon \leq \kappa$. Suppose $\kappa \to_* (\kappa)^{\delta+\epsilon}_2$ holds. Let $R \subseteq [\kappa]^{\delta}_* \times [\kappa]^{\epsilon}_*$ be such that for all $f \in [\kappa]^{\delta}_*$, $R_f = \{g \in [\kappa]^{\epsilon}_* : R(f, g)\} \in \mu^{\kappa}_{\epsilon}$. Then there is a club $C \subseteq \kappa$ so that for $f \in [C]^{\delta}_*$, $[C \setminus \sup(f) + 1]^{\epsilon}_* \subseteq R_f$.

Proof. If $h \in [\kappa]^{\delta+\epsilon}_*$, then let $h^0 \in [\kappa]^{\delta}_*$ and $h^1 \in [\kappa]^{\epsilon}_*$ be defined by $h^0 = h \restriction \delta$ and $h^1 = \mathsf{drop}(h, \delta)$. Define a partition $P : [\kappa]^{\delta+\epsilon}_* \to 2$ by $P(h) = 0$ if and only if $R(h^0, h^1)$. By $\kappa \to_* (\kappa)^{\delta+\epsilon}_2$, there is a club $C \subseteq \omega_1$ which is homogeneous for P. Fix an $f \in [C]^{\delta}_*$. Since $R_f \in \mu^{\kappa}_{\epsilon}$, there is a club $D \subseteq C$

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so that for all $g \in [D]^\epsilon_*$, $R(f,g)$ holds. Pick a $g \in [D]^\epsilon_*$ with $\sup(f) < g(0)$, and let $h \in [C]^{\delta+\epsilon}_*$ be defined so that $h^0 = f$ and $h^1 = g$. Then $P(h) = 0$. Thus, C is homogeneous for P taking value 0. Now, fix an $f \in [C]^\delta_*$. Take any $g \in [C \setminus \sup(f)+1]^\epsilon_*$. Let $h \in [C]^{\delta+\epsilon}_*$ be defined so that $h^0 = f$ and $h^1 = g$. $P(h) = 0$ implies that $R(f,g)$. Thus, $[C \setminus \sup(f)+1]^\epsilon_* \subseteq R_f$. □

If κ is a cardinal, then let club_κ denote the collection of club subsets of κ.

Fact 2.14. ([2]) Let κ be a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$ and $1 \le \epsilon < \kappa$. Suppose $R \subseteq [\kappa]^\epsilon_* \times \mathsf{club}_\kappa$ is a relation which is $\subseteq$-downward closed in the club coordinate in the sense that for all $\ell \in [\kappa]^\epsilon$ and all clubs $C \subseteq D$, if $R(\ell, D)$, then $R(\ell, C)$. Then there is a club $C \subseteq \kappa$ so that for all $\ell \in [C]^\epsilon_* \cap \mathrm{dom}(R)$, $R(\ell, C \setminus (\sup(\ell+1))$.

Fact 2.14 will not be used here. Fact 2.14 implies Fact 2.13; however, it requires $\kappa \to_* (\kappa)^\kappa_2$. Fact 2.14 is generally not true for weak partition cardinals which are not strong partition cardinals. For instance, under AD, Fact 2.14 fails at ω_2. Fact 2.14 gives slightly easier proof in the case of strong partition cardinals, but the paper seeks to prove these results for weak partition cardinals so Fact 2.13 must be used in a more indirect way.

Fact 2.15. (Everywhere wellordered measure witness uniformization) Let κ be a cardinal, $\epsilon < \kappa$ and assume $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$. If $R \subseteq \kappa \times [\kappa]^\epsilon_*$ has the property that for all $\alpha < \kappa$, $R_\alpha \in \mu^\kappa_\epsilon$, then there is a club $C \subseteq \kappa$ so that for all $\alpha < \kappa$, $[C \setminus \mathsf{next}^\omega_C(\alpha)+1]^\epsilon_* \subseteq R_\alpha$.

Proof. Define a new relation $S \subseteq \kappa \times [\kappa]^\epsilon_*$ by $S(\alpha, f)$ if and only if for all $\beta \le \alpha$, $R(\beta, f)$. Note that $S_\alpha \in \mu^\kappa_\epsilon$ since $S_\alpha = \bigcap_{\beta \le \alpha} R_\beta$, $R_\beta \in \mu^\kappa_\epsilon$ for each $\beta \le \alpha$, and μ^κ_ϵ is κ-complete by Fact 2.7. Applying Theorem 2.13, there is a club $C \subseteq \kappa$ so that for all $\alpha \in [C]^1_*$, $[C \setminus \alpha+1]^\epsilon_* \subseteq S_\alpha$.

Let $\alpha < \kappa$. Note that $\mathsf{next}^\omega_C(\alpha)$ is an element of C of cofinality ω and thus $\mathsf{next}^\omega_C(\alpha) \in [C]^1_*$. Thus, $[C \setminus \mathsf{next}^\omega_C(\alpha)+1]^\epsilon_* \subseteq S_{\mathsf{next}^\omega_C(\alpha)}$. Since $S_{\mathsf{next}^\omega_C(\alpha)} \subseteq R_\alpha$, $[C \setminus \mathsf{next}^\omega_C(\alpha)+1]^\epsilon_* \subseteq R_\alpha$. □

The axiom of determinacy AD provides a rich theory with an abundance of partition cardinals possessing desirable structures. For each $n \in \omega$, let $\boldsymbol{\delta}^1_n$ be the supremum of the ranks of prewellorderings on $\mathbb{R}$ which belong to the pointclass $\boldsymbol{\Delta}^1_n$. Under AD, $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_2 = \omega_2$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. Similarly, let $\boldsymbol{\delta}^2_1$ be the supremum of the ranks of prewellorderings on $\mathbb{R}$ which belong to the pointclass $\boldsymbol{\Delta}^2_1$. If $A \subseteq \mathbb{R}$, then let $\boldsymbol{\delta}_A$ be the least Σ_1-stable ordinals of $L(A,\mathbb{R})$, which is the least ordinal δ so that $L_\delta(A,\mathbb{R}) \prec_1 L(A,\mathbb{R})$. It is the case that $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})} = \boldsymbol{\delta}_\emptyset$.

Fact 2.16. Assume AD.

1. (Martin; [10] Theorem 12.2, [3] Corollary 4.27) $\omega_1 \to_* (\omega_1)^{\omega_1}_{<\omega_1}$
2. (Martin–Paris; [10] Corollary 13.5, [3] Theorem 5.19 and Corollary 6.17) $\omega_2 \to_* (\omega_2)^{<\omega_2}_2$. $\neg(\omega_2 \to_* (\omega_2)^{\omega_2}_2)$.
3. (Martin; [10] Theorem 11.2; Theorem [9] 2.36) For any $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1} \to_* (\boldsymbol{\delta}^1_{2n+1})^{<\omega_1}_2$.
4. ([8]) $\boldsymbol{\delta}^1_{2n+1} \to_* (\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}_{<\boldsymbol{\delta}^1_{2n+1}}$.
5. (Kunen; [10] Theorem 15.3) For all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+2} \to_* (\boldsymbol{\delta}^1_{2n+2})^{<\omega_1}_2$.
6. ([8]) $\boldsymbol{\delta}^1_{2n+2} \to_* (\boldsymbol{\delta}^1_{2n+2})^{<\boldsymbol{\delta}^1_{2n+2}}_2$. $\neg(\boldsymbol{\delta}^1_{2n+2} \to_* (\boldsymbol{\delta}^1_{2n+2})^{\boldsymbol{\delta}^1_{2n+2}}_2)$.
7. ([12]) For any $A \subseteq \mathbb{R}$, $\boldsymbol{\delta}_A \to_* (\boldsymbol{\delta}_A)^{\boldsymbol{\delta}_A}_{<\boldsymbol{\delta}_A}$.
8. ([12]) Assuming $\mathsf{DC}_\mathbb{R}$, $\boldsymbol{\delta}^2_1 \to_* (\boldsymbol{\delta}^2_1)^{\boldsymbol{\delta}^2_1}_{<\boldsymbol{\delta}^2_1}$.

Remark 2.17. Jackson [8] established the partition relations for the projective ordinals $\boldsymbol{\delta}^1_n$ for $1 \le n < \omega$ by first analyzing the measures on the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ which seem to require $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$. Kechris [11] showed that if AD holds, then $L(\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$. Thus, Jackson argument of [8] applied in $L(\mathbb{R})$ gives a good coding system for $(\boldsymbol{\delta}^1_{2n+1})^{L(\mathbb{R})}$ which belongs to $L(\mathbb{R})$. $\boldsymbol{\delta}^1_{2n+1} = (\boldsymbol{\delta}^1_{2n+1})^{L(\mathbb{R})}$ and a good coding system in $L(\mathbb{R})$ is still a good coding system in the original determinacy universe by

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the Moschovakis coding lemma. This shows that $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal using only the assumption of AD.

3. Almost everywhere short length continuity

Definition 3.1. Let κ be a cardinal and $\epsilon \leq \kappa$. Define an equivalence relation on ${}^{\epsilon}\kappa$ by $f\ E_0\ g$ if and only if there is an $\alpha < \epsilon$ so that for all β with $\alpha \leq \beta < \epsilon$, $f(\beta) = g(\beta)$.

Suppose $\Phi : [\kappa]^{\epsilon}_{*} \to \mathrm{ON}$ is a function. Φ is E_0-invariant μ^{κ}_{ϵ}-almost everywhere if and only if there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^{\epsilon}_{*}$, if $f\ E_0\ g$, then $\Phi(f) = \Phi(g)$.

Definition 3.2. Let κ be a cardinal and $\epsilon \leq \kappa$. Define $\sqsubseteq$ on $[\kappa]^{\epsilon}_{*}$ by $g \sqsubseteq f$ if and only if $g \in [\mathcal{C}_f]^{\epsilon}_{*}$.

(This notion depends on κ and ϵ. Implicitly, $g \sqsubseteq f$ implies f and g are functions of the correct type.)

Lemma 3.3. *Suppose κ is a cardinal, $\epsilon < \kappa$ is an additively indecomposable ordinal with $\mathrm{cof}(\epsilon) = \omega$, and $\kappa \to_{*} (\kappa)^{\epsilon \cdot \epsilon}_{2}$ holds. Let $\Phi : [\kappa]^{\epsilon}_{*} \to \mathrm{ON}$ be a function which is E_0-invariant μ^{κ}_{ϵ}-almost everywhere. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^{\epsilon}_{*}$, if $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.*

Proof. Since Φ is E_0-invariant μ^{κ}_{ϵ}-almost everywhere, let $C_0 \subseteq \kappa$ be a club so that for all $f, g \in [C_0]^{\epsilon}_{*}$, if $f\ E_0\ g$, then $\Phi(f) = \Phi(g)$. Define $P_0 : [C_0]^{\epsilon}_{*} \to 2$ by $P(f) = 0$ if and only if for all $g \sqsubseteq f$, $\Phi(g) = \Phi(f)$. By $\kappa \to_{*} (\kappa)^{\epsilon \cdot \epsilon}_{2}$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P_0.

The claim is that C_1 is homogeneous for P_0 taking value 0. For the sake of obtaining a contradiction, suppose C_1 is homogeneous for P_0 taking value 1. Define $P_1 : [C_1]^{\epsilon}_{*} \to 2$ by $P_1(f) = 0$ if and only if there exists a $g \sqsubseteq f$ so that $\Phi(g) > \Phi(f)$. By $\kappa \to_{*} (\kappa)^{\epsilon}_{2}$, there is a club $C_2 \subseteq C_1$ which is homogeneous for P_1.

Case 1: Suppose C_2 is homogeneous for P_1 taking value 1. Let $Z = \{\Phi(f) : f \in [C_2]^{\epsilon}_{*}\}$. Z has a minimal element since Z is a nonempty set of ordinals. Pick $f \in [C_2]^{\epsilon}_{*}$ with $\Phi(f) = \min(Z)$. Note that $P_0(f) = 1$ and $P_1(f) = 1$ imply that there exists a $g \sqsubseteq f$ so that $\Phi(g) < \Phi(f)$. However, because $g \in [C_2]^{\epsilon}_{*}$ since C_2 is a club, $\Phi(g) \in Z$ and $\Phi(g) < \Phi(f) = \min(Z)$ which is a contradiction.

Case 2: Suppose C_2 is homogeneous for P_1 taking value 0. For any function $h : \epsilon \cdot \epsilon \to \kappa$, define $\mathsf{main}(h) : \epsilon \to \kappa$ by $\mathsf{main}(h)(\alpha) = \sup\{h(\epsilon \cdot \alpha + \beta) : \beta < \epsilon\}$. Define $P_2 : [C_2]^{\epsilon \cdot \epsilon}_{*} \to 2$ by $P_2(h) = 0$ if and only if there is an $f \in [\mathcal{C}_h]^{\epsilon}_{*}$ so that $\mathsf{main}(h) \sqsubseteq f$ and $\Phi(f) < \Phi(\mathsf{main}(h))$. (Recall $\mathcal{C}_h$ is the closure $f[\epsilon \cdot \epsilon)]$.) By $\kappa \to_{*} (\kappa)^{\epsilon \cdot \epsilon}_{2}$, there is a club $C_3 \subseteq C_2$ which is homogeneous for P_2 and consists entirely of indecomposable ordinals. Let $C^1_3 = \{\alpha \in C_3 : \mathsf{enum}_{C_3}(\alpha) = \alpha\}$. Let $f \in [C^1_3]^{\epsilon}_{*}$ with $f(0) > \epsilon$. Since $P_1(f) = 0$, there exists some $g \sqsubseteq f$ such that $\Phi(f) < \Phi(g)$. As C^1_3 is a club, $g \in [C^1_3]^{\epsilon}_{*}$. Because g has the correct type, let $G : \epsilon \times \omega \to \mathrm{ON}$ witness that g has uniform cofinality ω. Since g is discontinuous everywhere, by modifying G if necessary, one may assume that for all $\alpha < \epsilon$, $\sup(g \restriction \alpha) < G(\alpha, 0)$. Since $\mathrm{cof}(\epsilon) = \omega$, let $\rho : \omega \to \epsilon$ be an increasing cofinal sequence through ϵ. For each $\eta < \epsilon$, let $\varpi(\eta)$ be the least n so that $\eta < \rho(n)$.

Fix $\alpha < \epsilon$. Let $\iota^{\alpha}_0 = G(\alpha, 0)$. Let $\iota^{\alpha}_{n+1} = \max\{\mathsf{next}^{\omega \cdot (\rho(n)+1)}_{C_3}(\iota^{\alpha}_n), G(\alpha, n+1)\}$. Suppose inductively, one has shown $\iota^{\alpha}_n < g(\alpha)$. Then since $\iota^{\alpha}_n < g(\alpha)$ and $\omega \cdot (\rho(n) + 1) < g(\alpha)$, Fact 2.12 implies that $\iota^{\alpha}_{n+1} < g(\alpha)$. For each $\eta < \epsilon$, let $r^{\alpha}(\eta) = \mathsf{next}^{\omega \cdot (\eta+1)}_{C_3}(\iota^{\alpha}_{\varpi(\eta)})$. Note that $\iota^{\alpha}_{\varpi(\eta)} < \mathsf{next}^{\omega \cdot (\eta+1)}_{C_3}(\iota^{\alpha}_{\varpi(\eta)}) = r^{\alpha}(\eta) < \mathsf{next}^{\omega \cdot (\rho(\varpi(\eta))+1)}_{C_3}(\iota^{\alpha}_{\varpi(\eta)}) \leq \iota^{\alpha}_{\varpi(\eta)+1}$, $r^{\alpha} \in [C_3]^{\epsilon}_{*}$ and $\sup(r^{\alpha}) = \sup\{\iota^{\alpha}_n : n \in \omega\} = g(\alpha)$. Let F_{α} be the collection of $\gamma \in f[\epsilon]$ such that $\sup(g \restriction \alpha) < \gamma < g(\alpha)$ and there is no ordinal $\eta < \epsilon$ so that $\sup(r^{\alpha} \restriction \eta) = \gamma$. Note that $\mathrm{ot}(F_{\alpha}) < \epsilon$ since $g \sqsubseteq f$. Thus, $\mathrm{ot}(r^{\alpha}[\epsilon] \cup F_{\alpha}) = \epsilon$ since ϵ is additively indecomposable, $\mathrm{ot}(F_{\alpha}) < \epsilon$ and $\sup(F_{\alpha}) \leq g(\alpha) = \sup(r^{\alpha})$. Let $s^{\alpha} : \epsilon \to (r^{\alpha}[\epsilon] \cup F_{\alpha})$ be the unique increasing function which enumerates $r[\epsilon] \cup F_{\alpha}$. Note that s^{α} has the correct type and $\sup(s^{\alpha}) = g(\alpha)$. Define $h : \epsilon \cdot \epsilon \to C_3$ by $h(\epsilon \cdot \alpha + \eta) = s^{\alpha}(\eta)$ whenever $\alpha, \eta < \epsilon$. Note that $h \in [C_3]^{\epsilon \cdot \epsilon}_{*}$, $f \in [\mathcal{C}_h]^{\epsilon}_{*}$, $\mathsf{main}(h) = g \sqsubseteq f$ and $\Phi(\mathsf{main}(h)) = \Phi(g) > \Phi(f)$. So $P_2(h) = 0$ and thus C_3 is homogeneous for P_2 taking value 0.

Let $C^0_3 = C_3$. If C^n_3 has been defined, then let $C^{n+1}_3 = \{\alpha \in C^n_3 : \mathsf{enum}_{C^n_3}(\alpha) = \alpha\}$. For $\alpha \in C_3$, let $\varsigma(\alpha) = \sup\{n \leq \varpi(\alpha) + 1 : \alpha \in C^n_3\}$. Let Y be the collection of $f \in [C^1_3]^{\epsilon}_{*}$ with the property that for all $1 \leq n < \omega$, there exists an $\alpha < \epsilon$ so that for all $\beta \geq \alpha$, $\varsigma(f(\beta)) \geq n$. Let $Z = \{\Phi(f) : f \in Y\}$. Since Z is a nonempty set of ordinals, Z has a minimal element. Let $g \in Y$ be such that $\Phi(g) = \min(Z)$. Since g is of the correct type, let $G : \epsilon \times \omega \to \kappa$ witness that g has uniform cofinality ω. Since g is discontinuous, one may assume that for all $\alpha < \epsilon$, $\sup(g \restriction \alpha) < G(\alpha, 0)$.

Fix $\alpha < \epsilon$. Let $\iota_0^\alpha = G(\alpha, 0)$. Let $\iota_{n+1}^\alpha = \max\{\mathsf{next}^{\omega\cdot(\rho(n)+1)}_{C_3^{\varsigma(g(\alpha))-1}}(\iota_n^\alpha), G(\alpha, n+1)\}\}$. Suppose inductively it has been shown that $\iota_n^\alpha < g(\alpha)$. Then since $\iota_n^\alpha < g(\alpha)$, $\omega \cdot (\rho(n) + 1) < g(\alpha)$ and $g(\alpha) \in C_3^{\varsigma(g(\alpha))} = \{\gamma \in C_3^{\varsigma(g(\alpha))-1} : \mathsf{enum}_{C_3^{\varsigma(g(\alpha))-1}}(\gamma) = \gamma\}$, Fact 2.12 implies that $\iota_{n+1}^\alpha < g(\alpha)$. For each $\eta < \epsilon$, let $r^\alpha(\eta) = \mathsf{next}^{\omega\cdot(\eta+1)}_{C_3^{\varsigma(g(\alpha))-1}}(\iota^\alpha_{\varpi(\eta)})$. Note that $\iota^\alpha_{\varpi(\eta)} < r^\alpha(\eta) < \iota^\alpha_{\varpi(\eta)+1}$, $r^\alpha \in [C_3^{\varsigma(g(\alpha))-1}]^\epsilon_*$ and $\sup(r^\alpha) = \sup\{\iota_n^\alpha : n \in \omega\} = g(\alpha)$. Let $h : \epsilon \cdot \epsilon \to C_3$ be defined by $h(\epsilon \cdot \alpha + \eta) = r^\alpha(\eta)$ whenever $\alpha, \eta < \epsilon$. Note that $h \in [C_3]^{\epsilon\cdot\epsilon}_*$ and $\mathsf{main}(h) = g$. Since $P_2(h) = 0$, there is an $f \in [\mathcal{C}_h]^\epsilon_*$ so that $g = \mathsf{main}(h) \sqsubseteq f$ and $\Phi(f) < \Phi(\mathsf{main}(h)) = \Phi(g)$. For each $n \in \omega$, let $\delta_n < \epsilon$ be least ordinal δ so that for all α with $\delta \leq \alpha < \epsilon$, $\varsigma(g(\alpha)) - 1 \geq n$. For each $n \in \omega$, let $\eta_n < \epsilon$ be the least η so that $f(\eta) \geq g(\delta_n)$ which exists since $g = \mathsf{main}(h) \sqsubseteq f$. For all $\eta \geq \eta_n$, since $f \in [\mathcal{C}_h]^\epsilon_*$, there is a unique $\alpha \geq \delta_n$ so that $f(\eta) \in \mathcal{C}_{r^\alpha} \subseteq C_3^{\varsigma(g(\alpha))-1} \subseteq C_3^n$. Thus, it has been shown that for all $n \in \omega$, there is a $\eta_n < \epsilon$ so that for all η with $\eta_n \leq \eta < \epsilon$, $\varsigma(f(\eta)) \geq n$. In particular, $\varsigma(f(\eta_2)) \in C_3^2$. (There is a possibility that $f \notin Y$ since $f \notin [C_3^1]^\epsilon_*$ because an initial segment of f takes value in $C_3^0 \setminus C_3^1$. However, an initial segment of f can be swapped to obtain an element $k \in Y$. The details follow.) Let $\sigma \in [C_3^1]^{\eta_2}_*$ be defined by $\sigma(\nu) = \mathsf{next}^{\omega\cdot(\nu+1)}_{C_3^1}(0)$ for each $\nu < \eta_2$. Note that $\sup(\sigma) \leq \mathsf{next}^{\omega\cdot(\eta_2+1)}_{C_3^1}(0) < f(\eta_2)$ by Fact 2.12 since $\omega \cdot (\eta_2 + 1) < f(\eta_2)$, $f(\eta_2) \in C_3^2$ and $C_3^2 = \{\alpha \in C_3^1 : \mathsf{enum}_{C_3^1}(\alpha) = \alpha\}$. Let $k \in [C_3^1]^\epsilon_*$ be defined as $k = \sigma\hat{\ }\mathsf{drop}(f, \eta_2)$. Note that k also has the property that for all $1 \leq n < \omega$, there is an $\eta < \epsilon$ so that for all α with $\eta \leq \alpha < \epsilon$, $\varsigma(k(\alpha)) \geq n$. Thus, $k \in Y$ and $\Phi(k) \in Z$. Since $k\ E_0\ f$, one has that $\Phi(k) = \Phi(f) < \Phi(g) = \min(Z)$. Contradiction.

Since Case 1 and Case 2 both lead to contradictions, P_1 is a partition with no homogeneous club which is impossible since $\kappa \to_* (\kappa)^\epsilon_2$ holds. Thus, C_1 must be homogeneous for P_0 taking value 0.

Now, suppose $f, g \in [C_1]^\epsilon_*$ with $\sup(f) = \sup(g)$. Since ϵ is additively indecomposable, $\mathrm{ot}(f[\epsilon] \cup g[\epsilon]) = \epsilon$. Define $h \in [C_1]^\epsilon_*$ by recursion as follows: Let $h(0) = \min(f[\epsilon] \cup g[\epsilon])$. If $\beta < \epsilon$ and $h \restriction \beta$ has been defined, then let $h(\beta)$ be the least element of $f[\epsilon] \cup g[\epsilon]$ greater than $\sup(h \restriction \beta) + 1$. Note that h is increasing, discontinuous and can be shown to have uniform cofinality ω using the witnesses to f and g having uniform cofinality ω. Observe that $P_1(h) = 0$, $f \sqsubseteq h$ and $g \sqsubseteq h$ imply that $\Phi(f) = \Phi(h) = \Phi(g)$. The proof is complete. □

Definition 3.4. Let κ be cardinal, $\epsilon < \kappa$, $\kappa \to_* (\kappa)^\epsilon_2$ holds, $C \subseteq \kappa$ be a club and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ be a function. Say that Φ depends only on the supremum relative to C if and only if for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$. Say that Φ is $\sqsubseteq$-constant on $[C]^\epsilon_*$ if and only if for all $f \in [C]^\epsilon_*$, for all $g \sqsubseteq f$, $\Phi(f) = \Phi(g)$.

A property $\varphi(f)$ on $[\kappa]^\epsilon_*$ holds μ^κ_ϵ-almost everywhere if and only if there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $\varphi(f)$ holds. To express Φ is $\sqsubseteq$-constant μ^κ_ϵ-almost everywhere involves a formula $\varphi_0(f)$ which only involves f. To express Φ depends only on the supremum relative to C requires a formula $\varphi_1(f)$ which has C itself as a parameter. This causes some technical difficulties which can easily be resolved using the club uniformization principle, Fact 2.14, if $\kappa \to_* (\kappa)^\kappa_2$. However, at weak partition cardinals which are not strong partition cardinals, Fact 2.13 will need to be used together with the next result.

Fact 3.5. Suppose κ is a cardinal, $\epsilon < \kappa$ is an additively indecomposable ordinal, $\kappa \to_* (\kappa)^\epsilon_2$ and $C \subseteq \kappa$ is a club. Φ depends on supremum relative to C if and only if Φ is $\sqsubseteq$-constant on $[C]^\epsilon_*$.

Proof. Suppose Φ depends only on the supremum relative to C. Suppose $f \in [C]^\epsilon_*$ and $g \sqsubseteq f$. Then $\sup(f) = \sup(g)$ and thus $\Phi(f) = \Phi(g)$.

Suppose Φ is $\sqsubseteq$-constant on $[C]^\epsilon_*$. Let $f, g \in [C]^\epsilon_*$ be such that $\sup(f) = \sup(g)$. Since $\epsilon < \kappa$ is an additively indecomposable ordinal and $\sup(f) = \sup(g)$, $\mathrm{ot}(f[\epsilon] \cup g[\epsilon]) = \epsilon$. Let $h : \epsilon \to C$ be defined by induction as follows: Let $h(0) = \min(f[\epsilon] \cup g[\epsilon])$. If $\beta < \epsilon$ and $h \upharpoonright \beta$ has been defined, then let $h(\beta)$ be the least element of $f[\epsilon] \cup g[\epsilon]$ greater than $\sup(h \upharpoonright \beta)$. Then $f \sqsubseteq h$ and $g \sqsubseteq h$. Since Φ is $\sqsubseteq$-constant on $[C]^\epsilon_*$, one has that $\Phi(f) = \Phi(h) = \Phi(g)$. □

Theorem 3.6. *Suppose κ is a cardinal and $\epsilon < \kappa$ is a limit ordinal with $\mathrm{cof}(\epsilon) = \omega$. Let $\epsilon_0 < \epsilon$ and $\epsilon_1 \leq \epsilon$ be such that $\epsilon = \epsilon_0 + \epsilon_1$ and ϵ_1 is an additively indecomposable ordinal. Suppose $\kappa \to_* (\kappa)^\epsilon_2$ and $\kappa \to_* (\kappa)^{\epsilon_1 \cdot \epsilon_1}_2$ hold. Let $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ be a function which is E_0-invariant μ^κ_ϵ-almost everywhere. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.*

Proof. Since $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ is E_0-invariant μ^κ_ϵ-almost everywhere, let $C_0 \subseteq \kappa$ be a club so that for all $f, g \in [C_0]^\epsilon_*$, if $f \; E_0 \; g$, then $\Phi(f) = \Phi(g)$. For each $\sigma \in [C_0]^{\epsilon_0}_*$, define $\Phi_\sigma : [C_0 \setminus \sup(\sigma)+1]^{\epsilon_1}_* \to \mathrm{ON}$ by $\Phi_\sigma(\ell) = \Phi(\sigma\hat{\ }\ell)$. Note that Φ_σ is E_0-invariant on $[C_0 \setminus (\sup(\sigma) + 1)]^{\epsilon_1}_*$. Since $\mathrm{cof}(\epsilon) = \omega$ implies that $\mathrm{cof}(\epsilon_1) = \omega$, $\kappa \to_* (\kappa)^{\epsilon_1 \cdot \epsilon_1}_2$ and Lemma 3.3 imply there is a club $D \subseteq C_0 \setminus (\sup(\sigma) + 1)$ so that for all $\ell, \iota \in [D]^{\epsilon_1}_*$, if $\sup(\ell) = \sup(\iota)$, then $\Phi_\sigma(\ell) = \Phi_\sigma(\iota)$.

Define $R \subseteq [C_0]^{\epsilon_0}_* \times [\kappa]^{\epsilon_1}_*$ by $R(\sigma, \ell)$ if and only if for all $\iota \sqsubseteq \ell$, $\Phi_\sigma(\iota) = \Phi_\sigma(\ell)$. By the observation of the previous paragraph, for each $\sigma \in [C_0]^{\epsilon_0}_*$, there is a club D so that Φ_σ depends only on supremum relative to D. By Fact 3.5, Φ_σ is $\sqsubseteq$-constant on $[D]^{\epsilon_1}_*$. Thus, $[D]^{\epsilon_1}_* \subseteq R_\sigma$. This shows that for all $\sigma \in [C_0]^{\epsilon_0}_*$, $R_\sigma \in \mu^\kappa_{\epsilon_1}$. By $\kappa \to_* (\kappa)^\epsilon_2$ and Fact 2.13, there is a club $C_1 \subseteq C_0$ so that for all $\sigma \in [C_1]^{\epsilon_0}_*$, $[C_1 \setminus (\sup(\sigma) + 1)]^{\epsilon_1}_* \subseteq R_\sigma$.

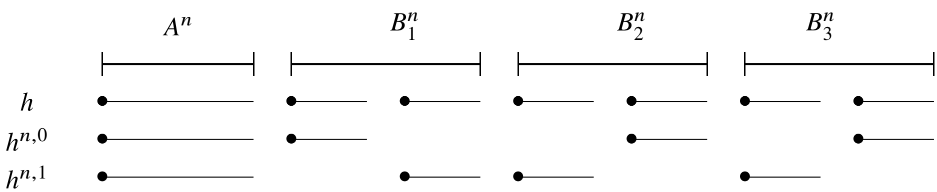
Let $\tau \in [C_1]^{\epsilon_0}_*$ be defined by $\tau(\alpha) = \mathsf{enum}_{C_1}(\omega \cdot \alpha + \omega)$ for each $\alpha < \epsilon_0$. Let $C_2 = C_1 \setminus \sup(\tau) + 1$. Now, suppose $f, g \in [C_2]^\epsilon_*$ and $\sup(f) = \sup(g)$. Let $\sigma_0, \sigma_1 \in [C_2]^{\epsilon_0}_*$ and $\ell_0, \ell_1 \in [C_2]^{\epsilon_1}_*$ be such that $f = \sigma_0\hat{\ }\ell_0$ and $g = \sigma_1\hat{\ }\ell_1$. Let $f' = \tau\hat{\ }\ell_0$ and $g' = \tau\hat{\ }\ell_1$. Note that $f \; E_0 \; f'$ and $g \; E_0 \; g'$. Since Φ is E_0-invariant, $\Phi(f) = \Phi(f')$ and $\Phi(g) = \Phi(g')$. However, Φ_τ is $\sqsubseteq$-constant on $[C_2]^{\epsilon_1}_*$ and so by Fact 3.5, Φ_τ depends only on supremum relative to C_2. Since $\sup(\ell_0) = \sup(\ell_1)$, $\Phi_\tau(\ell_0) = \Phi_\tau(\ell_1)$. In summary, $\Phi(f) = \Phi(f') = \Phi_\tau(\ell_0) = \Phi_\tau(\ell_1) = \Phi(g') = \Phi(g)$. Thus, C_2 is the desired club which completes the proof. □

Theorem 3.7. *Suppose κ is a cardinal and $\epsilon < \kappa$ with $\mathrm{cof}(\epsilon) = \omega$. Let $\epsilon_0 < \epsilon$ and $\epsilon_1 \leq \epsilon$ be such that $\epsilon = \epsilon_0 + \epsilon_1$ and ϵ_1 is an additively indecomposable ordinal. Suppose $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ and $\kappa \to_* (\kappa)^{\epsilon_1 \cdot \epsilon_1}_2$ hold. For any function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $f \upharpoonright \delta = g \upharpoonright \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.*

Proof. Since $\mathrm{cof}(\epsilon) = \omega$, let $\rho : \omega \to \epsilon$ be a cofinal increasing sequence through ϵ with $\rho(0) = \epsilon_0$. For $n \in \omega$, let $A^n = \{(0, 0, \alpha) : \alpha < \rho(n)\}$ and for each $1 \leq m < \omega$, $B^n_m = \{(m, i, \alpha) : i \in 2 \wedge \rho(n+m-1) \leq \alpha < \rho(n+m)\}$. Let $L^n = A^n \cup \bigcup_{1 \leq m < \omega} B^n_m$, and note that $L^n \subseteq \omega \times 2 \times \epsilon$. Let $\mathcal{L}^n = (L^n, \prec)$ where $\prec$ is the lexicographic ordering on $\omega \times 2 \times \epsilon$. Since $\rho(0) = \epsilon_0$ and ϵ_1 is additively indecomposable, $\mathrm{ot}(\mathcal{L}^n) = \epsilon_0 + \epsilon_1 = \epsilon$. For any function $h \in [\kappa]^{\mathcal{L}^n}_*$ and $i \in 2$, let $h^{n,i} \in [\kappa]^\epsilon_*$ be defined by

$$h^{n,i}(\alpha) = \begin{cases} h(m, 0, \alpha) & m = 0 \wedge \alpha < \rho(n) \\ h(m, i, \alpha) & m = 1 \wedge \rho(n) \leq \alpha < \rho(n+1) \\ h(m, 1-i, \alpha) & m > 1 \wedge \rho(n+m-1) \leq \alpha < \rho(n+m) \end{cases}.$$

The following picture indicates the relation between h, $h^{n,0}$ and $h^{n,1}$.



In other words, $h^{n,0}$ and $h^{n,1}$ are extracted from h in a manner so that $h^{n,0}$ and $h^{n,1}$ share the same k^{th}-block for $k < n$ (i.e., the functions agree before $\rho(n)$), the n^{th}-block of $h^{n,0}$ comes before the n^{th}-block of $h^{n,1}$, and for $k > n$, the k^{th}-block of $h^{n,1}$ comes before the k^{th}-block of $h^{n,0}$.

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If $f, g \in [\kappa]^{\epsilon}_{*}$, then say that the pair (f, g) has type n if and only if the following holds.

- For all $\alpha < \rho(n)$, $f(\alpha) = g(\alpha)$.
- $\sup(f \restriction \rho(n+1)) < g(\rho(n))$.
- For all $m \geq n+1$, $\sup(g \restriction \rho(m+1)) < f(\rho(m))$.

Observe that (f, g) has type n if and only if there is an $h \in [\kappa]^{\mathcal{L}^n}_{*}$ so that $h^{n,0} = f$ and $h^{n,1} = g$.

For each $n \in \omega$, let $P^n : [\kappa]^{\mathcal{L}^n}_{*} \to 3$ be defined by

$$P^n(h) = \begin{cases} 0 & \Phi(h^{n,0}) = \Phi(h^{n,1}) \\ 1 & \Phi(h^{n,0}) < \Phi(h^{n,1}) \\ 2 & \Phi(h^{n,0}) > \Phi(h^{n,1}) \end{cases}.$$

By the fact that $\operatorname{ot}(\mathcal{L}^n) = \epsilon$ and Fact 2.6, $\kappa \to_{*} (\kappa)^{\epsilon}_{3}$ implies that for each $n \in \omega$, there is a club $C \subseteq \kappa$ and an $i_n \in 3$ so that for all $h \in [C]^{\mathcal{L}^n}_{*}$, $P(h) = i_n$. For each $n \in \omega$, let $K_n = \{h \in [\kappa]^{\epsilon}_{*} : P^n(h) = i_n\}$ where $\mathcal{L}^n$ is identified with ϵ. For each $n \in \omega$, $K_n \in \mu^{\kappa}_{\epsilon}$. Since $\kappa \to_{*} (\kappa)^{\epsilon+\epsilon}_{2}$ implies μ^{κ}_{ϵ} is κ-complete by Fact 2.7, there is a club $C \subseteq \kappa$ so that $[C]^{\epsilon}_{*} \subseteq K_n$ for all $n \in \omega$. Thus, for all $n \in \omega$ and all $h \in [C]^{\epsilon}_{*}$, $P^n(h) = i_n$. By thinning C, one may assume C consists entirely of indecomposable ordinals and $\omega \cdot \epsilon < \min(C)$. Let $C^0 = C$. If C^n has been defined, then let $C^{n+1} = \{\alpha \in C^n : \mathsf{enum}_{C^n}(\alpha) = \alpha\}$. For each α with $\alpha \geq \epsilon_0 = \rho(0)$, let $\varsigma(\alpha)$ be the unique n so that $\rho(n) \leq \alpha < \rho(n+1)$.

(Case 1) For all $m \in \omega$, there exists an $n \geq m$ so that $i_n = 1$.

Let $\langle n_j : j \in \omega \rangle$ be an increasing enumeration of $\{n \in \omega : i_n = 1\}$. Let $\iota_n = \mathsf{enum}_{C^2}(n)$, the n^{th}-element of C^2. Let $\tau \in [C]^{\epsilon_0}_{*}$ be defined by $\tau(\alpha) = \mathsf{next}^{\omega \cdot (\alpha+1)}_{C}(0)$. Note that $\sup(\tau) < \iota_0$ since $\iota_0 \in C^2$, $\omega \cdot (\epsilon_0 + 1) < \iota_0$, and Fact 2.12. A sequence $\langle f_j : j \in \omega \rangle$ will be constructed so that all $j \in \omega$, the pair (f_{j+1}, f_j) has type n_j. This will be accomplished by recursively constructing the sequence while maintaining the following properties:

1. For all $j \in \omega$, $f_j \in [C]^{\epsilon}_{*}$.
2. For all $j \in \omega$, $f_j \restriction \epsilon_0 = \tau$.
3. For all $j, n \in \omega$, $\iota_n < f_j(\rho(n)) < \sup(f_j \restriction \rho(n+1)) < \iota_{n+1}$.
4. For all $j \in \omega$ and $\alpha < \rho(n_j)$, $f_j(\alpha) = f_{j+1}(\alpha)$.
5. For all $j \in \omega$, $\sup(f_{j+1} \restriction \rho(n_j + 1)) < f_j(\rho(n_j))$.
6. For all $j \in \omega$ and $n > n_j$, $\sup(f_j \restriction \rho(n+1)) < f_{j+1}(\rho(n))$.
7. For all $j \in \omega$ and $\alpha \geq \rho(n_j)$, $f_j(\alpha) \in C^1$.

Define f_0 by $f_0(\alpha) = \tau(\alpha)$ if $\alpha < \epsilon_0$ and $f_0(\alpha) = \mathsf{next}^{\omega \cdot (\alpha+1)}_{C^1}(\iota_{\varsigma(\alpha)})$ if $\epsilon_0 \leq \alpha < \epsilon$. Since $\omega \cdot (\rho(n+1)+1) < \iota_{n+1}$, $\iota_n < \iota_{n+1}$ and $C^2 = \{\alpha \in C_1 : \mathsf{enum}_{C^1}(\alpha) = \alpha\}$, Fact 2.12 implies that $\sup(f_0 \restriction \rho(n+1)) < \mathsf{next}^{\omega \cdot (\rho(n+1)+1)}_{C^1}(\iota_n) < \iota_{n+1}$.

Suppose f_j has been defined. Define f_{j+1} as follows: If $\alpha < \rho(n_j)$, let $f_{j+1}(\alpha) = f_j(\alpha)$. If $\rho(n_j) \leq \alpha < \rho(n_j+1)$, let $f_{j+1}(\alpha) = \mathsf{next}^{\omega \cdot (\alpha+1)}_{C^0}(\iota_{n_j})$. Observe that since $f_j(\rho(n_j)) \in C^1$ by (7), $\omega \cdot (\rho(n_j+1)) < f_j(\rho(n_j))$, $\iota_{n_j} < f_j(\rho(n_j))$ by (3) and $C^1 = \{\alpha \in C^0 : \mathsf{enum}_{C^0}(\alpha) = \alpha\}$, Fact 2.12 implies that $\sup(f_{j+1} \restriction \rho(n_j + 1)) < f_j(\rho(n_j))$. For α with $\rho(n_j + 1) \leq \alpha < \epsilon$, let $f_{j+1}(\alpha) = \mathsf{next}^{\omega \cdot \alpha}_{C^1}(\sup(f_j \restriction \rho(\varsigma(\alpha)+1)))$. For all $n > n_j$, $\sup(f_j \restriction \rho(n+1)) < f_{j+1}(\rho(n)) < \sup(f_{j+1} \restriction \rho(n+1)) < \iota_{n+1}$ since $\sup(f_{j+1} \restriction \rho(n+1)) < \mathsf{next}^{\omega \cdot (\rho(n+1)+1)}_{C^1}(\sup(f_j \restriction \rho(n+1))) < \iota_{n+1}$ because $\omega \cdot (\rho(n+1)+1) < \iota_{n+1}$, $\sup(f_j \restriction \rho(n+1)) < \iota_{n+1} \in C^2$, $C^2 = \{\alpha \in C_1 : \mathsf{enum}_{C^1}(\alpha) = \alpha\}$ and Fact 2.12. This shows that f_{j+1} has been constructed with the desired relations between f_j and f_{j+1}.

By (1), (2), (4), (5) and (6), $f_j, f_{j+1} \in [C]^{\epsilon}_{*}$ and (f_{j+1}, f_j) has type n_j. Thus, for each $j \in \omega$, there is a function $h_j \in [C]^{\mathcal{L}^{n_j}}_{*}$ so that $h_j^{n_j,0} = f_{j+1}$ and $h_j^{n_j,1} = f_j$. Since for all $j \in \omega$, C is homogeneous for P^{n_j} taking value 1, $P^{n_j}(h_j) = 1$. This implies $\Phi(f_{j+1}) = \Phi(h_j^{n_j,0}) < \Phi(h_j^{n_j,1}) = \Phi(f_j)$. Thus, $\langle \Phi(f_j) : j \in \omega \rangle$ is an infinite descending sequence of ordinals. This shows Case 1 is impossible.

(Case 2) For all $m \in \omega$, there exists an $n \geq m$ so that $i_n = 2$.

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Let $\langle n_j : j \in \omega\rangle$ be an increasing enumeration of $\{n \in \omega : i_n = 2\}$. For each $k \in \omega$, let $\varpi(k) = |\{n < k : i_n = 2\}$. For $\alpha < \epsilon_0 = \rho(0)$, let $\tau_0(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C^0}(0)$. Let $\iota_0 = \mathsf{next}^{\omega}_{C^0}(\sup(\tau_0))$. If ι_n has been defined, then let $\iota_{n+1} = \mathsf{next}^{\omega\cdot(\rho(n+1)+1)}_{C^{\varpi(n+1)}}(\iota_n)$. A sequence $\langle f_j : j \in \omega\rangle$ will be constructed so that for all $j \in \omega$, the pair (f_j, f_{j+1}) has type n_j. This will be accomplished by maintaining the following properties throughout the construction:

1. For all $j \in \omega$, $f_j \in [C]^{\epsilon}_*$.
2. For all $j \in \omega$, $f_j \restriction \epsilon_0 = \tau$.
3. For all $j, n \in \omega$, $\iota_n < f_j(\rho(n)) < \sup(f_j \restriction \rho(n+1)) < \iota_{n+1}$.
4. For all $j \in \omega$ and $\alpha < \rho(n_j)$, $f_j(\alpha) = f_{j+1}(\alpha)$.
5. For all $j \in \omega$, $\sup(f_j \restriction \rho(n_j + 1)) < f_{j+1}(\rho(n_j))$.
6. For all $j \in \omega$, $n > n_j$, $\sup(f_{j+1} \restriction \rho(n+1)) < f_j(\rho(n))$.
7. For all $j \in \omega$ and $\alpha \geq \rho(n_j + 1)$, $f_j(\alpha) \in C^{\varpi(\varsigma(\alpha))-j}$.

For $\alpha < \epsilon_0$, let $f_0(\alpha) = \tau(\alpha)$. For α with $\epsilon_0 = \rho(0) \leq \alpha < \epsilon$, let $f_0(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C^{\varpi(\varsigma(\alpha))}}(\iota_{\varsigma(\alpha)})$. Note that for each $n \in \omega$, $\sup(f_0 \restriction \rho(n+1)) \leq \mathsf{next}^{\omega\cdot(\rho(n+1))}_{C^{\varpi(n)}}(\iota_n) \leq \mathsf{next}^{\omega\cdot(\rho(n+1))}_{C^{\varpi(n+1)}}(\iota_n) < \mathsf{next}^{\omega\cdot(\rho(n+1)+1)}_{C^{\varpi(n+1)}}(\iota_n) = \iota_{n+1}$.

Suppose f_j has been defined. Define f_{j+1} as follows: If $\alpha < \rho(n_j)$, let $f_{j+1}(\alpha) = f_j(\alpha)$. If $\rho(n_j) \leq \alpha < \rho(n_j + 1)$, let $f_{j+1}(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C^0}(\sup(f_j \restriction \rho(n_j + 1)))$. Observe that $\sup(f_{j+1} \restriction \rho(n_j + 1)) < \iota_{n_j+1}$ since $\iota_{n_j+1} \in C^{j+1}$, $\sup(f_j \restriction \rho(n_j + 1)) < \iota_{n_j+1}$ by (3), $\omega \cdot (\rho(n_j + 1) + 1) < \iota_{n_j+1}$ and by Fact 2.12. For α with $\rho(n_j + 1) \leq \alpha < \epsilon$, let $f_{j+1}(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C^{\varpi(\varsigma(\alpha))-j-1}}(\iota_{\varsigma(\alpha)})$. (Observe for all α with $\rho(n_j + 1) \leq \alpha < \epsilon$, $\varpi(\varsigma(\alpha)) \geq j + 1$.) Note that for each $n > n_j$, $\sup(f_{j+1} \restriction \rho(n+1)) \leq \mathsf{next}^{\omega\cdot(\rho(n+1)+1)}_{C^{\varpi(n)}-j-1}(\iota_n) < f_j(\rho(n))$ since $\omega\cdot(\rho(n+1)+1) < f_j(\rho(n))$, $\iota_n < f(\rho(n))$, $f_j(\rho(n)) \in C^{\varpi(n)-j}$ and by Fact 2.12. This completes the construction of f_{j+1} with the desired relation between f_j and f_{j+1}.

By (1), (2), (4), (5) and (6), $f_j, f_{j+1} \in [C]^{\epsilon}_*$ and (f_j, f_{j+1}) has type n_j. Thus, for each $j \in \omega$, there is a function $h_j \in [C]^{\mathcal{L}^{n_j}}_*$ so that $h^{n_j,0}_j = f_j$ and $h^{n_j,1}_j = f_{j+1}$. Since for all $j \in \omega$, C is homogeneous for P^{n_j} taking value 2, $P^{n_j}(h_j) = 2$. This implies $\Phi(f_j) = \Phi(h^{n_j,0}) > \Phi(h^{n_j,1}) = \Phi(f_{j+1})$. $\langle \Phi(f_j) : j \in \omega\rangle$ is an infinite descending sequence of ordinals. This shows Case 2 is impossible.

The failure of both Case 1 and Case 2 implies that the following Case 3 must hold.

(Case 3) There exists an $m^* \in \omega$ so that for all $n \geq m^*$, $i_n = 0$.

Fix $\ell \in [C]^{\rho(m^*)}_*$. Define $\Phi_\ell : [C]^{\epsilon_1}_* \to \mathrm{ON}$ by $\Phi_\ell(v) = \Phi(\ell\hat{\ }v)$. It will be shown that Φ_ℓ is E_0-invariant $\mu^{\kappa}_{\epsilon_1}$-almost everywhere. Suppose $v, w \in [C^2]^{\epsilon_1}_*$ and $v \; E_0 \; w$. Let $f_{m^*} = \ell\hat{\ }v$ and $g_{m^*} = \ell\hat{\ }w$. Since $v \; E_0 \; w$, let $n^* \geq m^*$ be such that for all $\alpha \geq \rho(n^*)$, $f_{m^*}(\alpha) = g_{m^*}(\alpha)$. For $j \geq m^*$, let $\iota^f_j = f_{m^*}(\rho(j))$ and $\iota^g_j = g_{m^*}(\rho(j))$. Note that $\iota^f_j, \iota^g_j \in C^2$ for all $j \geq m^*$. One will define two finite sequences $\langle f_j : m^* \leq j \leq n^*\rangle$ and $\langle g_j : m^* \leq j \leq n^*\rangle$ with the following properties:

1. For all $m^* \leq j \leq n^*$, $f_j, g_j \in [C]^{\epsilon}_*$. For all $m^* \leq j \leq n^*$ and $\rho(j) \leq \alpha < \epsilon$, $f_j(\alpha), g_j(\alpha) \in C^1$.
2. For all $m^* \leq j \leq n^*$ and $m^* \leq k < \omega$, $\sup(f_j \restriction \rho(k)) < \iota^f_k$ and $\sup(g_j \restriction \rho(k)) < \iota^g_k$.
3. For all $m^* \leq j \leq n^*$, $f_j \restriction \rho(m^*) = \ell = g_j \restriction \rho(m^*)$.
4. For all $m^* \leq j \leq n^*$, $f_j \restriction \rho(j) = g_j \restriction \rho(j)$.
5. For all $m^* \leq j \leq n^*$ and α with $\rho(n^*) \leq \alpha < \epsilon$, $f_j(\alpha) = g_j(\alpha)$.
6. For all $m^* \leq j < n^*$, $\sup(f_{j+1} \restriction \rho(j+1)) < f_j(\rho(j))$ and $\sup(g_{j+1} \restriction \rho(j+1)) < g_j(\rho(j))$.
7. For all $m^* \leq j < n^*$ and $j + 1 \leq k < \omega$, $\sup(f_j \restriction \rho(k+1)) < f_{j+1}(\rho(k))$ and $\sup(g_j \restriction \rho(k+1)) < g_{j+1}(\rho(k))$.

Note that f_{m^*} and g_{m^*} have already been defined above. Suppose $m^* \leq j < n^*$ and f_j and g_j have already been defined with the above properties. For $\alpha < \rho(j)$, let $f_{j+1}(\alpha)$ and $g_{j+1}(\alpha)$ be $f_j(\alpha) = g_j(\alpha)$ by (4) and therefore $\sup(f_{j+1} \restriction \rho(j)) = \sup(g_{j+1} \restriction \rho(j))$. For $\rho(j) \leq \alpha < \rho(j+1)$, define $f_{j+1}(\alpha)$ and $g_{j+1}(\alpha)$ to be $\mathsf{next}^{\omega\cdot(\alpha+1)}_{C^0}(\sup(f_{j+1} \restriction \rho(j))) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C_0}(\sup(g_{j+1} \restriction \rho(j)))$. Thus, $f_{j+1} \restriction \rho(j+1) = g_{j+1} \restriction \rho(j+1)$. Note that since $f_j(\rho(j)) \in C^1$, $g_j(\rho(j)) \in C^1$ and

$\omega\cdot(\rho(j+1)+1) < \min\{f_j(\rho(j)), g_j(\rho(j))\}$, Fact 2.12 implies that $\sup(f_{j+1} \upharpoonright \rho(j+1)) = \sup(g_{j+1} \upharpoonright \rho(j+1)) < \min\{f_j(\rho(j)), g_j(\rho(j))\}$. For $\alpha \geq \rho(j+1)$, let $f_{j+1}(\alpha) = \mathsf{next}_{C^1}^{\omega\cdot(\alpha+1)}(\sup(f_j \upharpoonright \rho(\varsigma(\alpha))))$ and $g_{j+1}(\alpha) = \mathsf{next}_{C^1}^{\omega\cdot(\alpha+1)}(\sup(g_j \upharpoonright \rho(\varsigma(\alpha))))$. Since for all $k \geq j+1$, $\sup(f_j \upharpoonright \rho(k)) < \iota_k^f$ and $\sup(g_j \upharpoonright \rho(k)) < \iota_k^g$ by (2), $\omega\cdot(\rho(k)+1) < \min\{\iota_f^g, \iota_k^f\}$ and $\iota_k^f, \iota_k^g \in C^2$, Fact 2.12 implies that $\sup(f_{j+1} \upharpoonright \rho(k)) < \iota_k^f$ and $\sup(g_{j+1} \upharpoonright \rho(k)) < \iota_k^f$. By (5), for any $k \geq n^*$, $\sup(f_j \upharpoonright \rho(k)) = \sup(g_j \upharpoonright \rho(k))$. This implies that for all $\alpha \geq \rho(n^*)$, $f_{j+1}(\alpha) = g_{j+1}(\alpha)$. This completes the construction of f_{j+1} and g_{j+1}.

By (1), (3), (4), (5), (6) and (7), (f_{j+1}, f_j) and (g_{j+1}, g_j) are of type j for each j such that $m^* \leq j < n^*$. For each j so that $m^* \leq j < n^*$, let $h_j, p_j \in [C]_*^{\mathcal{L}^j}$ be such that $h_j^{j,0} = f_{j+1}$, $h_j^{j,1} = f_j$, $p_j^{j,0} = g_{j+1}$ and $p_j^{j,1} = g_j$. Since for all $j \geq m^*$, C is homogeneous for P^j taking value 0, one has that for all $m^* \leq j < n^*$, $P^j(h_j) = 0$ and $P^j(p_j) = 0$. Thus, $\Phi(f_{j+1}) = \Phi(f_j)$ and $\Phi(g_{j+1}) = \Phi(g_j)$. Also, by (3), (4) and (5), $f_{n^*} = g_{n^*}$. Putting these together, one has that

$$\Phi_\ell(v) = \Phi(\ell\hat{\ }v) = \Phi(f_{m^*}) = \Phi(f_{m^*+1}) = \ldots = \Phi(f_{n^*}) = \Phi(g_{n^*}) = \ldots = \Phi(g_{m^*+1}) = \Phi(g_{m^*}) = \Phi(\ell\hat{\ }w) = \Phi_\ell(w).$$

It has been shown that for all $v, w \in [C]_*^{\epsilon_1}$, if $v\ E_0\ w$, then $\Phi_\ell(v) = \Phi_\ell(w)$. This shows that Φ_ℓ is E_0-invariant $\mu_{\epsilon_1}^\kappa$-almost everywhere.

Define a relation $R \subseteq [C]_*^{\rho(m^*)} \times [\kappa]_*^{\epsilon_1}$ by $R(\ell, v)$ if and only if for all $w \sqsubseteq v$, $\Phi_\ell(w) = \Phi_\ell(v)$. Since it was shown above that for each $\ell \in [C]^{\rho(m^*)}$, Φ_ℓ is E_0-invariant $\mu_{\epsilon_1}^\kappa$-almost everywhere and $\kappa \to_* (\kappa)^{\epsilon_1\cdot\epsilon_1}$ holds, Theorem 3.7 implies that there is a club $D \subseteq \kappa$ so that for all $v, w \in [D]_*^{\epsilon_1}$, if $\sup(v) = \sup(w)$, then $\Phi_\ell(v) = \Phi_\ell(w)$. In particular, for any $v \in [D]_*^{\epsilon_1}$ and any $w \sqsubseteq v$, $\sup(v) = \sup(w)$ and thus $\Phi_\ell(v) = \Phi_\ell(w)$. This shows $[D]_*^{\epsilon_1} \subseteq R_\ell$. For all $\ell \in [C]_*^{\rho(m^*)}$, $R_\ell \in \mu_{\epsilon_1}^\kappa$. By Fact 2.13, there is a club $E \subseteq C$ so that for all $\ell \in [E]^{\rho(m^*)}$, $[E \setminus (\sup(\ell)+1)]_*^{\epsilon_1} \subseteq R_\ell$.

Let $\delta = \rho(m^*)$. Suppose $f, g \in [E]_*^\epsilon$, $\sup(f) = \sup(g)$ and $f \upharpoonright \delta = g \upharpoonright \delta$. Let $\ell = f \upharpoonright \delta = g \upharpoonright \delta$. There exists $v, w \in [E]_*^{\epsilon_1}$ so that $f = \ell\hat{\ }v$ and $g = \ell\hat{\ }w$. Φ_ℓ is $\sqsubseteq$-constant on $[E \setminus (\sup(\ell)+1)]_*^{\epsilon_1}$ and so by Fact 3.5, Φ_ℓ depends on supremum relative to $E \setminus (\sup(\ell)+1)$. Since $v, w \in [E \setminus (\sup(\ell)+1)]_*^{\epsilon_1}$ and $\sup(v) = \sup(w)$, $\Phi(f) = \Phi(\ell\hat{\ }v) = \Phi_\ell(v) = \Phi_\ell(w) = \Phi(\ell\hat{\ }w) = \Phi(g)$. It has been shown that there is a δ so that for all $f, g \in [E]_*^\epsilon$, if $\sup(f) = \sup(g)$ and $f \upharpoonright \delta = g \upharpoonright \delta$, then $\Phi(f) = \Phi(g)$. □

Corollary 3.8. *Suppose κ is a cardinal, $\epsilon < \kappa$ is a limit ordinal with $\mathrm{cof}(\epsilon) = \omega$ and $p \in \omega$. Let $\epsilon_0 < \epsilon$ and $\epsilon_1 \leq \epsilon$ be such that $\epsilon = \epsilon_0 + \epsilon_1$ and ϵ_1 is an additively indecomposable ordinal. Assume $\kappa \to_* (\kappa)_2^{\epsilon+\epsilon+p}$ and $\kappa \to_* (\kappa)_2^{\epsilon_1\cdot\epsilon_1+p}$. Let $\Phi : [\kappa]_*^{\epsilon+p} \to \mathrm{ON}$ be a function. Then there is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]_*^{\epsilon+p}$, if $f \upharpoonright \delta = g \upharpoonright \delta$, $\sup(f \upharpoonright \epsilon) = \sup(g \upharpoonright \epsilon)$ and for all $i < p$, $f(\epsilon+i) = g(\epsilon+i)$, then $\Phi(f) = \Phi(g)$.*

Proof. The argument is similar to Theorem 3.7 where all partitions now include p elements at the top. □

Theorem 3.9. *Suppose κ is a cardinal, $\epsilon < \omega_1$ and $\kappa \to_* (\kappa)_2^{\epsilon\cdot\epsilon}$ holds. Let $\Phi : [\kappa]_*^\epsilon \to \mathrm{ON}$. Then there is a club $C \subseteq \kappa$ and finitely many ordinals $\delta_0, \ldots, \delta_k \leq \epsilon$ so that for all $f, g \in [C]_*^\epsilon$, if for all $0 \leq i \leq k$, $\sup(f \upharpoonright \delta_i) = \sup(g \upharpoonright \delta_i)$, then $\Phi(f) = \Phi(g)$.*

Proof. The sequences of ordinals $\delta_0, \ldots, \delta_k$ will be defined by recursion. Let $\delta_0 = \epsilon$. Suppose $\epsilon = \delta_0 > \ldots > \delta_i$ have been defined so that there exists a club $D \subseteq \kappa$ with the property that for all $f, g \in [C]_*^\epsilon$, if $f \upharpoonright \delta_i = g \upharpoonright \delta_i$ and for all $0 \leq j < i$, $\sup(f \upharpoonright \delta_j) = \sup(g \upharpoonright \delta_j)$, then $\Phi(f) = \Phi(g)$. Fix such a club D. If δ_i is a successor ordinal, then let δ_{i+1} be the predecessor of δ_i. If δ_i is a limit, then $\mathrm{cof}(\delta_i) = \omega$ since $\delta_i \leq \epsilon < \omega_1$. Define $\Psi : [D]^{\delta_i+i} \to \mathrm{ON}$ as follows: Suppose $\ell \in [D]^{\delta_i}$ and $\gamma_0 > \ldots > \gamma_{i-1} > \sup(\ell)$ in D. Let $\Psi(\ell, \gamma_{i-1}, \ldots, \gamma_0) = \Phi(f)$ where f is any element of $[D]_*^\epsilon$ so that $\sup(f \upharpoonright \delta_i) = \ell$, $\sup(f \upharpoonright \delta_j) = \gamma_j$ for each $j < i$. $\Psi(\ell, \gamma_{i-1}, \ldots, \gamma_0)$ is well defined and independent of the choice of f with the above property by the induction hypothesis on $\delta_0, \ldots, \delta_i$ and D. By Corollary 3.8, there is a $\delta < \delta_i$ and a club $C \subseteq D$ so that for all $\ell, \iota \in [C]_*^{\delta_i}$, $\gamma_{i-1} < \ldots < \gamma_0$

in C, if $\sup(\ell) = \sup(\iota) < \gamma_{i-1}$ and $\ell \restriction \delta = \iota \restriction \delta$, then $\Psi(\ell, \gamma_{i-1}, ..., \gamma_0) = \Psi(\iota, \gamma_{i-1}, ..., \gamma_0)$. Let δ_{i+1} be the least δ with this property. By definition of Ψ, it has been shown that there is a club C and ordinals $\epsilon = \delta_0 > ... > \delta_{i+1}$ so that for all $f, g \in [C]^\epsilon_*$, if $f \restriction \delta_{i+1} = g \restriction \delta_{i+1}$ and for all $j < i + 1$, $\sup(f \restriction \delta_j) = \sup(g \restriction \delta_j)$, then $\Phi(f) = \Phi(g)$. By the wellfoundedness of the ordinals, there is some k so that at stage k, $\delta_k = 0$. Then the finite sequence $\epsilon = \delta_0 > \delta_1 > ... > \delta_k = 0$ has the property that there is a club C so that for all $f, g \in [C]^\epsilon_*$, if for all $j \leq k$, $\sup(f \restriction \delta_j) = \sup(g \restriction \delta_j)$, then $\Phi(f) = \Phi(g)$. □

The rest of this section will put the earlier result in context and provide some additional examples especially under AD.

Corollary 3.10. *Assume* AD. *Suppose κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming* $\mathsf{DC}_{\mathbb{R}}$*). If $\epsilon < \omega_1$ and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, then there is a club $C \subseteq \kappa$ and finitely many ordinals $\beta_0 < \beta_1 < ... < \beta_{p-1} \leq \epsilon$ (where $p \in \omega$) so that for all $f, g \in [C]^\epsilon_*$, if for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$, then $\Phi(f) = \Phi(g)$.*

Assume AD. *Suppose κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming* $\mathsf{DC}_{\mathbb{R}}$*). If $\epsilon < \kappa$ with $\mathrm{cof}(\epsilon) = \omega$ and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, then there is a club $C \subseteq \kappa$ and a $\delta < \epsilon$ so that for all $f, g \in [C]^\epsilon_*$, if $\sup(f) = \sup(g)$ and $f \restriction \delta = g \restriction \delta$, then $\Phi(f) = \Phi(g)$.*

Proof. This follows from Fact 2.16, Theorem 3.7 and Theorem 3.9. □

The next result shows that the assumption that $\mathrm{cof}(\epsilon) = \omega$ is necessary in Theorem 3.7.

Fact 3.11. Suppose $\zeta < \kappa$ are two cardinals such that $\zeta \to_* (\zeta)^2_2$, $\kappa \to_* (\kappa)^\zeta_2$ and $\mathsf{DC}_{[\kappa]^\zeta}$ hold. Then the ultrapower $\prod_\zeta \kappa/\mu^\zeta_1$ is a wellordering (and hence an ordinal) and there is a function $\Phi : [\kappa]^\zeta_* \to \prod_\zeta \kappa/\mu^\zeta_1$ so that for all clubs $C \subseteq \kappa$ and all $\delta < \zeta$, there are functions $f, g \in [C]^\zeta_*$ with $\sup(f) = \sup(g)$, $f \restriction \delta = g \restriction \delta$ and $\Phi(f) \neq \Phi(g)$.

Proof. The partition relation $\zeta \to_* (\zeta)^2_2$ implies that μ^ζ_1 is a ζ-complete ultrafilter on ζ and thus $\mathrm{cof}(\zeta) = \zeta > \omega$. $\mathsf{DC}_{[\kappa]^\zeta}$ implies that the ultrapower $\prod_\zeta \kappa/\mu^\zeta_1$ is a wellordering which can be identified as an ordinal. If $f : \zeta \to \kappa$, then let $[f]_{\mu^\zeta_1}$ denote the element of the ultrapower represented by the function f. Define $\Phi : [\kappa]^\zeta_* \to \prod_\zeta \kappa/\mu^\zeta_1$ by $\Phi(f) = [f]_{\mu^\zeta_1}$. Let $\delta < \zeta$ and $C \subseteq \kappa$ be a club. Let $\ell \in [C]^\delta_*$. Let $\iota_0, \iota_1 \in [C]^\zeta_*$ be defined by $\iota_0(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_C(\sup(\ell))$ and $\iota_1(\alpha) = \mathsf{next}^{\omega\cdot(\alpha+2)}_C(\sup(\ell))$. Let $f = \ell\hat{}\iota_0$ and $g = \ell\hat{}\iota_1$. Then $f, g \in [C]^\zeta_*$, $\sup(f) = \sup(g)$, $f \restriction \delta = \ell = g \restriction \delta$ and $\Phi(f) < \Phi(g)$ since $\{\alpha < \zeta : f(\alpha) < g(\alpha)\} \supseteq \{\alpha < \zeta : \alpha \geq \delta\} \in \mu^\zeta_1$. □

Fact 3.12. Assume AD. There is a function $\Phi : [\omega_2]^{\omega_1}_* \to \omega_3$ so that for all clubs $C \subseteq \kappa$ and $\delta < \omega_1$, there are functions $f, g \in [C]^{\omega_1}_*$ so that $\sup(f) = \sup(g)$, $f \restriction \delta = g \restriction \delta$ and $\Phi(f) \neq \Phi(g)$.

Proof. Under AD, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\prod_{\omega_1} \omega_2/\mu^{\omega_1}_1 = \omega_3$. As in Fact 3.11, the map $\Phi : [\omega_2]^{\omega_1} \to \omega_3$ defined by $\Phi(f) = [f]_{\mu^{\omega_1}_1}$ has the desired property. □

The restriction that $\epsilon < \omega_1$ is also necessary in Theorem 3.9 according to the following example.

Example 3.13. Suppose $\zeta < \kappa$ are two cardinals such that $\zeta \to_* (\zeta)^2_2$, $\kappa \to (\kappa)^\zeta_2$ and $\mathsf{DC}_{[\kappa]^\zeta}$ hold. Then there is a function $\Psi : [\kappa]^{\zeta+\omega}_* \to \prod_\zeta \kappa/\mu^\zeta_1$ so that for all clubs $C \subseteq \kappa$ and all finite set of ordinals $\beta_0 < ... < \beta_{p-1} \leq \zeta + \omega$ (where $p \in \omega$), there are functions $f, g \in [C]^{\zeta+\omega}_*$ so that for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$ and $\Psi(f) \neq \Psi(g)$.

Assume AD. There is a function $\Psi : [\omega_2]^{\omega_1+\omega}_* \to \omega_3$ so that for all clubs $C \subseteq \omega_2$ and all finite sets of ordinals $\beta_0 < \beta_1 < ... < \beta_{p-1} \leq \omega_1 + \omega$ (where $p \in \omega$), there are functions $f, g \in [C]^{\omega_1+\omega}_*$ so that for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$ and $\Psi(f) \neq \Psi(g)$.

Proof. For the first statement, let $\Psi : [\kappa]^{\zeta+\omega}_* \to \prod_\zeta \kappa/\mu^\zeta_1$ be defined by $\Psi(f) = \Phi(f \restriction \zeta)$ where Φ is the function from the proof of Fact 3.11.

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For the second statement, let $\Psi : [\omega_2]^{\omega_1+\omega}_* \to \omega_3$ be defined by $\Psi(f) = \Phi(f \restriction \omega_1)$ where Φ is the function from the proof of Fact 3.12. For a slightly more interesting example, one can also use $\Upsilon(f) = \Psi(f) + \sup(f) = \Phi(f \restriction \omega_1) + \sup(f)$. Note that since $\mathrm{cof}(\omega_1 + \omega) = \omega$, Theorem 3.7 does apply to Υ and indeed, for all $f, g \in [\omega_2]^{\omega_1+\omega}_*$, if $\sup(f) = \sup(g)$ and $f \restriction \omega_1 = g \restriction \omega_1$, then $\Upsilon(f) = \Upsilon(g)$. □

Consider a function $\Phi : [\omega_2]^\epsilon \to \mathrm{ON}$ where $\omega_1 \leq \epsilon < \omega_2$ but $\mathrm{cof}(\epsilon) > \omega$. Neither Theorem 3.7 nor Theorem 3.9 is applicable since $\epsilon \geq \omega_1$ and $\mathrm{cof}(\epsilon) \neq \omega$. Moreover, Fact 3.12 gives an example of a function $\Phi : [\omega_2]^{\omega_1}_* \to \omega_3$ which fails to satisfy the short length continuity property under AD. Remarkably under AD, if one demands the function Φ takes image in ω_2 rather than ω_3, then the short length continuity properties do hold even if $\mathrm{cof}(\epsilon) = \omega_1$. This result is possible under AD because ω_2 has an ultrapower representation as $\prod_{\omega_1} \omega_1/\mu_1^{\omega_1}$ which can be studied using the Kunen tree analysis.

Fact 3.14. ([6]) Assume AD. Suppose $\epsilon < \omega_2$ (including the possibility $\mathrm{cof}(\epsilon) = \omega_1$) and $\Phi : [\omega_2]^\epsilon_* \to \omega_2$. Then there is a club $C \subseteq \omega_2$ and finitely many ordinals $\beta_0 < \beta_1 < ... < \beta_{p-1} \leq \epsilon$ (where $p \in \omega$) so that for all $f, g \in [C]^\epsilon_*$, if for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$, then $\Phi(f) = \Phi(g)$.

4. Applications of short length continuity

Fact 4.1. If κ is a cardinal and $\epsilon \leq \kappa$ with $\mathrm{cof}(\kappa) \geq \epsilon$, then $|{}^\epsilon\kappa| = |[\kappa]^\epsilon| = |[\kappa]^\epsilon_*|$.

Proof. For $f \in {}^\epsilon\kappa$, define by recursion $\Phi(f)$ as follows: $\Phi(f)(0) = f(0)$. If $\beta < \epsilon$ and $\Phi(f) \restriction \beta$ has been defined, then $\sup(\Phi(f) \restriction \beta) < \kappa$ since $\mathrm{cof}(\kappa) \geq \epsilon$ and so let $\Phi(f)(\beta) = \sup(\Phi(f) \restriction \beta) + f(\beta)$. $\Phi(f) \in [\kappa]^\epsilon$. $\Phi : {}^\epsilon\kappa \to [\kappa]^\epsilon$ is an injection and thus $|{}^\epsilon\kappa| \leq |[\kappa]^\epsilon| \leq |{}^\epsilon\kappa|$.

Let $A = \{\omega \cdot (\alpha + 1) : \alpha < \kappa\}$. Suppose $f \in [A]^\epsilon$. For each $\alpha < \epsilon$, let $\gamma_\alpha < \kappa$ be such that $f(\alpha) = \omega \cdot (\gamma_\alpha + 1)$. Define $F : \epsilon \times \omega \to \kappa$ by $F(\alpha, n) = \omega \cdot \gamma_\alpha + n$. F witnesses that f has uniform cofinality ω. Fix $\beta < \epsilon$. Let $\zeta = \sup\{\gamma_\alpha + 1 : \alpha < \beta\}$. Note that $\zeta \leq \gamma_\beta$. Then $\sup(f \restriction \beta) \leq \omega \cdot \zeta \leq \omega \cdot \gamma_\beta < \omega \cdot (\gamma_\beta + 1) = f(\beta)$. This shows that f is discontinuous everywhere. Hence, f has the correct type. Thus, it has been shown that $[A]^\epsilon = [A]^\epsilon_*$. Since $|A| = \kappa$, $|[\kappa]^\epsilon_*| \leq |[\kappa]^\epsilon| = |[A]^\epsilon| = |[A]^\epsilon_*| \leq |[\kappa]^\epsilon_*|$. □

The following application of the almost everywhere short length continuity shows that infinite exponent partition spaces are not wellorderable assuming suitable partition properties. (More optimal results are known. For instance, the correct type partition relation $\kappa \to_* (\kappa)^2_2$ implies $[\kappa]^\omega$ is not wellorderable. The ordinary partition relation $\kappa \to (\kappa)^\omega_2$ also implies $[\kappa]^\omega$ is not wellorderable. It is not clear if $\kappa \to_* (\kappa)^2_2$ implies $\kappa \to (\kappa)^\omega_2$ or $\kappa \to (\kappa)^\omega_2$ implies $\kappa \to_* (\kappa)^2_2$.)

Theorem 4.2. *Suppose κ is a cardinal so that $\kappa \to_* (\kappa)^{\omega\cdot\omega}_2$. $[\kappa]^\omega$ is not wellorderable and thus for all $\epsilon, \delta \in \mathrm{ON}$ with $\omega \leq \epsilon$ and $\kappa \leq \delta$, ${}^\epsilon\delta$ and $\mathscr{P}(\delta)$ are not wellorderable.*

Proof. Suppose $[\kappa]^\omega$ was wellorderable. Then there is an injection $\Phi : [\kappa]^\omega_* \to \mathrm{ON}$. By Theorem 3.7, there is a club $C \subseteq \kappa$ and an $n < \omega$ so that for all $f, g \in [C]^\omega_*$, if $f \restriction n = g \restriction n$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$. Pick any $f, g \in [C]^\omega_*$ so that $f \restriction n = g \restriction n$, $\sup(f) = \sup(g)$ and $f(n) \neq g(n)$. Then $f \neq g$ and $\Phi(f) = \Phi(g)$. Φ is not an injection. Contradiction.

If $\omega \leq \epsilon$ and $\kappa \leq \delta$, then $[\kappa]^\omega$ injects to ${}^\epsilon\delta$ and $\mathscr{P}(\delta)$. Thus, ${}^\epsilon\delta$ and $\mathscr{P}(\delta)$ cannot be wellorderable. □

The following cardinality computation was proved in [5] for ω_1 using AD and $\mathsf{DC}_\mathbb{R}$.

Fact 4.3. ([5] Theorem 2.9) Assuming AD, $\neg(|{}^{<\omega_1}\omega_1| \leq |{}^\omega(\omega_\omega)|)$.

([5] Theorem 4.4) Assuming AD and $\mathsf{DC}_\mathbb{R}$, there is no injection of ${}^{<\omega_1}\omega_1$ into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals.

The arguments in [5] used many techniques of determinacy (often specific to ω_1). The techniques seem difficult to generalize to the higher projective ordinals $\boldsymbol{\delta}^1_n$ and have no analog at strong partition cardinals which are limit cardinals like $\boldsymbol{\delta}^2_1$. The following result generalizes Fact 4.3 purely from the weak partition relation.

Theorem 4.4. *Suppose κ is a cardinal so that $\kappa \to_* (\kappa)^{<\kappa}_2$. Then for all $\chi < \kappa$, there is no injection of ${}^{<\kappa}\kappa$ into ${}^{\chi}\mathrm{ON}$, the class of χ-length sequences of ordinals. In particular, for all $\chi < \kappa$, $|{}^{\chi}\kappa| < |{}^{<\kappa}\kappa|$.*

Proof. Suppose there is an injection $\Phi' : {}^{<\kappa}\kappa \to {}^{\chi}\mathrm{ON}$. By Fact 4.1, $|{}^{<\kappa}\kappa| = |[\kappa]^{<\kappa}_*|$ and thus one has an injection $\Phi : [\kappa]^{<\kappa}_* \to {}^{\chi}\mathrm{ON}$. For each $\epsilon < \kappa$ and $\gamma < \chi$, let $\Phi^{\epsilon}_{\gamma} : [\kappa]^{\epsilon}_* \to \mathrm{ON}$ be defined by $\Phi^{\epsilon}_{\gamma}(f) = \Phi(f)(\gamma)$. By Theorem 3.7, for each $\gamma < \chi$ and $\epsilon \in [\kappa]^1_*$ (equivalently, $\epsilon < \kappa$ and $\mathrm{cof}(\epsilon) = \omega$), there is a club C and a $\delta < \epsilon$ so that for all $f, g \in [C]^{\epsilon}_*$, if $\sup(f) = \sup(g)$ and $f \upharpoonright \delta = g \upharpoonright \delta$, then $\Phi^{\epsilon}_{\gamma}(f) = \Phi^{\epsilon}_{\gamma}(g)$. Let $\delta^{\epsilon}_{\gamma}$ be the least such $\delta < \epsilon$. For each $\gamma < \chi$, define $\Lambda_\gamma : [\kappa]^1_* \to \kappa$ by $\Lambda_\gamma(\epsilon) = \delta^{\epsilon}_{\gamma}$. Note that for all $\epsilon \in [\kappa]^1_*$, $\Lambda_\gamma(\epsilon) < \epsilon$ and so by Fact 2.10 or Fact 2.11, there is a unique $\delta_\gamma < \kappa$ so that $\Lambda^{-1}_\gamma[\{\delta_\gamma\}] \in \mu^{\kappa}_1$. Since $\kappa \to_* (\kappa)^2_2$ implies κ is regular and $\chi < \kappa$, let $\delta^* = \sup\{\delta_\gamma + 1 : \gamma < \chi\}$ and observe $\delta^* < \kappa$.

Note that for all $\gamma < \chi$, $\Lambda^{-1}_\gamma[\delta^*] \in \mu^{\kappa}_1$ since $\delta_\gamma \in \delta^*$. By Fact 2.11, μ^{κ}_1 is κ-complete and thus $\bigcap_{\gamma<\chi} \Lambda^{-1}_\gamma[\delta^*] \in \mu^{\kappa}_1$. There is a club $E \subseteq \kappa$ with E consisting entirely of indecomposable ordinals so that $[E]^1_* \subseteq \bigcap_{\gamma<\chi} \Lambda^{-1}_\gamma[\delta^*]$. Fix an $\epsilon^* > \delta^*$ with $\epsilon^* \in [E]^1_*$ and observe that ϵ^* is an additively indecomposable ordinal with $\mathrm{cof}(\epsilon^*) = \omega$.

(See Remark 4.5 for some context for the argument of this next paragraph.) For any $\gamma < \chi$ and $\iota \in [\kappa]^{\delta^*}_*$, let $\Phi^{\epsilon^*}_{\gamma,\iota} : [\kappa]^{\epsilon^*}_* \to \kappa$ be defined by $\Phi^{\epsilon^*}_{\gamma,\iota}(\ell) = \Phi^{\epsilon^*}_{\gamma}(\iota\hat{}\ell)$. For each $\gamma < \chi$, let A_γ be the set of $f \in [\kappa]^{\epsilon^*}_*$ so that for all $\ell \sqsubseteq \mathsf{drop}(f, \delta^*)$, $\Phi^{\epsilon^*}_{\gamma}(f) = \Phi^{\epsilon^*}_{\gamma, f\upharpoonright\delta^*}(\ell)$. Fix a $\gamma < \chi$. $\Lambda_\gamma(\epsilon^*) = \delta_\gamma \leq \delta^*$ implies that there is a club $F \subseteq \kappa$ so that for all $f, g \in [F]^{\epsilon^*}_*$, if $\sup(f) = \sup(g)$ and $f \upharpoonright \delta^* = g \upharpoonright \delta^*$, then $\Phi^{\epsilon^*}_{\gamma}(f) = \Phi^{\epsilon^*}_{\gamma}(g)$. In particular, if $f \in [F]^{\epsilon^*}_*$, then for any $\ell \sqsubseteq \mathsf{drop}(f, \delta^*)$, $\Phi^{\epsilon^*}_{\gamma, f\upharpoonright\delta^*}(\ell) = \Phi_\gamma(f)$. This shows that $[F]^{\epsilon^*}_* \subseteq A_\gamma$ and hence $A_\gamma \in \mu^{\kappa}_{\epsilon^*}$. Since $\kappa \to_* (\kappa)^{\epsilon^*+\epsilon^*}_2$ implies $\mu^{\kappa}_{\epsilon^*}$ is κ-complete, $\bigcap_{\gamma<\chi} A_\gamma \in \mu^{\kappa}_{\epsilon^*}$. Thus, there is a club $G \subseteq \kappa$ so that $[G]^{\epsilon^*}_* \subseteq \bigcap_{\gamma<\chi} A_\gamma$. (Such a club G could also be obtain by an application of Fact 2.15.)

Fix a $\gamma < \chi$. Suppose $f, g \in [G]^{\epsilon^*}_*$ with $\sup(f) = \sup(g)$ and $f \upharpoonright \delta^* = g \upharpoonright \delta^*$. Let $\iota = f \upharpoonright \delta^* = g \upharpoonright \delta^*$. Note that $\Phi^{\epsilon^*}_{\gamma,\iota}$ is $\sqsubseteq$-constant on $[G]^{\epsilon^*}_*$. Fact 3.5 implies that $\Phi^{\epsilon^*}_{\gamma,\iota}$ depends only on supremum relative to G. Thus, $\Phi^{\epsilon^*}_{\gamma}(f) = \Phi^{\epsilon^*}_{\gamma,\iota}(\mathsf{drop}(f, \delta^*)) = \Phi^{\epsilon^*}_{\gamma,\iota}(\mathsf{drop}(g, \delta^*)) = \Phi^{\epsilon^*}_{\gamma}(g)$ since $\sup(\mathsf{drop}(f, \delta^*)) = \sup(\mathsf{drop}(g, \delta^*))$. So it has been shown that for all $\gamma < \chi$, for all $f, g \in [G]^{\epsilon^*}_*$, if $\sup(f) = \sup(g)$ and $f \upharpoonright \delta^* = g \upharpoonright \delta^*$, then $\Phi^{\epsilon^*}_{\gamma}(f) = \Phi^{\epsilon^*}_{\gamma}(g)$.

Let $f, g \in [G]^{\epsilon^*}_*$ be such that $\sup(f) = \sup(g)$, $f \upharpoonright \delta^* = g \upharpoonright \delta^*$ and $f \neq g$. By the property of G from above, for all $\gamma < \chi$, $\Phi^{\epsilon^*}_{\gamma}(f) = \Phi^{\epsilon^*}_{\gamma}(g)$. This implies $\Phi(f) = \Phi(g)$. This is impossible since $\Phi : [\kappa]^{<\kappa}_* \to {}^{\chi}\mathrm{ON}$ was assumed to be an injection. □

Remark 4.5. In the proof of Theorem 4.4, an indirect argument was used to obtain the club G and establish its properties by appealing to the κ-completeness of $\mu^{\kappa}_{\epsilon^*}$. This argument could be circumvented if one had the ability to make a χ-length choice of clubs given by each instance of Theorem 3.7 applied to $\Phi^{\epsilon^*}_{\gamma}$ with $\gamma < \chi$.

The short length continuity result is also used in [6] to show $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$ under AD. There, this indirect argument was not necessary since ω-many clubs could be chosen by $\mathsf{AC}^{\mathbb{R}}_{\omega}$ and the Moschovakis coding lemma, which follows from AD. [2] investigated the everywhere wellordered club uniformization principle at κ, which is the assertion that for every relation $R \subseteq \kappa \times \mathsf{club}_\kappa$ which is $\subseteq$-downward closed in the club_κ-coordinate, there is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}_\kappa$ so that for all $\alpha \in \mathrm{dom}(R)$, $R(\alpha, \Phi(\alpha))$. This selection principle would also suffice. [2] showed that this principle holds at κ under AD if κ is the prewellordering ordinal of a pointclass possessing suitable definable boundedness properties. [2] also showed that if κ is a strong partition cardinal, then the everywhere wellordered club uniformization principle at κ is equivalent to $\kappa \to_* (\kappa)^{\kappa}_{<\kappa}$. However, Theorem 4.4 does not presuppose that κ is a strong partition cardinal, AD or any other conditions beyond κ being a weak partition cardinal.

Corollary 4.6. *Assume AD. Suppose κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming $\mathsf{DC}_{\mathbb{R}}$). Then for any $\chi < \kappa$, $|{}^{\chi}\kappa| < |{}^{<\kappa}\kappa|$ and ${}^{<\kappa}\kappa$ does not inject into ${}^{\chi}\mathrm{ON}$.*

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Section 5 will investigate almost everywhere monotonicity. The remainder of this section will establish almost everywhere monotonicity for functions $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ when $\epsilon < \omega_1$ and satisfies suitable partition relations. This will use Theorem 3.9 to reduce to the almost everywhere monotonicity of functions $\Phi : [\kappa]^p_* \to \mathrm{ON}$ when p is finite which will be established next.

Fact 4.7. Suppose κ is a cardinal, $p \in \omega$, $\kappa \to_* (\kappa)^{p+1}_2$ and $\Phi : [\kappa]^p_* \to \mathrm{ON}$ is a function. Then there is a club $C \subseteq \kappa$ so that for $f, g \in [C]^p_*$, if for all $n < p$, $f(n) \leq g(n)$, then $\Phi(f) \leq \Phi(g)$.

Proof. Let $k < p$. For $h \in [\kappa]^{p+1}_*$, let $h^{k,0}, h^{k,1} \in [\kappa]^p_*$ be defined by

$$h^{k,0}(n) = \begin{cases} h(n) & n \leq k \\ h(n+1) & k < n < p \end{cases} \qquad h^{k,1}(n) = \begin{cases} h(n) & n < k \\ h(n+1) & k \leq n < p \end{cases}.$$

Define $P^k : [\kappa]^{p+1}_* \to 2$ by $P^k(h) = 0$ if and only if $\Phi(h^{k,0}) \leq \Phi(h^{k,1})$. By $\kappa \to_* (\kappa)^{p+1}_2$, let $C \subseteq \kappa$ be homogeneous for P^k. Suppose C is homogeneous for P taking value 1. Fix $\ell \in [C]^{\omega+(p-k-1)}$. Define $f_i \in [C]^p_*$ as follows.

$$f_i(n) = \begin{cases} \ell(n) & n < k \\ \ell(k+i) & n = k \\ \ell(\omega + n - k - 1) & k < n < p. \end{cases}$$

Note that for all $i \in \omega$, there is an $h_i \in [C]^{p+1}_*$ so that $h^{k,0} = f_i$ and $h^{k,1} = f_{i+1}$. $P(h_i) = 1$ implies that $\Phi(f_{i+1}) = \Phi(h_i^{k,1}) < \Phi(h_i^{k,0}) = \Phi(f_i)$. Thus, $\langle \Phi(f_i) : i \in \omega \rangle$ is an infinite descending sequence of ordinals. Thus, C must be homogeneous for P^k taking value 0.

For each $k < p$, let $C_k \subseteq \kappa$ be a club which is homogeneous for P^k taking value 0. Let $C = \bigcap_{k<p} C_k$. Suppose $f, g \in [C]^p_*$ is such that for all $n < p$, $f(n) \leq g(n)$. If $f = g$, then it is clear that $\Phi(f) \leq \Phi(g)$. Suppose $f \neq g$. Let $k_0 < \ldots < k_q$ with $q < p$ enumerate $\{k < p : f(k) < g(k)\}$. For $0 \leq i \leq q$, let

$$c_i(n) = \begin{cases} f(n) & n < k_{q-i} \\ g(n) & k_{q-i} \geq n \end{cases}.$$

Observe the following hold.

1. For all $i \leq q$, $c_i \in [C]^p_*$.
2. $c_q = g$.
3. There is an $h_0 \in [C]^{p+1}_*$ so that $h_0^{k_q,0} = f$ and $h_0^{k_q,1} = c_0$.
4. For each $0 < i \leq q$, there is an $h_i \in [C]^{p+1}_*$ so that $h_i^{k_{q-i},0} = c_{i-1}$ and $h_i^{k_{q-i},1} = c_i$.

These properties and the fact that $P^{k_{q-i}}(h_i) = 0$ for each $i \leq q$ imply that $\Phi(f) \leq \Phi(c_0) \leq \Phi(c_1) \leq \ldots \leq \Phi(c_q) = \Phi(g)$. □

The next result will be improved in Section 5.

Theorem 4.8. *Suppose κ is a cardinal, $\epsilon < \omega_1$, $\kappa \to_* (\kappa)^{\epsilon \cdot \epsilon}_2$ holds and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

Proof. By Theorem 3.9, there is a club C_0 and finitely many ordinals $\beta_0 < \beta_1 < \ldots < \beta_{p-1} \leq \epsilon$ so that for all $f, g \in [C_0]^\epsilon_*$, if for all $i < p$, $\sup(f \restriction \beta_i) = \sup(g \restriction \beta_i)$, then $\Phi(f) = \Phi(g)$. Let C_1 be the club of limit points of C_0. Define $\Psi : [C_1]^p_* \to \mathrm{ON}$ by $\Psi(\ell) = \Phi(f)$ for any $f \in [C_0]^\epsilon_*$ so that for all $i < p$, $\sup(f \restriction \beta_i) = \ell(i)$. Note $\Psi(\ell)$ is well defined and independent of the choice of f. By Fact 4.7, there is a club $C_2 \subseteq C_1$ so that for all $\ell, \iota \in [C_2]^p_*$, if for all $i < p$, $\ell(i) \leq \iota(i)$, then $\Psi(\ell) \leq \Psi(\iota)$.

Now, suppose $f, g \in [C_2]^\epsilon_*$ so that for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$. Let $\ell_f(i) = \sup(f \restriction \beta_i)$ and $\ell_g(i) = \sup(g \restriction \beta_i)$. Note that ℓ_f, ℓ_g are discontinuous since p is finite and $\beta_0 < \ldots < \beta_{p-1}$. Since

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$\epsilon < \omega_1$ and $f, g \in [C_2]^\epsilon_*$, it follows that $\ell_f(i)$ and $\ell_g(i)$ have cofinality ω, and thus $\ell_f, \ell_g \in [C_2]^p_*$. By definition of Ψ, $\Phi(f) = \Psi(\ell_f)$ and $\Phi(g) = \Psi(\ell_g)$. Note that for all $i < p$, $\ell_f(i) \leq \ell_g(i)$. By the choice of club C_2, $\Psi(\ell_f) \leq \Psi(\ell_g)$. Thus, $\Phi(f) = \Psi(\ell_f) \leq \Psi(\ell_g) = \Phi(g)$. □

Corollary 4.9. *Assume* AD. *If κ is ω_1, ω_2, $\boldsymbol{\delta}^1_n$ for any $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ for some $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming* $\mathsf{DC}_{\mathbb{R}}$*), then for any $\epsilon < \omega_1$ and function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, there exists a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

Proof. This follows from Theorem 4.8. □

5. Almost everywhere monotonicity

The next result shows that if a partition relation fails at κ, then there is a corresponding failure of almost everywhere monotonicity at κ. Thus, partition relations are necessary for the almost everywhere monotonicity property.

Fact 5.1. Suppose κ is a cardinal and $\epsilon \leq \kappa$ is such that $\kappa \to_* (\kappa)^\epsilon_2$ fails. Then there is a function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$ so that for all club $C \subseteq \kappa$, there exist $f, g \in [C]^\epsilon_*$ so that for all $\alpha \leq \epsilon$, $f(\alpha) \leq g(\alpha)$ and $\Phi(g) < \Phi(f)$.

Proof. Let $P : [\kappa]^\epsilon_* \to 2$ be such that for all club $C \subseteq \kappa$, there exists functions $h_0, h_1 \in [C]^\epsilon_*$ so that $P(h_0) = 0$ and $P(h_1) = 1$. P fails the monotonicity property.

Suppose $C \subseteq \kappa$ is a club. As noted above, there is some $f \in [C]^\epsilon_*$ with $P(f) = 1$. Let $D \subseteq C$ be a club with the property that for all $h \in [D]^\kappa_*$, $f(\alpha) \leq h(\alpha)$ for all $\alpha < \epsilon$. As noted above, there is some $g \in [D]^\epsilon_*$ so that $\Phi(g) = 0$. Then $f, g \in [C]^\epsilon_*$, for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$ and $\Phi(g) = 0 < 1 = \Phi(f)$. □

The following lemma considers pairs (f, g) possessing property (2) and (3) stated below in order to simplify the construction of the relevant functions of the correct type. Theorem 5.3 will reduce the general case to this lemma.

Lemma 5.2. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. For any function $\Phi : [\kappa]^\kappa_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\kappa_*$, if f and g have the property that for all $\alpha < \kappa$,*

1. *$f(\alpha) \leq g(\alpha)$,*
2. *there is no limit ordinal $\beta < \kappa$ so that $\sup(f \restriction \beta) = g(\alpha)$,*
3. *and there is no limit ordinal $\beta < \kappa$ so that $\sup(g \restriction \beta) = f(\alpha)$,*

then $\Phi(f) \leq \Phi(g)$.

Proof. Let $\mathcal{I} : \kappa \to \kappa$ be an increasing and discontinuous function whose image consists of indecomposable ordinals. For any $h \in [\kappa]^\kappa_*$, let $\mathsf{main}(h) \in [\kappa]^\kappa_*$ be defined by $\mathsf{main}(h)(\alpha) = h(\mathcal{I}(\alpha))$. (Observe that $\mathsf{main}(h)$ is an increasing function of the correct type since h is an increasing function of the correct type.) Define $P : [\kappa]^\kappa_* \to 2$ by $P(h) = 0$ if and only if for all $p \in [h[\kappa]]^\kappa_*$, $\Phi(\mathsf{main}(h)) \leq \Phi(\mathsf{main}(p))$. By $\kappa \to_* (\kappa)^\kappa_2$, let $C_0 \subseteq \kappa$ be a club homogeneous for P. Let $Z = \{\Phi(\mathsf{main}(h)) : h \in [C_0]^\kappa_*\}$ which has a minimal element since it is a nonempty set of ordinals. Let $h^* \in [C_0]^\kappa_*$ be such that $\Phi(\mathsf{main}(h^*)) = \min(Z)$. If $p \in [h^*[\kappa]]^\kappa_*$, then $p \in [C_0]^\kappa_*$ and thus $\Phi(\mathsf{main}(p)) \in Z$ and $\Phi(\mathsf{main}(h^*)) = \min(Z) \leq \Phi(\mathsf{main}(p))$. This shows $P(h^*) = 0$. Since $h^* \in [C_0]^\kappa_*$, C_0 must be homogeneous for P taking value 0. By choosing a subclub of C_0 if necessary, one may assume that C_0 consists entirely of indecomposable ordinals and for all $f \in [C_0]^\kappa_*$, for all $\alpha < \kappa$, $\mathcal{I}(\alpha) < f(\alpha)$ (which is possible since $\mathcal{I}$ and f are discontinuous). Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Since C_1 is a subclub of C_0, C_1 also consists entirely of indecomposable ordinals.

Now, fix $f, g \in [C_1]^\kappa_*$ with properties (1), (2) and (3). One will construct simultaneously by recursion two functions $h \in [C_0]^\kappa_*$ and $p \in [h[\kappa]]^\kappa_*$ so that $\mathsf{main}(h) = f$ and $\mathsf{main}(p) = g$. The construction will recursively define at each stage longer initial segments of the final two objects, h and p.

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Suppose $\alpha < \kappa$ and the following holds:

(a) For each $\beta < \alpha$, $h \upharpoonright \mathcal{I}(\beta)+1$ has been defined, is a function of the correct type and $h(\mathcal{I}(\beta)) = f(\beta)$.
(b) For each $\beta < \alpha$, $\sigma_\beta \leq \beta + 1$ has been defined. If $\beta_0 \leq \beta_1 < \alpha$, then $\sigma_{\beta_0} \leq \sigma_{\beta_1}$.
(c) For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $p \upharpoonright \mathcal{I}(\eta) + 1$ has been defined, is a function of the correct type and $p(\mathcal{I}(\eta)) = g(\eta)$.
(d) For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $g(\eta) \leq f(\beta) < g(\sigma_\beta)$.

Let $\iota_\alpha = \sup\{\sigma_\beta : \beta < \alpha\}$. Let $\delta_0 = \sup\{\mathcal{I}(\beta) + 1 : \beta < \alpha\}$ and $\tau_0 = \sup\{\mathcal{I}(\beta) + 1 : \beta < \iota_\alpha\}$. Since $\iota_\alpha \leq \alpha$, one has $\tau_0 \leq \delta_0$. Note that, since $\mathcal{I}$ is discontinuous and takes value among indecomposable ordinals, $\tau_0 \leq \delta_0 \leq \sup(\mathcal{I} \upharpoonright \alpha) + 1 < \mathcal{I}(\alpha)$. Properties (a) and (c) imply that $h \upharpoonright \delta_0$ and $p \upharpoonright \tau_0$ have been defined. Note $\sup(h \upharpoonright \delta_0) = \sup(f \upharpoonright \alpha) < f(\alpha)$ since f is discontinuous. Also, $\sup(p \upharpoonright \tau_0) = \sup(g \upharpoonright \iota_\alpha)$.

If α is a successor ordinal with $\alpha = \alpha^* + 1$, then $\iota_\alpha = \sigma_{\alpha*}$. By property (d), $\sup(g \upharpoonright \iota_\alpha) = \sup(g \upharpoonright \sigma_{\alpha^*}) \leq f(\alpha^*) = \sup(f \upharpoonright \alpha) < g(\sigma_{\alpha^*}) = g(\iota_\alpha)$. Suppose α is a limit ordinal and $\langle \sigma_\beta : \beta < \alpha \rangle$ is not eventually constant. Property (d) implies that $\sup(g \upharpoonright \iota_\alpha) \leq \sup(f \upharpoonright \alpha) \leq \sup\{g(\sigma_\beta) : \beta < \alpha\} = \sup(g \upharpoonright \iota_\alpha) < g(\iota_\alpha)$ by the discontinuity of g. Suppose α is a limit ordinal and $\langle \sigma_\beta : \beta < \alpha \rangle$ is eventually constant. Then $\sup\{g(\sigma_\beta) : \beta < \alpha\} = g(\iota_\alpha)$. Then by property (d) and property (2) for the strict inequality, $\sup(g \upharpoonright \iota_\alpha) \leq \sup(f \upharpoonright \alpha) < g(\iota_\alpha)$. The following property $(*)$ has been established in all cases: $\sup(g \upharpoonright \iota_\alpha) \leq \sup(f \upharpoonright \alpha) < g(\iota_\alpha)$.

Let $A = \{\beta < \alpha : \sup(f \upharpoonright \alpha) < g(\beta) < f(\alpha)\}$.

(Case A) If $A = \emptyset$.

Then let $\tau = \tau_0$ and $\delta = \delta_0$.

(Case B) $A \neq \emptyset$.

Note $\iota_\alpha < \alpha$ since if $\iota_\alpha = \alpha$, then by $(*)$ and the discontinuity of f, $\sup(g \upharpoonright \alpha) = \sup(g \upharpoonright \iota_\alpha) \leq \sup(f \upharpoonright \alpha) < f(\alpha)$. However, $\sup(g \upharpoonright \alpha) < f(\alpha)$ implies $A = \emptyset$ which is a contradiction. Note that by $(*)$, $\iota_\alpha = \min(A)$. Let $\xi = \mathrm{ot}(A) \leq \alpha$, and observe that $A = \{\iota_\alpha + \eta : \eta < \xi\}$. Recall δ_0 and τ_0 have already been defined above. Note that $\sup(h \upharpoonright \delta_0) = \sup(f \upharpoonright \alpha) < g(\iota_\alpha)$. For $0 < \nu < \xi$, suppose that for all $\eta < \nu$, the following holds:

- $\epsilon_\eta = \delta_0 + \mathcal{I}(\iota_\alpha + \eta)$ and $\mu_\eta = \tau_0 + \mathcal{I}(\iota_\alpha + \eta)$ have been defined.
- $h \upharpoonright \epsilon_\eta + 1$ and $p \upharpoonright \mu_\eta + 1$ have been defined.
- $h(\epsilon_\eta) = g(\iota_\alpha + \eta) = p(\mu_\eta)$.

Let $\delta_\nu = \sup\{\epsilon_\eta + 1 : \eta < \nu\}$ and $\tau_\nu = \sup\{\mu_\eta + 1 : \eta < \nu\}$. Note that $\delta_\nu = \sup\{\delta_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 : \eta < \nu\} < \delta_0 + \mathcal{I}(\iota_\alpha + \nu)$ and $\tau_\nu = \sup\{\tau_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 : \eta < \nu\} < \tau_0 + \mathcal{I}(\iota_\alpha + \nu)$ since $\mathcal{I}$ is discontinuous. The above assumptions imply that $h \upharpoonright \delta_\nu$ and $p \upharpoonright \tau_\nu$ are defined, and $\sup(h \upharpoonright \delta_\nu) = \sup(g \upharpoonright (\iota_\alpha + \nu)) = \sup(p \upharpoonright \tau_\nu)$.

Fix ν with $0 \leq \nu < \xi$. Let $\epsilon_\nu = \delta_\nu + \mathcal{I}(\iota_\alpha + \nu)$ and $\mu_\nu = \tau_\nu + \mathcal{I}(\iota_\alpha + \nu)$. Since $\delta_\nu < \delta_0 + \mathcal{I}(\iota_\alpha + \nu)$, $\tau_\nu < \tau_0 + \mathcal{I}(\iota_\alpha + \nu)$ and $\mathcal{I}(\iota_\alpha + \nu)$ is indecomposable, $\epsilon_\nu = \delta_\nu + \mathcal{I}(\iota_\alpha + \nu) = \delta_0 + \mathcal{I}(\iota_\alpha + \nu)$ and $\mu_\nu = \tau_\nu + \mathcal{I}(\iota_\alpha + \nu) = \tau_0 + \mathcal{I}(\iota_\alpha + \nu)$. For $\beta < \mathcal{I}(\iota_\alpha + \nu)$, let $h(\delta_\nu + \beta) = p(\tau_\nu + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \upharpoonright \delta_\nu))$. This defines $h \upharpoonright \epsilon_\nu$ and $p \upharpoonright \mu_\nu$ with $\sup(h \upharpoonright \epsilon_\nu) = \sup(p \upharpoonright \mu_\nu) \leq \mathsf{next}_{C_0}^{\mathcal{I}(\iota_\alpha+\nu)}(\sup(h \upharpoonright \delta_\nu)) < g(\iota_\alpha + \nu)$ by Fact 2.12 since $\sup(h \upharpoonright \delta_0) < g(\iota_\alpha)$ (in the case $\nu = 0$), $\sup(h \upharpoonright \delta_\nu) = \sup(g \upharpoonright \tau_\nu) = \sup(g \upharpoonright (\iota_\alpha + \nu)) < g(\iota_\alpha + \nu)$ (in the case $0 < \nu < \xi$) and $\mathcal{I}(\iota_\alpha + \nu) < g(\iota_\alpha + \nu)$. Let $h(\epsilon_\nu) = p(\mu_\nu) = g(\iota_\alpha + \nu)$.

Let $\tau = \sup\{\mu_\nu + 1 : \nu < \xi\}$ and $\delta = \sup\{\epsilon_\nu + 1 : \nu < \xi\}$. Note $p \upharpoonright \tau$ and $h \upharpoonright \delta$ have been defined so that $\sup(p \upharpoonright \tau) = \sup(h \upharpoonright \delta) = \sup\{g(\gamma) : \gamma \in A\} < f(\alpha)$ by property (3). Since $\mathcal{I}$ is discontinuous, $\iota_\alpha + \nu < \alpha$ for all $\nu < \xi$, $\delta_0 < \mathcal{I}(\alpha)$ and $\mathcal{I}(\alpha)$ is indecomposable, one has $\delta = \sup\{\epsilon_\nu + 1 : \nu < \xi\} = \sup\{\delta_0 + \mathcal{I}(\iota_\alpha + \nu) + 1 : \nu < \xi\} \leq \delta_0 + \sup(\mathcal{I} \upharpoonright \alpha) + 1 < \delta_0 + \mathcal{I}(\alpha) = \mathcal{I}(\alpha)$.

In either Case A or Case B, ordinals τ and δ have been defined with $\tau \leq \delta < \mathcal{I}(\alpha)$ and $\sup(h \upharpoonright \delta) < f(\alpha)$. Let $\ell = \min(\kappa \setminus A)$.

(Case I) $g(\ell) > f(\alpha)$.

Let $\sigma_\alpha = \iota_\alpha$ if Case A held, and let $\sigma_\alpha = \ell$ if Case B held. For $\beta < \mathcal{I}(\alpha)$, let $h(\delta + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \upharpoonright \delta))$. Note that $\sup(h \upharpoonright \mathcal{I}(\alpha)) \leq \mathsf{next}_{C_0}^{\mathcal{I}(\alpha)}(\sup(h \upharpoonright \delta)) < f(\alpha)$ by Fact 2.12 since $f(\alpha) \in C_1$, $\sup(h \upharpoonright \delta) < f(\alpha)$ and $\mathcal{I}(\alpha) < f(\alpha)$. Let $h(\mathcal{I}(\alpha)) = f(\alpha)$.

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(Case II) $g(\ell) = f(\alpha)$.

For $\beta < \mathcal{I}(\ell)$, let $h(\delta + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \restriction \delta))$.

(Case II.1) $\ell = \alpha$.

Then $h \restriction \mathcal{I}(\alpha)$ and $p \restriction \mathcal{I}(\alpha)$ have been defined with $\sup(h \restriction \mathcal{I}(\alpha)) = \sup(p \restriction \mathcal{I}(\alpha)) \leq \mathsf{next}_{C_0}^{\mathcal{I}(\alpha)}(\sup(h \restriction \delta)) < f(\alpha)$ by Fact 2.12 since $f(\alpha) \in C_1$, $\sup(h \restriction \delta) < f(\alpha)$ and $\mathcal{I}(\alpha) < f(\alpha)$. Let $h(\mathcal{I}(\alpha)) = p(\mathcal{I}(\alpha)) = f(\alpha) = g(\alpha)$. Let $\sigma_\alpha = \alpha + 1$.

(Case II.2) $\ell < \alpha$.

Then $h \restriction (\delta + \mathcal{I}(\ell))$ and $p \restriction (\delta + \mathcal{I}(\ell))$ have been defined with $\sup(p \restriction (\delta + \mathcal{I}(\ell))) = \sup(h \restriction (\delta + \mathcal{I}(\ell))) < \mathsf{next}_{C_0}^{\mathcal{I}(\ell)}(\sup(h \restriction \delta)) < f(\alpha)$ by Fact 2.12 since $f(\alpha) \in C_1$, $\sup(h \restriction \mathcal{I}(\ell)) < f(\alpha)$ and $\mathcal{I}(\ell) < \mathcal{I}(\alpha) < f(\alpha)$. For $\beta < \mathcal{I}(\alpha)$, let $h(\delta + \mathcal{I}(\ell) + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \restriction (\delta + \mathcal{I}(\ell))))$.

This defines $h \restriction \mathcal{I}(\alpha)$ with $\sup(h \restriction \mathcal{I}(\alpha)) \leq \mathsf{next}_{C_0}^{\mathcal{I}(\alpha)}(\sup(h \restriction (\delta + \mathcal{I}(\ell)))) < f(\alpha)$ by Fact 2.12 since $f(\alpha) \in C_1$, $\sup(f \restriction (\delta + \mathcal{I}(\ell))) < f(\alpha)$ and $\mathcal{I}(\alpha) < f(\alpha)$. Let $h(\mathcal{I}(\alpha)) = f(\alpha)$ and $p(\mathcal{I}(\ell)) = g(\ell) = f(\alpha)$. Let $\sigma_\alpha = \ell + 1$.

This completes the construction of the desired objects satisfying properties (a), (b), (c) and (d). Let $h = \bigcup\{h \restriction \mathcal{I}(\alpha) : \alpha < \kappa\}$ and $p \in \bigcup\{p \restriction \sigma_\alpha : \alpha < \kappa\}$. By construction, h and p are increasing functions of the correct type. (To verify these functions have uniform cofinality ω, note that an ordinal of the form $\mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\gamma)$ with $\beta, \gamma < \kappa$ is a uniform limit of an ω-sequence from C_0). Then $h \in [C_0]^\kappa_*$, $p \in [h[\kappa]]^\kappa_*$, $\mathsf{main}(h) = f$ and $\mathsf{main}(p) = g$. Since $P(h) = 0$, one has $\Phi(f) = \Phi(\mathsf{main}(h)) \leq \Phi(\mathsf{main}(p)) = \Phi(g)$. □

Theorem 5.3. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$. For any function $\Phi : [\kappa]^\kappa_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\kappa_*$, if for all $\alpha < \kappa$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

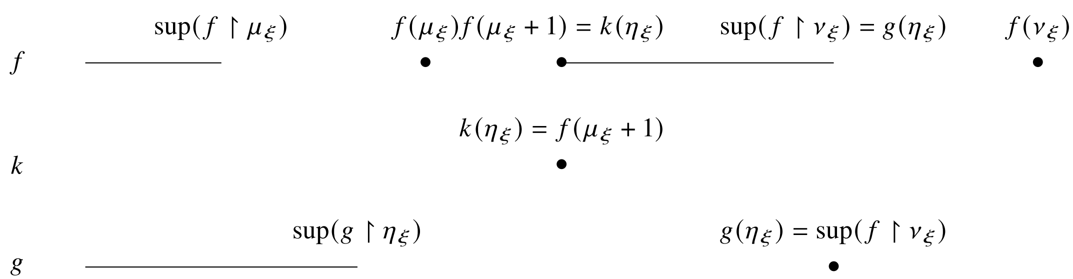
Proof. Let $C_0 \subseteq \kappa$ be a club satisfying the property from Lemma 5.2. One may assume that C_0 consists entirely of indecomposable ordinals. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Suppose $f, g \in [C_1]^\kappa_*$ and for all $\alpha < \kappa$, $f(\alpha) \leq g(\alpha)$. Let conditions (1), (2) and (3) refer to the conditions from Lemma 5.2.

First, one will construct a $k \in [C_1]^\kappa_*$ so that the pair (f, k) satisfies condition (1) and (2) and the pair (k, g) satisfies condition (1), (2) and (3). Let $\zeta \leq \kappa$ and $\langle \eta_\xi : \xi < \zeta\rangle$ and $\langle \nu_\xi : \xi < \zeta\rangle$ be two increasing sequences so that the following holds.

(a) For all $\xi < \zeta$, ν_ξ is a limit ordinal.
(b) For all $\xi < \zeta$, $\sup(f \restriction \nu_\xi) = g(\eta_\xi)$.
(c) For all $\nu < \kappa$ and $\eta < \kappa$, if ν is a limit ordinal and $\sup(f \restriction \nu) = g(\eta)$, then there is a $\xi < \zeta$ so that $\nu = \nu_\xi$ and $\eta = \eta_\xi$.

These objects refer to the areas in which the pair (f, g) fails to satisfy condition (2). (Note $\zeta = 0$ if there are no failures.) Observe that for all $\xi < \zeta$, $g(\eta_\xi) < f(\nu_\xi)$ implies that $\eta_\xi < \nu_\xi$ since for all $\alpha < \kappa$, $f(\alpha) \leq g(\alpha)$. For each $\xi < \zeta$, let μ_ξ be the least γ so that $\sup(g \restriction \eta_\xi) \leq f(\gamma)$. Note that $\eta_\xi \leq \mu_\xi < \mu_\xi + 1 < \nu_\xi$ since for all $\alpha < \kappa$, $f(\alpha) \leq g(\alpha)$, $\eta_\xi < \nu_\xi$ and ν_ξ is a limit ordinal.

Define $k : \kappa \to C_1$ as follows: Let $\alpha < \kappa$. If $\alpha \neq \eta_\xi$ for any $\xi < \zeta$, then let $k(\alpha) = g(\alpha)$. If there is a $\xi < \zeta$ so that $\alpha = \eta_\xi$, then let $k(\alpha) = f(\mu_\xi + 1)$. The following illustrates the construction.



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Since for all α, $k(\alpha) \in f[\kappa] \cup g[\kappa]$, one can construct a witness $K : \kappa \times \omega \to \kappa$ to k having uniform cofinality ω by using witnesses $F : \kappa \times \omega \to \kappa$ and $G : \kappa \times \omega \to \kappa$ to f and g, respectively, having uniform cofinality ω.

If for all $\xi < \zeta$, $\alpha \neq \eta_\xi$, then $f(\alpha) \leq g(\alpha) = k(\alpha)$. If $\alpha = \eta_\xi$ for some $\xi < \zeta$, then $f(\alpha) = f(\eta_\xi) \leq f(\mu_\xi) < f(\mu_\xi + 1) = k(\alpha)$. So for all $\alpha < \kappa$, $f(\alpha) \leq k(\alpha)$. The pair (f, k) satisfies condition (1).

If for all $\alpha < \zeta$, $\alpha \neq \eta_\xi$, then $k(\alpha) = g(\alpha)$. If there is a $\xi < \zeta$ so that $\alpha = \eta_\xi$, then $k(\alpha) = f(\mu_\xi + 1) < \sup(f \restriction \nu_\xi) = g(\eta_\xi) = g(\alpha)$. Thus, for all $\alpha < \kappa$, $k(\alpha) \leq g(\alpha)$ and hence condition (1) holds for the pair (k, g).

Let property $(*)$ for (k, g) assert that for all $\alpha < \kappa$, $\sup(g \restriction \alpha) < k(\alpha)$. If for all $\xi < \zeta$, $\alpha \neq \eta_\xi$, then $\sup(g \restriction \alpha) < g(\alpha) = k(\alpha)$ since g is discontinuous. If there is a $\xi < \zeta$ so that $\alpha = \eta_\xi$, then $\sup(g \restriction \alpha) = \sup(g \restriction \eta_\xi) \leq f(\mu_\xi) < f(\mu_\xi + 1) = k(\alpha)$. It has been shown that property $(*)$ for (k, g) holds.

By property $(*)$ for (k, g) and condition (1) for the pair (k, g), for any $\alpha < \beta < \kappa$, $k(\alpha) \leq g(\alpha) \leq \sup(g \restriction \beta) < k(\beta)$. This shows that k is increasing. Also, by property $(*)$ for (k, g) and condition (1) for the pair (k, g), for all $\alpha < \kappa$, $\sup(k \restriction \alpha) \leq \sup(g \restriction \alpha) < k(\alpha)$. This shows that k is discontinuous. It has been shown that k is an increasing function of the correct type into C_1, that is, $k \in [C_1]^\kappa_*$.

If for all $\xi < \zeta$, $\alpha \neq \eta_\xi$, then there is no limit ordinal ν so that $\sup(f \restriction \nu) = g(\alpha) = k(\alpha)$. If there is a $\xi < \zeta$ so that $\alpha = \eta_\xi$, then $k(\alpha) = f(\mu_\xi + 1)$ and there is no limit ordinal ν so that $\sup(f \restriction \nu) = k(\alpha) = f(\mu_\xi + 1)$ since f is a strictly increasing function. The pair (f, k) satisfies condition (2).

Let property $(**)$ for (k, g) assert that for all limit ordinals ν, $\sup(k \restriction \nu) = \sup(g \restriction \nu)$. Fix a limit ordinal ν. By condition (1) for the pair (k, g), $\sup(k \restriction \nu) \leq \sup(g \restriction \nu)$. By property $(*)$ for (k, g), for each $\eta < \nu$, $\sup(g \restriction \eta) < k(\eta)$. Thus, since ν is a limit ordinal, $\sup(g \restriction \nu) \leq \sup(k \restriction \nu)$. Property $(**)$ for (k, g) has been established.

Suppose condition (2) for (k, g) fails. Then there is an $\alpha < \kappa$ and a limit ordinal ν so that $\sup(k \restriction \nu) = g(\alpha)$. Then by property $(**)$ for (k, g), $\sup(g \restriction \nu) = \sup(k \restriction \nu) = g(\alpha)$ which is impossible since g is an increasing and discontinuous function. Condition (2) for the pair (k, g) has been shown.

Suppose condition (3) for the pair (k, g) fails. Then there is ordinal α and a limit ordinal ν so that $\sup(g \restriction \nu) = k(\alpha)$. Then by property $(**)$ for (k, g), $\sup(k \restriction \nu) = \sup(g \restriction \nu) = k(\alpha)$. This is impossible since k is an increasing and discontinuous function. Condition (3) for the pair (k, g) has been shown.

Since C_0 has the property stated in Lemma 5.2, (k, g) satisfies condition (1), (2) and (3) and $k, g \in [C_1]^\kappa_*$, one has that $\Phi(k) \leq \Phi(g)$.

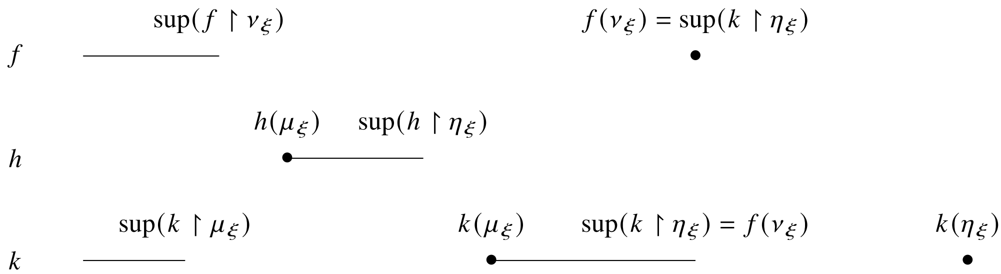
Next, one will construct an $h \in [C_0]^\kappa_*$ so that the pair (f, h) satisfies condition (1), (2) and (3) and the pair (h, k) satisfies condition (1), (2) and (3). Let $\zeta \leq \kappa$ and $\langle \eta_\xi : \xi < \zeta \rangle$ and $\langle \nu_\xi : \xi < \zeta \rangle$ be two increasing sequences so that the following holds.

(a) For all $\xi < \zeta$, η_ξ is a limit ordinal.
(b) For all $\xi < \zeta$, $\sup(k \restriction \eta_\xi) = f(\nu_\xi)$.
(c) For all $\eta < \kappa$ and $\nu < \kappa$, if η is a limit ordinal and $\sup(k \restriction \eta) = f(\nu)$, then there is a $\xi < \zeta$ so that $\eta = \eta_\xi$ and $\nu = \nu_\xi$.

These objects refer to the areas in which the pair (f, k) fails to satisfy condition (3). Note that $\eta_\xi \leq \nu_\xi$ because if $\nu_\xi < \eta_\xi$, then there is a γ with $\nu_\xi < \gamma < \eta_\xi$ since η_ξ is a limit and thus $f(\nu_\xi) \leq k(\nu_\xi) < k(\gamma) < \sup(k \restriction \eta_\xi) = f(\nu_\xi)$ which is a contradiction. Let μ_ξ be the least $\gamma < \eta_\xi$ so that $\sup(f \restriction \nu_\xi) < k(\gamma)$.

Define $h : \kappa \to C_0$ as follows: If $\alpha < \kappa$ and there is no $\xi < \zeta$ so that $\mu_\xi \leq \alpha < \eta_\xi$, then let $h(\alpha) = k(\alpha)$. If $\alpha < \kappa$ and there is a $\xi < \zeta$ so that $\mu_\xi \leq \alpha < \eta_\xi$, then let $h(\alpha) = \mathsf{next}_{C_0}^{\omega \cdot ((\alpha - \mu_\xi)+1)}(\sup(f \restriction \nu_\xi))$.

The following illustrates the construction.



Since k has the correct type, there is a function $K : \kappa \times \omega \to \kappa$ witnessing k has uniform cofinality ω. If $\alpha < \kappa$ and there is no $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$, then $h(\alpha) = k(\alpha)$ and K can be used to produce an ω-sequence whose limit is $h(\alpha)$. If $\alpha < \kappa$ and there is a $\xi < \zeta$ with $\mu_\xi \leq \alpha < \nu_\xi$, then $h(\alpha) = \mathsf{next}_{C_0}^{\omega\cdot((\alpha-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)$ is a uniform limit of an ω-sequence from C_0. From these observations, a witness to h having uniform cofinality ω can be constructed.

Note that for each $\xi < \zeta$, $\eta_\xi \leq \nu_\xi \leq \sup(f \restriction \nu_\xi) < k(\mu_\xi)$. Since $k(\mu_\xi)$ is indecomposable, $\omega \cdot ((\eta_\xi - \mu_\xi) + 1) < k(\mu_\xi)$. By Fact 2.12, for all α such that $\mu_\xi \leq \alpha < \eta_\xi$, $h(\alpha) = \mathsf{next}_{C_0}^{\omega\cdot((\alpha-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) < \mathsf{next}_{C_0}^{\omega\cdot((\eta_\xi-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) < k(\mu_\xi)$. In particular, the following property $(***)$ holds: $\sup(h \restriction \eta_\xi) \leq \mathsf{next}_{C_0}^{\omega\cdot((\eta_\xi-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) < k(\mu_\xi)$.

If α is such that there is no $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$, then $h(\alpha) = k(\alpha)$. If $\alpha < \kappa$ and there is a $\xi < \zeta$ so that $\mu_\xi \leq \alpha < \eta_\xi$, then $h(\alpha) < k(\mu_\xi) \leq k(\alpha)$ by $(***)$. It has been shown that for all $\alpha < \kappa$, $h(\alpha) \leq k(\alpha)$ and thus condition (1) holds for the pair (h, k).

Suppose $\alpha < \beta < \kappa$. If there are no $\xi_1 < \zeta$ and $\xi_2 < \zeta$ with $\mu_{\xi_1} \leq \alpha < \eta_{\xi_1}$ and $\mu_{\xi_2} \leq \beta < \eta_{\xi_2}$, then $h(\alpha) = k(\alpha) < k(\beta) = h(\beta)$. Suppose there is a $\xi_1 < \zeta$ with $\mu_{\xi_1} \leq \alpha < \eta_{\xi_1}$ and no $\xi_2 < \zeta$ with $\mu_{\xi_2} \leq \beta < \eta_{\xi_2}$. Then $\mu_{\xi_1} < \eta_{\xi_1} \leq \beta$. By $(***)$, $h(\alpha) < k(\mu_{\xi_1}) < k(\eta_{\xi_1}) \leq k(\beta) = h(\beta)$. Suppose there is no $\xi_1 < \zeta$ with $\mu_{\xi_1} \leq \alpha < \eta_{\xi_1}$ and there is a $\xi_2 < \zeta$ with $\mu_{\xi_2} \leq \beta < \eta_{\xi_2}$. Note that $\alpha < \mu_{\xi_2}$. Then by the definition of μ_{ξ_2}, $h(\alpha) = k(\alpha) < \sup(f \restriction \nu_{\xi_2}) < h(\beta)$. Now, suppose there exist $\xi_1 < \zeta$ and $\xi_2 < \zeta$ so that $\mu_{\xi_1} \leq \alpha < \eta_{\xi_1}$ and $\mu_{\xi_2} \leq \beta < \eta_{\xi_2}$. If $\xi_1 = \xi_2$, then let $\xi = \xi_1 = \xi_1$ and observe $h(\alpha) = \mathsf{next}_{C_0}^{\omega\cdot((\alpha-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) < \mathsf{next}_{C_0}^{\omega\cdot((\beta-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) = h(\beta)$. Suppose $\xi_1 \neq \xi_2$ and thus $\xi_1 < \xi_2$. By $(***)$ and the definitions of μ_ξ, η_ξ and ν_ξ, $h(\alpha) < k(\mu_{\xi_1}) < \sup(k \restriction \eta_{\xi_1}) = f(\nu_{\xi_1}) \leq \sup(f \restriction \nu_{\xi_2}) < h(\beta)$. Thus, in all cases, it has been shown that if $\alpha < \beta < \kappa$, then $h(\alpha) < h(\beta)$ and thus h is an increasing function.

Suppose $\alpha < \kappa$ is such that there is no $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$. Then by property (1) for the pair (h, k) and the discontinuity of k, $\sup(h \restriction \alpha) \leq \sup(k \restriction \alpha) < k(\alpha) = h(\alpha)$. Suppose there is a $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$. First, suppose $\alpha = \mu_\xi$. Then by condition (1) for the pair (h, k) and the definition of μ_ξ, $\sup(h \restriction \alpha) = \sup(h \restriction \mu_\xi) \leq \sup(k \restriction \mu_\xi) \leq \sup(f \restriction \nu_\xi) < h(\mu_\xi) = h(\alpha)$. Suppose $\mu_\xi < \alpha < \eta_\xi$. Then $\sup(h \restriction \alpha) \leq \mathsf{next}_{C_0}^{\omega\cdot(\alpha-\mu_\xi)}(\sup(f \restriction \nu_\xi)) < \mathsf{next}_{C_0}^{\omega\cdot((\alpha-\mu_\xi)+1)}(\sup(f \restriction \nu_\xi)) = h(\alpha)$. Thus, in all cases, $\sup(h \restriction \alpha) < h(\alpha)$. This shows h is discontinuous everywhere. It has been established that h is an increasing function of the correct type through C_0 (that is, $h \in [C_0]^\kappa_*$).

If $\alpha < \kappa$ and there is no $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$, then $\sup(h \restriction \alpha) < h(\alpha) = k(\alpha)$ since h is discontinuous. Suppose $\alpha < \kappa$ and there is a $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$. Suppose there is no $\bar{\xi} < \zeta$ so that $\mu_{\bar{\xi}} \leq \eta_\xi < \eta_{\bar{\xi}}$, then $\sup(h \restriction \eta_\xi) < k(\mu_\xi) \leq k(\alpha) < \sup(k \restriction \eta_\xi) = f(\nu_\xi) < k(\eta_\xi) = h(\eta_\xi)$. In particular, $\sup(h \restriction \eta_\xi) < k(\alpha) < h(\eta_\xi)$. Suppose there is a $\bar{\xi} < \zeta$ so that $\mu_{\bar{\xi}} \leq \eta_\xi < \eta_{\bar{\xi}}$. Then $\sup(h \restriction \eta_\xi) < k(\mu_\xi) \leq k(\alpha) < \sup(k \restriction \eta_\xi) = f(\nu_\xi) \leq \sup(f \restriction \nu_{\bar{\xi}}) < h(\mu_{\bar{\xi}}) \leq h(\eta_\xi)$. (This implies that $\mu_{\bar{\xi}} = \eta_\xi$.) One has $\sup(h \restriction \eta_\xi) < k(\alpha) < h(\eta_\xi)$. This shows that for all $\alpha < \kappa$, there is no limit ordinal γ so that $\sup(h \restriction \gamma) = k(\alpha)$. Therefore, condition (2) holds for the pair (h, k).

If $\alpha < \kappa$ and there is no $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$, then $\sup(k \restriction \alpha) < k(\alpha) = h(\alpha) < k(\alpha + 1)$. If $\alpha < \kappa$ and there is a $\xi < \zeta$ with $\mu_\xi \leq \alpha < \eta_\xi$, then $\sup(k \restriction \mu_\xi) \leq \sup(f \restriction \nu_\xi) < h(\mu_\xi) \leq h(\alpha) <$

$\sup(h \upharpoonright \eta_\xi) < k(\mu_\xi)$ using $(***)$. It has been shown that for all $\alpha < \kappa$, there is no limit ordinal γ so that $\sup(k \upharpoonright \gamma) = h(\alpha)$. Condition (3) has been shown for the pair (h, k).

Since C_0 has the property of Lemma 5.2, (h, k) satisfies condition (1), (2) and (3) and $h, k \in [C_0]^\kappa_*$, one has that $\Phi(h) \leq \Phi(k)$.

Suppose $\alpha < \kappa$ and there is no $\xi < \zeta$ such that $\mu_\xi \leq \alpha < \eta_\xi$. Then $f(\alpha) \leq k(\alpha) = h(\alpha)$ since the pair (f, k) satisfies condition (1). Suppose $\alpha < \kappa$ and there is a $\xi < \zeta$ such that $\mu_\xi \leq \alpha < \eta_\xi$. Since $\alpha < \eta_\xi \leq \nu_\xi$, one has that $f(\alpha) \leq \sup(f \upharpoonright \nu_\xi) < h(\mu_\xi) \leq h(\alpha)$. It has been shown that for all $\alpha < \kappa$, $f(\alpha) \leq h(\alpha)$ and so condition (1) holds for the pair (f, h).

Suppose $\alpha < \kappa$ and there is no $\xi < \zeta$ such that $\mu_\xi \leq \alpha < \eta_\xi$, then there is no limit ordinal γ so that $\sup(f \upharpoonright \gamma) = k(\alpha) = h(\alpha)$ since the pair (f, k) satisfies condition (2). Suppose $\alpha < \kappa$ and there is a $\xi < \zeta$ such that $\mu_\xi \leq \alpha < \eta_\xi$, then $\sup(f \upharpoonright \nu_\xi) < h(\mu_\xi) \leq h(\alpha) < \sup(h \upharpoonright \eta_\xi) < k(\mu_\xi) < \sup(k \upharpoonright \eta_\xi) = f(\nu_\xi)$. So for all $\alpha < \kappa$, there is no limit ordinal γ so that $\sup(f \upharpoonright \gamma) = h(\alpha)$. Condition (2) holds for the pair (f, h).

Suppose $\alpha < \kappa$. First, suppose there is no $\xi < \zeta$ so that $\alpha = \nu_\xi$. There is a unique ρ so that $\sup(k \upharpoonright \rho) < f(\alpha) \leq k(\rho)$. Suppose there is no $\bar{\xi} < \zeta$ so that $\mu_{\bar{\xi}} \leq \rho < \eta_{\bar{\xi}}$. Then $\sup(h \upharpoonright \rho) \leq \sup(k \upharpoonright \rho) < f(\alpha) \leq k(\rho) = h(\rho)$. Suppose there is a $\bar{\xi} < \zeta$ so that $\mu_{\bar{\xi}} \leq \rho < \eta_{\bar{\xi}}$. Then $f(\alpha) \leq \sup(f \upharpoonright \nu_{\bar{\xi}})$. Therefore, $\sup(h \upharpoonright \rho) \leq \sup(k \upharpoonright \rho) < f(\alpha) \leq \sup(f \upharpoonright \nu_{\bar{\xi}}) < h(\mu_{\bar{\xi}}) \leq h(\rho)$. (This implies $\mu_{\bar{\xi}} = \rho$.) Hence, $\sup(h \upharpoonright \rho) < f(\alpha) < h(\rho)$. Now, suppose there is a $\xi < \zeta$ so that $\alpha = \nu_\xi$. Suppose that there is no $\bar{\xi} < \zeta$ so that $\mu_{\bar{\xi}} \leq \eta_\xi < \eta_{\bar{\xi}}$. Then $\sup(h \upharpoonright \eta_\xi) < k(\mu_\xi) < \sup(k \upharpoonright \eta_\xi) = f(\nu_\xi) = f(\alpha) < k(\eta_\xi) = h(\eta_\xi)$ using $(***)$ here. Thus, $\sup(h \upharpoonright \eta_\xi) < f(\alpha) < h(\eta_\xi)$. Suppose there is a $\bar{\xi}$ so that $\mu_{\bar{\xi}} \leq \eta_\xi < \eta_{\bar{\xi}}$. Then $f(\alpha) \leq \sup(f \upharpoonright \nu_{\bar{\xi}})$. Therefore, $\sup(h \upharpoonright \eta_\xi) < k(\mu_\xi) \leq \sup(k \upharpoonright \eta_\xi) = f(\nu_\xi) = f(\alpha) \leq \sup(f \upharpoonright \nu_{\bar{\xi}}) < h(\mu_{\bar{\xi}}) \leq h(\eta_\xi)$. (This implies $\mu_{\bar{\xi}} = \eta_\xi$.) Hence, $\sup(h \upharpoonright \eta_\xi) < f(\alpha) < h(\eta_\xi)$. In all cases, it has been shown that for all $\alpha < \kappa$, there is no limit ordinal γ so that $\sup(h \upharpoonright \gamma) = f(\alpha)$. The pair (f, h) satisfies condition (3).

Since C_0 has the property of Lemma 5.2, (f, h) satisfies condition (1), (2) and (3) and $f, h \in [C_0]^\kappa_*$, one has that $\Phi(f) \leq \Phi(h)$.

In conclusion $\Phi(f) \leq \Phi(h) \leq \Phi(k) \leq \Phi(g)$. □

Corollary 5.4. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$, $\epsilon \leq \kappa$ and $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$. Then there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

Proof. Define $\Phi' : [\kappa]^\kappa_* \to \mathrm{ON}$ by $\Phi'(f) = \Phi(f \upharpoonright \epsilon)$. The result follows by applying Theorem 5.3 to Φ'. □

Corollary 5.5. *Assume* AD. *Suppose κ is ω_1, $\boldsymbol{\delta}^1_{2n+1}$ for $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ where $A \subseteq \mathbb{R}$ or $\boldsymbol{\delta}^2_1$ (assuming* $\mathsf{DC}_\mathbb{R}$*). For any $\epsilon \leq \kappa$ and any function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

A suitable modification can be used to investigate almost everywhere monotonicity for weak partition cardinals which may not be strong partition cardinals.

Lemma 5.6. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)^{<\kappa}_2$. For any $\epsilon < \kappa$ and function $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, there is a club $C \subseteq \kappa$ so that for all $f, g \in [C]^\epsilon_*$, if f and g have the following properties:*

1. *For all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$.*
2. *For all $\alpha < \epsilon$, there is no limit ordinal $\beta \leq \epsilon$ so that $\sup(f \upharpoonright \beta) = g(\alpha)$ (where $\sup(f \upharpoonright \epsilon) = \sup(f)$).*
3. *For all $\alpha < \epsilon$, there is no limit ordinal $\beta < \epsilon$ so that $\sup(g \upharpoonright \beta) = f(\alpha)$.*

then $\Phi(f) \leq \Phi(g)$.

Proof. Let $\mathcal{I} : \epsilon + \epsilon \to \kappa$ be an increasing and discontinuous function whose image consists of indecomposable ordinals. Let $\epsilon^0 = \sup\{\mathcal{I}(\alpha) + 1 : \alpha < \epsilon\}$ and $\epsilon^1 = \sup\{\mathcal{I}(\alpha) + 1 : \alpha < \epsilon + \epsilon\}$. Note that $\epsilon^0 + \epsilon^1 = \epsilon^1$. Suppose $h \in [\kappa]^{\epsilon^1}_*$. Let $h^0 \in [\kappa]^{\epsilon^0}_*$ be defined by $h^0 = h \upharpoonright \epsilon^0$. If $\ell \in [\kappa]^{\epsilon^0}$, then let $\mathsf{main}(\ell) \in [\kappa]^\epsilon$ be defined by $\mathsf{main}(\ell)(\alpha) = \ell(\mathcal{I}(\alpha))$.

Define $P : [\kappa]^{\epsilon^1}_* \to 2$ by $P(h) = 0$ if and only if for all $p \in [h[\epsilon^1]]^{\epsilon^0}_*$, $\Phi(\mathsf{main}(h^0)) \leq \Phi(\mathsf{main}(p))$. By $\kappa \to_* (\kappa)^{\epsilon^1}_2$, there is a club $C_0 \subseteq \kappa$ which is homogeneous for P. Let $Z = \{\Phi(\mathsf{main}(\ell)) : \ell \in [C_0]^{\epsilon^0}_*\}$ which has a minimal element since it is a nonempty set of ordinals. Let $\ell^* \in [C_0]^{\epsilon^0}_*$ be such that $\Phi(\ell^*) = \min(Z)$. Let $\mathfrak{h} \in [C_0]^{\epsilon^1}_*$ be defined by

$$\mathfrak{h}(\alpha) = \begin{cases} \ell^*(\alpha) & \alpha < \epsilon^0 \\ \mathsf{next}_{C_0}^{\omega\cdot((\alpha-\epsilon^0)+1)}(\sup(\ell^*)) & \epsilon^0 \leq \alpha < \epsilon^1 \end{cases}.$$

Note that $\mathfrak{h} \in [C_0]^{\epsilon^1}_*$ and $\mathfrak{h}^0 = \ell^*$. If $p \in [h[\epsilon^1]]^{\epsilon^0}_*$, then $p \in [C_0]^{\epsilon^0}_*$. Thus, $\Phi(\mathsf{main}(p)) \in Z$ and $\Phi(\mathfrak{h}^0) = \Phi(\mathsf{main}(\ell^*)) = \min(Z) \leq \Phi(\mathsf{main}(p))$. This shows $P(\mathfrak{h}) = 0$. Since $\mathfrak{h} \in [C_0]^{\epsilon^1}_*$, C_0 must be homogeneous for P taking value 0. (Note only $\kappa \to_* (\kappa)^{\epsilon^1}_2$ is needed rather than the full weak partition relation.) By choosing a subclub of C_0, one may assume C_0 consists of indecomposable ordinals and $\epsilon^1 < \min(C_0)$. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$.

Fix $f, g \in [C_1]^{\epsilon}_*$ with properties (1), (2) and (3). One will construct by recursion two functions $h \in [C_0]^{\epsilon^1}_*$ and $p \in [h[\epsilon^1]]^{\epsilon^0}_*$ so that $\mathsf{main}(h^0) = f$ and $\mathsf{main}(p) = g$. The construction and verification are quite similar to Lemma 5.2, so some details of the verification will be omitted.

Suppose $\alpha < \epsilon$ and the following holds:

(a) For each $\beta < \alpha$, $h \restriction \mathcal{I}(\beta)+1$ has been defined, is a function of the correct type and $h(\mathcal{I}(\beta)) = f(\beta)$.
(b) For each $\beta < \alpha$, $\sigma_\beta \leq \beta + 1$ has been defined. If $\beta_0 \leq \beta_1 < \alpha$, then $\sigma_{\beta_0} \leq \sigma_{\beta_1}$.
(c) For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $p \restriction \mathcal{I}(\eta) + 1$ has been defined, is a function of the correct type and $p(\mathcal{I}(\eta)) = g(\eta)$.
(d) For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $g(\eta) \leq f(\beta) < g(\sigma_\beta)$.

Let $\iota_\alpha = \sup\{\sigma_\beta : \beta < \alpha\}$, $\delta_0 = \sup\{\mathcal{I}(\beta) + 1 : \beta < \alpha\}$ and $\tau_0 = \sup\{\mathcal{I}(\beta) + 1 : \beta < \iota_\alpha\}$. Observe $\sup(g \restriction \iota_\alpha) \leq \sup(f \restriction \alpha) < g(\iota_\alpha)$.

Let $A = \{\beta < \alpha : \sup(f \restriction \alpha) < g(\beta) < f(\alpha)\}$.

(Case A) If $A = \emptyset$.

Then let $\tau = \tau_0$ and $\delta = \delta_0$.

(Case B) $A \neq \emptyset$.

One must have $\iota_\alpha < \alpha$ and $\iota_\alpha = \min(A)$. Let $\xi = \mathrm{ot}(A) \leq \alpha$, and observe that $A = \{\iota_\alpha + \eta : \eta < \xi\}$. δ_0 and τ_0 have already been defined above with $\sup(h \restriction \delta_0) = \sup(f \restriction \alpha) < g(\iota_\alpha)$. For $0 < \nu < \xi$, suppose that for all $\eta < \nu$, the following holds:

- $\epsilon_\eta = \delta_0 + \mathcal{I}(\iota_\alpha + \eta)$ and $\mu_\eta = \tau_0 + \mathcal{I}(\iota_\alpha + \eta)$ have been defined.
- $h \restriction \epsilon_\eta + 1$ and $p \restriction \mu_\eta + 1$ have been defined.
- $h(\epsilon_\eta) = g(\iota_\alpha + \eta) = p(\mu_\eta)$.

Let $\delta_\nu = \sup\{\epsilon_\eta + 1 : \eta < \nu\}$ and $\tau_\nu = \sup\{\mu_\eta + 1 : \eta < \nu\}$. Note that $\delta_\nu < \delta_0 + \mathcal{I}(\iota_\alpha + \nu)$ and $\tau_\nu < \tau_0 + \mathcal{I}(\iota_\alpha + \nu)$. $h \restriction \delta_\nu$ and $p \restriction \tau_\nu$ are defined with $\sup(h \restriction \delta_\nu) = \sup(g \restriction (\iota_\alpha + \nu)) = \sup(g \restriction \tau_\nu)$.

Fix ν with $0 \leq \nu < \xi$. Let $\epsilon_\nu = \delta_\nu + \mathcal{I}(\iota_\alpha + \nu) = \delta_0 + \mathcal{I}(\iota_\alpha + \nu)$ and $\mu_\nu = \tau_\nu + \mathcal{I}(\iota_\alpha + \nu) = \tau_0 + \mathcal{I}(\iota_\alpha + \nu)$. For $\beta < \mathcal{I}(\iota_\alpha + \nu)$, let $h(\delta_\nu + \beta) = p(\tau_\nu + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \restriction \delta_\nu))$. Let $h(\epsilon_\nu) = p(\mu_\nu) = g(\iota_\alpha + \nu)$.

Let $\tau = \sup\{\mu_\xi + 1 : \nu < \xi\}$ and $\delta = \sup\{\epsilon_\nu + 1 : \nu < \xi\}$. Note $p \restriction \tau$ and $h \restriction \tau$ have been defined so that $\sup(p \restriction \tau) = \sup(h \restriction \delta) = \sup(g(\gamma) : \gamma \in A\} < f(\alpha)$ by property (3). Observe also that $\delta < \mathcal{I}(\alpha)$.

Now, in either Case A or Case B, ordinals τ and δ have been defined with $\tau \leq \delta < \mathcal{I}(\alpha)$ and $\sup(h \restriction \delta) < f(\alpha)$. Let $\ell = \min(\kappa \setminus A)$.

(Case I) $g(\ell) > f(\alpha)$.

Let $\sigma_\alpha = \iota_\alpha$ if Case A held, and let $\sigma_\alpha = \ell$ if Case B held. For $\beta < \mathcal{I}(\alpha)$, let $h(\delta + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \restriction \delta))$. Let $h(\mathcal{I}(\alpha)) = f(\alpha)$.

(Case II) $g(\ell) = f(\alpha)$.

For $\beta < \mathcal{I}(\ell)$, let $h(\delta + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \restriction \delta))$.

(Case II.1) $\ell = \alpha$.

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Let $h(\mathcal{I}(\alpha)) = p(\mathcal{I}(\alpha)) = f(\alpha) = g(\alpha)$. Let $\sigma_\alpha = \alpha + 1$.

(Case II.2) $\ell < \alpha$.

For $\beta < \mathcal{I}(\alpha)$, let $h(\delta + \mathcal{I}(\ell) + \beta) = \mathsf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup(h \upharpoonright (\delta + \mathcal{I}(\ell))))$. Let $h(\mathcal{I}(\alpha)) = f(\alpha)$ and $p(\mathcal{I}(\ell)) = g(\ell) = f(\alpha)$. Let $\sigma_\alpha = \ell + 1$.

Let $\varpi \leq \epsilon$ be largest such that $\sup(g \upharpoonright \varpi) \leq \sup(f)$. Let $\varsigma = \sup\{\mathcal{I}(\alpha) + 1 : \alpha < \varpi\}$. After ϵ-many stages, $h \upharpoonright \epsilon^0$ and $p \upharpoonright \varsigma$ have been defined. Note $\mathsf{main}(h \upharpoonright \epsilon^0) = f$.

(Case 1) $\sup(f) = \sup(g)$.

Then $\varpi = \epsilon$, $\varsigma = \epsilon^0$ and $p \in [C_0]_*^{\epsilon^0}$ has been completely defined with $\mathsf{main}(p) = g$.

For $\alpha < \epsilon^1$, let $h(\epsilon^0 + \alpha) = \mathsf{next}_{C_0}^{\omega\cdot(\alpha+1)}(\sup(h \upharpoonright \epsilon^0))$. This completes the construction of $h \in [C_0]_*^{\epsilon^1}$.

(Case 2) $\sup(f) < \sup(g)$.

Then $\varpi < \epsilon$ and $\varsigma < \epsilon^0$. Suppose $\nu < \epsilon$ and the following holds.

(i) For all $\eta < \epsilon + \nu$, $h \upharpoonright \mathcal{I}(\eta) + 1$ has been defined and is a function of the correct type.
(ii) For all $\eta < \varpi + \nu$, $p \upharpoonright \mathcal{I}(\eta) + 1$ has been defined and is a function of the correct type.
(iii) For $\eta < \epsilon$, $h(\mathcal{I}(\epsilon + \eta)) = g(\varpi + \eta) = p(\mathcal{I}(\varpi + \eta))$.

Suppose $\nu < \epsilon$. Let $\lambda_\nu = \sup\{\mathcal{I}(\eta) + 1 : \eta < \epsilon + \nu\}$ and $\rho_\nu = \sup\{\mathcal{I}(\eta) + 1 : \eta < \varpi + \nu\}$. (Note that $\lambda_0 = \epsilon^0$ and $\rho_0 = \varsigma$.) These assumptions imply that $h \upharpoonright \lambda_\nu$ and $p \upharpoonright \rho_\nu$ are defined. For $\alpha < \mathcal{I}(\varpi + \nu)$, let $h(\lambda_\nu + \alpha) = p(\rho_\nu + \alpha) = \mathsf{next}_{C_0}^{\omega\cdot(\alpha+1)}(\sup(h \upharpoonright \lambda_\nu))$. This defines $h \upharpoonright (\lambda_\nu + \mathcal{I}(\varpi + \nu))$ and $p \upharpoonright \mathcal{I}(\varpi + \nu)$. For all $\alpha < \mathcal{I}(\epsilon + \nu)$, let $h(\lambda_\nu + \mathcal{I}(\varpi + \nu) + \alpha) = \mathsf{next}_{C_0}^{\omega\cdot(\alpha+1)}(\sup(h \upharpoonright (\lambda_\nu + \mathcal{I}(\varpi + \nu))))$. This defines $h \upharpoonright \mathcal{I}(\epsilon + \nu)$ since $\lambda_\nu + \mathcal{I}(\varpi + \nu) + \mathcal{I}(\epsilon + \nu) = \mathcal{I}(\epsilon + \nu)$ since $\mathcal{I}(\epsilon + \nu)$ is indecomposable. Let $h(\mathcal{I}(\epsilon + \nu)) = p(\mathcal{I}(\varpi + \nu)) = g(\varpi + \nu)$. This defines $h \upharpoonright \mathcal{I}(\epsilon + \nu) + 1$ and $p \upharpoonright \mathcal{I}(\varpi + \nu) + 1$.

The construction of $h \in [C_0]_*^{\epsilon^1}$ and $p \in [h[\epsilon^1]]_*^{\epsilon^0}$ has been completed so that $\mathsf{main}(h^0) = f$ and $\mathsf{main}(p) = g$. Since $h \in [C_0]_*^{\epsilon^1}$, $p \in [h[\epsilon^1]]_*^{\epsilon_0}$ and $P(h) = 0$, one has $\Phi(f) = \Phi(\mathsf{main}(h^0)) \leq \Phi(\mathsf{main}(p)) = \Phi(g)$. □

Theorem 5.7. *Suppose κ is a cardinal satisfying $\kappa \to_* (\kappa)_2^{<\kappa}$. For any $\epsilon < \kappa$ and function $\Phi : [\kappa]_*^\epsilon \to$ ON, there is a club $C \subseteq \kappa$ so that for all $f, g \in [\kappa]_*^\epsilon$, if for all $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

Proof. Let $C_0 \subseteq \kappa$ be a club consisting of indecomposable ordinals with the properties from Lemma 5.6. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$.

Let $\zeta \leq \epsilon$ and $\langle \eta_\xi : \xi < \zeta \rangle$ and $\langle \nu_\xi : \xi < \zeta \rangle$ be two increasing sequences with the following property.

(a) For all $\xi < \zeta$, ν_ξ is a limit ordinal.
(b) For all $\xi < \zeta$, $\sup(f \upharpoonright \nu_\xi) = g(\eta_\xi)$.
(c) For all $\nu < \epsilon$ and $\eta < \epsilon$, if ν is a limit ordinal and $\sup(f \upharpoonright \nu) = g(\eta)$, then there is a $\xi < \zeta$ so that $\nu = \nu_\xi$ and $\eta = \eta_\xi$.

(Note it is possible that $\nu_\xi = \epsilon$ when $\sup(f \upharpoonright \epsilon) = \sup(f) = g(\eta_\xi)$.) These indicate the region in which the pair (f, g) fails to satisfy condition (2) of Lemma 5.6. For each $\xi < \zeta$, let μ_ξ be the least γ so that $\sup(g \upharpoonright \eta_\xi) \leq f(\gamma)$.

Define $k \in [C_1]_*^\epsilon$ as follows: If $\alpha < \epsilon$ and there is no $\xi < \zeta$ so that $\alpha = \eta_\xi$, then let $k(\alpha) = g(\alpha)$. If $\alpha < \epsilon$ and there is a $\xi < \zeta$ so that $\alpha = \eta_\xi$, then let $k(\alpha) = f(\mu_\xi + 1)$. As in Theorem 5.3, the pair (f, k) satisfies conditions (1) and (2) and the pair (k, g) satisfies conditions (1), (2) and (3) of Lemma 5.6. Therefore, $\Phi(k) \leq \Phi(g)$.

Let $\zeta \leq \epsilon$ and $\langle \eta_\xi : \xi < \zeta \rangle$ and $\langle \nu_\xi : \xi < \zeta \rangle$ be two increasing sequences so that the following hold.

(i) For all $\xi < \zeta$, η_ξ is a limit ordinal.
(ii) For all $\xi < \zeta$, $\sup(k \upharpoonright \eta_\xi) = f(\nu_\xi)$.
(iii) For all $\eta < \epsilon$ and $\nu < \epsilon$, if η is a limit ordinal and $\sup(k \upharpoonright \eta) = f(\nu)$, then there is a $\xi < \zeta$ so that $\eta = \eta_\xi$ and $\nu = \nu_\xi$.

Let μ_ξ be the least $\gamma < \eta_\xi$ so that $\sup(f \upharpoonright \nu_\xi) < k(\gamma)$.

Define $h \in [C_0]^\epsilon_*$ as follows: If $\alpha < \epsilon$ and there is no $\xi < \zeta$ so that $\mu_\xi \leq \alpha < \eta_\xi$, then let $h(\alpha) = k(\alpha)$. If $\alpha < \epsilon$ and there is a $\xi < \zeta$ so that $\mu_\xi \leq \alpha < \eta_\xi$, then let $h(\alpha) = \mathsf{next}^{\omega\cdot((\alpha-\mu_\xi)+1)}_{C_0}(\sup(f \restriction \nu_\xi))$. As before, the pairs (f, h) and (h, k) both satisfy conditions (1), (2) and (3) of Lemma 5.6. Thus, $\Phi(f) \leq \Phi(h) \leq \Phi(k)$.

This concludes that $\Phi(f) \leq \Phi(g)$. □

Corollary 5.8. *Assume* AD. *Suppose* κ *is* ω_1, ω_2, $\boldsymbol{\delta}^1_n$ *for* $1 \leq n < \omega$, $\boldsymbol{\delta}_A$ *where* $A \subseteq \mathbb{R}$ *or* $\boldsymbol{\delta}^2_1$ *(assuming* $\mathsf{DC}_\mathbb{R}$*). For any* $\epsilon < \kappa$ *and* $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$, *there is a club* $C \subseteq \kappa$ *so that for all* $f, g \in [C]^\epsilon_*$, *if for all* $\alpha < \epsilon$, $f(\alpha) \leq g(\alpha)$, *then* $\Phi(f) \leq \Phi(g)$.

6. A finite continuity property for long functions on ω_1

Expecting a function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ to satisfy an almost everywhere finite continuity in the sense that there are finitely many ordinals $\delta_0, ..., \delta_k < \omega_1$ so that $\Phi(f)$ only depends on $\sup(f \restriction \delta_i)$ is impossible.

Example 6.1. Let $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$ be defined by $\Psi(f) = f(f(0))$. For any finite set of ordinals $\delta_0, ..., \delta_{k-1} < \omega_1$, for any club C, there are $f, g \in [C]^{\omega_1}_*$ so that for all $i < k$, $\sup(f \restriction \delta_i) = \sup(g \restriction \delta_i)$ and $\Psi(f) \neq \Psi(g)$.

Proof. Let $\delta = \sup\{\delta_0, ..., \delta_{k-1}\}$. Pick $f, g \in [C]^{\omega_1}_*$ so that $f \restriction \delta = g \restriction \delta$, $f(0) = g(0) > \delta$ and $f(f(0)) \neq g(g(0))$. Then for all $i < k$, $\sup(f \restriction \delta_i) = \sup(g \restriction \delta_i)$ but $\Psi(f) = f(f(0)) \neq g(g(0)) = \Psi(g)$. □

Expecting a function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ to have finitely many functions $\Gamma_0, ..., \Gamma_{k-1}$ so that $\Phi(f)$ depends only on $\sup(f \restriction \Gamma_i(f))$ is also impossible by the following example. The concept of a closure point of a function will be very important in this section.

Definition 6.2. Let $f \in [\omega_1]^{\omega_1}_*$. An ordinal $\beta \in \omega$ is a closure point of f if and only if for all $\alpha < \beta$, $f(\alpha) < \beta$ (or equivalently) if and only if $\sup(f \restriction \beta) = \beta$. Let $\mathfrak{C}_f = \{\beta \in \omega_1 : \sup(f \restriction \beta) = \beta\}$ be the collection of closure points of f. $\mathfrak{C}_f$ is a club subset of ω_1.

Example 6.3. Let $\Psi : [\omega_1]^{\omega_1}_* \to \omega_1$ be defined by $\Psi(f) = \min(\mathfrak{C}_f)$, that is, $\Psi(f)$ is the least closure point of f. Then for any club $C \subseteq \omega_1$ and for any finite collection of functions $\Gamma_0, ..., \Gamma_{k-1} : [\omega_1]^{\omega_1}_* \to \omega_1$, there is an $f \in [C]^{\omega_1}_*$ and a $g \in [C]^{\omega_1}_*$ so that for all $i < k$, $\sup(f \restriction \Gamma_i(f)) = \sup(g \restriction \Gamma_i(f))$ and $\Phi(f) \neq \Phi(g)$.

Proof. Let $C_0 \subseteq C$ be a club consisting entirely of indecomposable ordinals. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Let C_2 be the club of limit points of C_1. Let $f \in [C_2]^{\omega_1}_*$. Then $\Psi(f) = \min(\mathfrak{C}_f) \in C_2$. Let $\gamma = \sup\{\Gamma_i(f) : i < k \wedge \Gamma_i(f) < \Psi(f)\}$, and note that $\gamma = 0$ if there are no $i < k$ with $\Gamma_i(f) < \Psi(f)$. Since $\Psi(f) \in C_2$ and $\sup(f \restriction \gamma) < \sup(f \restriction \Psi(f)) = \Psi(f)$, there exists a $\delta \in C_1$ with $\sup(f \restriction \gamma) < \delta < \Psi(f)$. Define $g \in [C_0]^{\omega_1}_*$ by

$$g(\alpha) = \begin{cases} f(\alpha) & \alpha < \gamma \vee \alpha \geq \Psi(f) \\ \mathsf{next}^{\omega\cdot(\alpha+1)}_{C_0}(\sup(f \restriction \gamma)) & \gamma \leq \alpha < \Psi(f) \end{cases}.$$

By Fact 2.12, g is indeed an increasing function. Moreover, since $\gamma < \delta$ and $\sup(f \restriction \gamma) < \delta$, Fact 2.12 also implies that $\sup(g \restriction \delta) = \delta$. Thus, $\delta \in \mathfrak{C}_g$. Therefore, $\Psi(g) = \min(\mathfrak{C}_g) \leq \delta < \Psi(f)$. However, for all $i < k$, $\sup(g \restriction \Gamma_i(f)) = \sup(f \restriction \Gamma_i(f))$. □

Motivated by this example, Theorem 6.18 will show that if one demands that closure points remains the same, then there will be finitely many functions $\Gamma_0, ..., \Gamma_{k-1}$ so that for $\mu^{\omega_1}_{\omega_1}$-almost all f, $\Phi(f)$ depends only on $\sup(f \restriction \Gamma_i(f))$.

The results of this section will be proved using the strong partition relation $\omega_1 \to_* (\omega_1)^{\omega_1}_*$ and an additional combinatorial principle called the almost everywhere short length club uniformization for ω_1. More specifically, a fine form of $\mu^{\omega_1}_{\omega_1}$-almost everywhere continuity is needed.

Let club_{ω_1} denote the collection of club subsets of ω_1. A relation $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ is said to $\subseteq$-downward closed in the club_{ω_1}-coordinate if and only if for all $\ell \in [\omega_1]^{<\omega_1}_*$, if $R(\ell, D)$ and $C \subseteq D$, then $R(\ell, C)$.

Definition 6.4. Almost everywhere short length club uniformization at ω_1 is the asserting that for all $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$ which is $\subseteq$-downward closed in the club_{ω_1}-coordinate, there is a club $C \subseteq \omega_1$ and a function $\Lambda : (\mathrm{dom}(R) \cap [C]^{<\omega_1}_*) \to \mathsf{club}_{\omega_1}$ so that for all $\ell \in \mathrm{dom}(R) \cap [C]^{<\omega_1}_*$, $R(\ell, \Lambda(\ell))$.

Almost everywhere short length club uniformization is established in [4] Theorem 3.10 under AD using techniques which are specific to ω_1. [2] gives a more general argument which holds for many other known strong partition cardinals under AD.

Fact 6.5. ([4] Theorem 3.10, [2]) Assume AD. The almost everywhere short length club uniformization at ω_1 holds.

The almost everywhere short length club uniformization at ω_1 combined with the strong partition relation $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ gives a simpler form of the almost everywhere club uniformization principle stated below.

Fact 6.6. ([2]) (Strong almost everywhere short length club uniformization for ω_1) Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and the almost everywhere short length club uniformization principle holds at ω_1. For all $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}_{\omega_1}$, there exists a club $C \subseteq \omega_1$ so that for all $\ell \in [C]^{<\omega_1}_* \cap \mathrm{dom}(R)$, $R(\ell, C \setminus \sup(\ell)+1)$.

[4] used Fact 6.5 to show that every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous $\mu^{\omega_1}_{\omega_1}$-almost everywhere.

Fact 6.7. Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and the almost everywhere short length club uniformization principle holds for ω_1. Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. There is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $g \upharpoonright \alpha = f \upharpoonright \alpha$, then $\Phi(f) = \Phi(g)$.

Here, an even finer form of continuity established in [2] from Fact 6.6 will be needed.

Definition 6.8. Let $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ and $C \subseteq \omega_1$ be a club. One says that $\ell \in [C]^{<\omega_1}_*$ is a continuity point for Φ relative to C if and only if for all $f, g \in [C]^{\omega_1}_*$ so that $f \upharpoonright |\ell| = \ell = g \upharpoonright |\ell|$, $\Phi(f) = \Phi(g)$.

Fact 6.9. ([2]) Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and the almost everywhere short length club uniformization principle holds for ω_1. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. Then there is a club $C \subseteq \omega_1$ so that for all $f \in [C]^{\omega_1}_*$ and $\alpha < \omega_1$, if $\Phi(f) < f(\alpha)$, then $f \upharpoonright \alpha$ is a continuity point for Φ relative to C.

Lemma 6.10. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and that the almost everywhere short length club uniformization principle holds at ω_1. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is a function so that for $\mu^{\omega_1}_{\omega_1}$-almost all f, $\Phi(f)$ is a successor ordinal. Then there is a club $C \subseteq \omega_1$ and a function $\Gamma^\Phi : [C]^{\omega_1}_* \to \omega_1$ so that the following holds:*

1. *For all $f \in [C]^{\omega_1}_*$, $\Gamma^\Phi(f) < \Phi(f)$.*
2. *For all $f \in [C]^{\omega_1}_*$, $\Gamma^\Phi(f) + 1 = \Phi(f)$.*
3. *For all $f \in [C]^{\omega_1}_*$, $f \upharpoonright \Gamma^\Phi(f)$ is a continuity point for Φ relative to C.*

Proof. Let $C_0 \subseteq \omega_1$ be a club consisting entirely of limit ordinals so that for all $f \in [C_0]^{\omega_1}_*$, $\Phi(f)$ is a successor ordinal. By Fact 6.9, there is a club $C_1 \subseteq C_0$ so that for all $f \in [C_1]^{\omega_1}_*$ and $\alpha < \omega_1$, if $\Phi(f) < f(\alpha)$, then $f \upharpoonright \alpha$ is a continuity point for Φ relative to C_1. Define $\Gamma^\Phi : [C_1]^{\omega_1}_* \to \omega_1$ by letting $\Gamma^\Phi(f)$ be the predecessor of $\Phi(f)$. Note that $\Gamma^\Phi(f) \leq \sup(f \upharpoonright \Gamma^\Phi(f)) < f(\Gamma^\Phi(f))$ since $\sup(f \upharpoonright \Gamma^\Phi(f)) \in C_1$ is a limit ordinal and by the discontinuity of *f*. Since $\Phi(f) = \Gamma^\Phi(f) + 1$, this implies that $\Phi(f) \leq f(\Gamma^\Phi(f))$. However, since $f \in [C_1]^{\omega_1}_*$ and C_1 consist entirely of limit ordinals, $\Phi(f) < f(\Gamma^\Phi(f))$ since $\Phi(f)$ is a successor ordinal. By Fact 6.9, $f \upharpoonright \Gamma^\Phi(f)$ is a continuity point for Φ relative to C_1. ◻

Lemma 6.11. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and that the almost everywhere short length club uniformization principle holds at ω_1. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is a function so that for $\mu^{\omega_1}_{\omega_1}$-almost all f, $\Phi(f)$ is* *a nonzero limit ordinal and* $\Phi(f) \notin \mathfrak{C}_f$ *(i.e.,* $\Phi(f)$ *is not a closure point of f). Then there is a club* $C \subseteq \omega_1$ *and a function* $\Gamma^\Phi : [C]^{\omega_1}_* \to \omega_1$ *so that the following holds:*

1. *For all* $f \in [C]^{\omega_1}_*$, $\Gamma^\Phi(f) < \Phi(f)$.
2. *For all* $f \in [C]^{\omega_1}_*$, $\mathrm{ot}(\{\alpha : \Gamma^\Phi(f) \leq \alpha < \Phi(f)\})$ *is an additively indecomposable ordinal.*
3. *For all* $f \in [C]^{\omega_1}_*$, $f \upharpoonright \Gamma^\Phi(f)$ *is a continuity point for* Φ *relative to* C.

Proof. Let $C_0 \subseteq \omega_1$ be a club so that for all $f \in [C_0]^{\omega_1}_*$, $\Phi(f)$ is a limit ordinal which is not a closure point of *f*. By Fact 6.9, let $C_1 \subseteq C_0$ be a club so that for all $\alpha < \omega_1$, if $\Phi(f) < f(\alpha)$, then $f \upharpoonright \alpha$ is a continuity point of Φ relative to C_1. Define $\Gamma : [C_1]^{\omega_1}_* \to \omega_1$ by $\Gamma(f)$ is the unique β such that $\sup(f \upharpoonright \beta) \leq \Phi(f) < f(\beta)$. For all $f \in [C_1]^{\omega_1}_*$, $f \upharpoonright \Gamma(f)$ is a continuity point for Φ relative to C_1. Note that $\Gamma(f) \leq \Phi(f)$. If $\Gamma(f) = \Phi(f)$, then since $\Phi(f)$ is a limit ordinal, $\Phi(f) \leq \sup(f \upharpoonright \Phi(f)) = \sup(f \upharpoonright \Gamma(f)) \leq \Phi(f)$. Thus, $\Phi(f) \in \mathfrak{C}_f$ which contradicts the assumption. It has been shown that $\Gamma(f) < \Phi(f)$ for all $f \in [C_1]^{\omega_1}_*$. For each $f \in [C_1]^{\omega_1}_*$, let $\delta_f = \mathrm{ot}(\{\alpha : \Gamma(f) \leq \alpha < \Phi(f)\})$. Note that δ_f is a limit ordinal since $\Phi(f)$ is a limit ordinal. Let ϵ_f be the least ordinal so that there exists an additively indecomposable ordinal ν_f with $\delta_f = \epsilon_f + \nu_f$. Note that $\epsilon_f < \delta_f$. Define $\Gamma^\Phi : [C_1]^{\omega_1}_* \to \omega_1$ by $\Gamma^\Phi(f) = \Gamma(f) + \epsilon_f$. Then for all $f \in [C_1]^{\omega_1}_*$, $\Gamma^\Phi(f) < \Phi(f)$, $f \upharpoonright \Gamma^\Phi(f)$ is a continuity point for Φ relative to C_1 and $\mathrm{ot}(\{\alpha : \Gamma^\Phi(f) \leq \alpha < \Phi(f)\}) = \nu_f$ which is an additively indecomposable ordinal. □

If $\Phi(f)$ is a function so that for $\mu^{\omega_1}_{\omega_1}$-almost all *f*, $\Phi(f) \in \mathfrak{C}_f$ (such as the function from Example 6.3), then $\Phi(f)$ is the least β so that $f \upharpoonright \beta$ is a continuity point for Φ. For such a function, condition (3) must be weakened otherwise the crucial condition (1) will not hold.

Fact 6.12. Assume κ is a cardinal, $\epsilon \leq \kappa$ and $\kappa \to_* (\kappa)^\epsilon_2$. Let $\Phi : [\kappa]^\epsilon_* \to \mathrm{ON}$. There is a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, for all $g \sqsubseteq f$, $\Phi(f) \leq \Phi(g)$.

Proof. This follows from Theorem 5.3; however, in this particular instance, the argument is much simpler. Let $P : [\kappa]^\epsilon_* \to 2$ be defined by $P(f) = 0$ if and only if for all $g \sqsubseteq f$, $\Phi(f) \leq \Phi(g)$. By $\kappa \to_* (\kappa)^\epsilon_2$, there is a club $C \subseteq \kappa$ which is homogeneous for P. Suppose C was homogeneous for P taking value 1. Let $Z = \{\Phi(f) : f \in [C]^\epsilon_*\}$. Let $\beta = \min(Z)$. Let $f \in [C]^{\omega_1}_*$ be such that $\Phi(f) = \beta$. Since $P(f) = 1$, there is a $g \sqsubseteq f$ so that $\Phi(g) < \Phi(f)$. Since $g \in [C]^\epsilon_*$, $\Phi(g) \in Z$. Then $\Phi(g) < \beta = \min(Z)$ which is a contradiction. C must be homogeneous for P taking value 0. □

Definition 6.13. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ and $f \in [\omega_1]^{\omega_1}_*$. Let $A^{\Phi,\beta}_f = \{g \in [\omega_1]^{\omega_1}_* : g \sqsubseteq f \wedge \mathfrak{C}_g = \mathfrak{C}_f \wedge g \upharpoonright \beta = f \upharpoonright \beta\}$. Let $\mathfrak{B}^{\Phi,\beta}_f = \{\Phi(g) : g \in A^{\Phi,\beta}_f\}$. Note that if $\beta_0 \leq \beta_1$, then $A^{\Phi,\beta_1}_f \subseteq A^{\Phi,\beta_0}_f$ and $\mathfrak{B}^{\Phi,\beta_1}_f \subseteq \mathfrak{B}^{\Phi,\beta_0}_f$.

Lemma 6.14. *Suppose* $\omega_1 \to_* (\omega_1)^{\omega_1}_2$. *For all* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$, *there is a club* $C \subseteq \omega_1$ *so that for all* $f \in [C]^{\omega_1}_*$, $\sup(\mathfrak{B}^{\Phi,0}_f) < \omega_1$.

Proof. Suppose $h \in [\omega_1]^{\omega_1}_*$ and for all $\alpha < \omega_1$, $h(\alpha)$ is an indecomposable ordinal. Let $\langle \gamma^h_\alpha : \alpha < \omega_1\rangle$ denote the increasing enumeration of $\mathfrak{C}_h$, the club of closure points of h, which are also indecomposable ordinals. Thus, for all $\alpha < \beta$, $\mathrm{ot}(\{\eta < \omega_1 : \gamma^h_\alpha \leq \eta < \gamma^h_\beta\}) = \gamma^h_\beta$. For $\alpha < \omega_1$, let $B^h_\alpha = \{(\eta, \zeta) : \gamma^h_\alpha \leq \eta < \gamma^h_{\alpha+1} \wedge \zeta = h(\eta)\}$. For $i \in 2$, $\bigcup_{\alpha<\omega_1} B^h_{2\alpha+i}$ is the graph of a partial function whose domain is a subset of ω_1. Denote this partial function by $\tilde{h}^i$. Let $\mathfrak{m}_{\mathrm{dom}(\tilde{h}^i)} : \mathrm{dom}(\tilde{h}^i) \to \omega_1$ be the Mostowski collapse of $\mathrm{dom}(\tilde{h}^i)$. Define $h^i(\alpha) = \tilde{h}^i \circ \mathfrak{m}^{-1}_{\mathrm{dom}(\tilde{h}^i)}$. Intuitively, h^0 and h^1 are the concatenations of h restricted to the even and odd, respectively, blocks determined by the sequence $\langle \gamma^h_\alpha : \alpha < \omega_1\rangle$ of closure points of h. Note that $\gamma^{h^i}_{\alpha+1} = \gamma^h_{2\alpha+1+i}$ and thus if α is a limit ordinal $\gamma^{h^i}_\alpha = \gamma^h_\alpha$.

Also, observe that if $f, g \in [\omega_1]^{\omega_1}_*$ have the property that for all $\alpha < \omega_1$, $f(\alpha)$ and $g(\alpha)$ are indecomposable ordinals, $\sup(B^f_\alpha) < \min(B^g_\alpha)$ and $\sup(B^g_\alpha) < \min(B^f_{\alpha+1})$, then there is an $h \in [\omega_1]^{\omega_1}_*$ so that $h^0 = f$ and $h^1 = g$. To see this: Let $\langle \gamma^*_\alpha : \alpha < \omega_1\rangle$ be the increasing enumeration of $\{\gamma^f_\alpha : \alpha < \omega_1\} \cup \{\gamma^g_\alpha : \alpha < \omega_1\}$. Note that for all limit ordinals α, $\gamma^*_{2\alpha} = \gamma^*_\alpha = \gamma^f_\alpha = \gamma^g_\alpha$. For each

α, $\gamma^*_{2\alpha+1} = \gamma^f_{\alpha+1}$ and $\gamma^*_{2\alpha+2} = \gamma^g_{\alpha+1}$ by the assumptions on f and g. Define h by recursion as follows. Suppose $h \upharpoonright \gamma^*_{2\alpha}$ has been defined. For each $\xi < \gamma^*_{2\alpha+1} = \gamma^f_{\alpha+1}$, let $h(\gamma^*_{2\alpha} + \xi) = f(\gamma^f_\alpha + \xi)$. This defines $h \upharpoonright \gamma^*_{2\alpha+1}$. For each $\xi < \gamma^*_{2\alpha+2} = \gamma^g_{\alpha+1}$, let $h(\gamma^*_{2\alpha+1} + \xi) = g(\gamma^g_\alpha + \xi)$. This defines $h \upharpoonright \gamma^*_{2\alpha+2}$. By recursion, this completes the definition of h. (The assumptions on f and g are needed to ensure h is an increasing and discontinuous function.) Note that for all $\alpha < \omega_1$, $\gamma^h_\alpha = \gamma^*_\alpha$. Therefore, $h^0 = f$ and $h^1 = g$.

Define $P : [\omega_1]^{\omega_1}_* \to 2$ by $P(h) = 0$ if and only if $\Phi(h^0) \leq \Phi(h^1)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C_0 \subseteq \omega_1$ which is homogeneous for P and consists entirely of indecomposable ordinals. For the sake of contradiction, suppose C_0 is homogeneous for P taking value 1. Pick any $h \in [C_0]^{\omega_1}_*$. For each $n \in \omega$, let $\tilde{g}_n$ denote the partial function whose graph is $\bigcup_{\alpha<\omega_1} B^h_{\omega\cdot\alpha+n}$. Let $\mathfrak{m}_{\mathrm{dom}(\tilde{g}_n)} : \mathrm{dom}(\tilde{g}_n) \to \omega_1$ be the Mostowski collapse of $\mathrm{dom}(\tilde{g}_n)$. Let $g_n = \tilde{g}_n \circ \mathfrak{m}^{-1}_{\mathrm{dom}(\tilde{g}_n)}$. Note that $B^{g_n}_\alpha = B^h_{\omega\cdot\alpha+n}$. Therefore, $\sup(B^{g_n}_\alpha) = \gamma^h_{\omega\cdot\alpha+n+1} < h(\omega\cdot\alpha+n+1) = \min(B^h_{\omega\cdot\alpha+n+1}) = \min(B^{g_{n+1}}_\alpha) < \gamma_{\omega\cdot\alpha+\omega} = \min(B^{g_0}_{\alpha+1}) \leq \min(B^{g_n}_{\alpha+1})$. By the previous observation, for each $n \in \omega$, there is an $h_n \in [C_0]^{\omega_1}_*$ so that $h^0_n = g_n$ and $h^1_n = g_{n+1}$. However, $P(h_n) = 1$ implies that $\Phi(g_{n+1}) = \Phi(h^1_n) < \Phi(h^0_n) = \Phi(g_n)$. $\langle \Phi(g_n) : n \in \omega\rangle$ is an infinite decreasing sequence of ordinals which is impossible. This shows that C_0 must be homogeneous for P taking value 0.

Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Let C_2 be the club of limit points of C_1. Let $f \in [C_2]^{\omega_1}_*$. Let $\gamma_0 = 0$. If α is a limit ordinal and for all $\beta < \alpha$, γ_β has been defined, then let $\gamma_\alpha = \sup\{\gamma_\beta : \beta < \alpha\}$. If α is a successor ordinal, then let $\gamma_\alpha = \mathsf{next}_{C_1}(\gamma^f_\alpha)$. Since $f(\gamma^f_\alpha) \in C_2$ and C_2 consists of the limit points of C_1, $\gamma_\alpha < f(\gamma^f_\alpha)$. Define $k \in [C_0]^{\omega_1}_*$ by recursion as follow: Suppose $\delta < \omega_1$ and $k \upharpoonright \gamma_\delta$ has been defined. For each $\alpha < \gamma_{\delta+1}$, let $k(\gamma_\delta + \alpha) = \mathsf{next}^{\omega\cdot(\alpha+1)}_{C_0}(\gamma^f_{\delta+1})$. Since $\gamma_{\delta+1}$ is indecomposable and $\gamma_{\delta+1} \in C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$, Fact 2.12 implies that this defines $k \upharpoonright \gamma_{\delta+1}$ and $\sup(k \upharpoonright \gamma_{\delta+1}) = \gamma_{\delta+1}$. Thus, $\gamma^k_\alpha = \gamma_\alpha$ for all $\alpha < \omega_1$. Observe that for each $\delta < \omega_1$, $\gamma^f_{\delta+1} < \mathsf{next}^\omega_{C_0}(\gamma^f_{\delta+1}) = k(\gamma_\delta) = \min(B^k_\delta) < \sup(B^k_\delta) = \gamma^k_{\delta+1} < f(\gamma^f_{\delta+1}) = \min(B^f_{\delta+1})$. Now, suppose $g \sqsubseteq f$ and $\mathfrak{C}_g = \mathfrak{C}_f$ (that is, for all $\alpha < \omega_1$, $\gamma^g_\alpha = \gamma^f_\alpha$). Then we have that for all $\delta < \omega_1$, $\sup(B^g_\delta) = \gamma^g_{\delta+1} = \gamma^f_{\delta+1} < \min(B^k_\delta) < \sup(B^k_\delta) = \gamma^k_{\delta+1} < f(\gamma^f_{\delta+1}) \leq g(\gamma^f_{\delta+1}) = g(\gamma^g_{\delta+1}) = \min(B^g_{\delta+1})$. By the observation above, there is an $h_g \in [C_0]^{\omega_1}_*$ so that $h^0_g = g$ and $h^1_g = k$. Then $P(h_g) = 0$ implies that $\Phi(g) \leq \Phi(k)$. It has been shown that for all $g \sqsubseteq f$ with $\mathfrak{C}_g = \mathfrak{C}_f$, $\Phi(g) \leq \Phi(k)$. Hence, $\sup(\mathfrak{B}^{\Phi,0}_f) \leq \Phi(k) < \omega_1$. □

Lemma 6.15. *Assume* DC, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ *and that the almost everywhere short length club uniformization principle holds at* ω_1. *Suppose* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ *is a function so that for* $\mu^{\omega_1}_{\omega_1}$*-almost all* f, $\Phi(f) \in \mathfrak{C}_f$. *Then there is a club* $C \subseteq \omega_1$ *and a function* $\Gamma^\Phi : [C]^{\omega_1}_* \to \omega_1$ *so that that the following holds:*

1. *For all* $f \in [C]^{\omega_1}_*$, $\Gamma^\Phi(f) < \Phi(f)$.
2. *For all* $f \in [C]^{\omega_1}_*$, $\mathrm{ot}(\{\alpha : \Gamma^\Phi(f) \leq \alpha < \Phi(\alpha)\})$ *is an additively indecomposable ordinal.*
3. *For all* $f \in [C]^{\omega_1}_*$, *for all* $g \sqsubseteq f$ *with* $\mathfrak{C}_f = \mathfrak{C}_g$ *and* $g \upharpoonright \Gamma^\Phi(f) = f \upharpoonright \Gamma^\Phi(f)$, *then* $\Phi(g) = \Phi(f)$.

Proof. By the assumption and Fact 6.9, there is a club C_0 so that the following holds.

(a) For all $f \in [C_0]^{\omega_1}_*$, if $\Phi(f) < f(\alpha)$, then $f \upharpoonright \alpha$ is a continuity point for Φ relative to C_0.
(b) $\Phi(f) \in \mathfrak{C}_f$.

For $f \in [C_0]^{\omega_1}_*$ and $\beta < \omega_1$, let $\Lambda(f,\beta) = \sup(\mathfrak{B}^{\Phi,\beta}_f) < \omega_1$ by Fact 6.14 (where $\mathfrak{B}^{\Phi,\beta}$ is defined in Definition 6.13). Observe that by condition (b), $\Lambda(f,\beta) \in \mathfrak{C}_f$. Let $Z_f = \{\Lambda(f,\beta) : \beta < \Phi(f)\}$. Let $\Gamma(f) = \min\{\beta < \Phi(f) : \Lambda(f,\beta) = \min(Z_f)\}$. The main property of Γ is that for all β with $\Gamma(f) \leq \beta < \Phi(f)$, $\Lambda(f,\beta) = \Lambda(f,\Gamma(f))$. Define $\Sigma\colon [C_0]^{\omega_1}_* \to \omega_1$ by $\Sigma(f) = \Lambda(f,\Gamma(f))$. Again, for all $f \in [C_0]^{\omega_1}_*$, $\Sigma(f) \in \mathfrak{C}_f$. Applying Fact 6.12 to Σ, Γ and Φ, there is a club $C_1 \subseteq C_0$ on which Σ, Γ and Φ are subsequence monotonic: that is, for all $f \in [C_1]^{\omega_1}_*$, for all $g \sqsubseteq f$, $\Phi(f) \leq \Phi(g)$, $\Sigma(f) \leq \Sigma(g)$ and $\Gamma(f) \leq \Gamma(g)$.

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Claim 1: For all $f \in [C_1]^{\omega_1}_*$, for all $g \sqsubseteq f$ with $\mathfrak{C}_f = \mathfrak{C}_g$ and $g \upharpoonright \Gamma(f) = f \upharpoonright \Gamma(f)$, $\Sigma(f) = \Sigma(g)$ and $\Gamma(f) = \Gamma(g)$.

To see Claim 1: Because $g \sqsubseteq f$ and subsequence monotonicity of Γ, $\Gamma(f) \leq \Gamma(g)$. Hence, $A^{\Phi,\Gamma(g)}_g \subseteq A^{\Phi,\Gamma(f)}_f$ and thus $\Sigma(g) \leq \Sigma(f)$. However, subsequence monotonicity of Σ and $g \sqsubseteq f$ imply $\Sigma(f) \leq \Sigma(g)$. Thus, $\Sigma(f) = \Sigma(g)$. Now, suppose $\Gamma(f) < \Gamma(g)$. This implies that there is an $h \sqsubseteq g$ with $\mathfrak{C}_h = \mathfrak{C}_g$, $h \upharpoonright \Gamma(f) = g \upharpoonright \Gamma(f) = f \upharpoonright \Gamma(f)$ and $\Phi(h) > \Sigma(g) = \Sigma(f)$. Thus, $h \in A^{\Phi,\Gamma(f)}_f$ and therefore $\Lambda(f, \Gamma(f)) \geq \Phi(h) > \Sigma(f) = \Lambda(f, \Gamma(f))$, which is a contradiction. One must have that $\Gamma(g) \leq \Gamma(f)$. Since it has already been observed above that $\Gamma(f) \leq \Gamma(g)$, $\Gamma(f) = \Gamma(g)$. This establishes Claim 1.

Define $P : [C_1]^{\omega_1}_* \to 2$ by $P(f) = 0$ if and only if $\Phi(f) = \Sigma(f)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C_2 \subseteq C_1$ which is homogeneous for P.

Claim 2: C_2 is homogeneous for P taking value 0.

To see Claim 2: Pick $f \in [C_2]^{\omega_1}_*$. Let $\rho : \omega \to \Sigma(f)$ be an increasing cofinal sequence with $\rho(0) = \Gamma(f)$. One will construct a sequence $\langle f_n : n \in \omega \rangle$ with the following properties:

1. $f_0 = f$. For all $n \in \omega$, $f_n \in [C_2]^{\omega_1}_*$, $f_{n+1} \sqsubseteq f_n$ and $\mathfrak{C}_{f_n} = \mathfrak{C}_{f_{n+1}} = \mathfrak{C}_f$.
2. For all $n \in \omega$, for all $\alpha \geq \Sigma(f)$, $f_n(\alpha) = f(\alpha)$.
3. For all $n \in \omega$, $f_{n+1} \upharpoonright \rho(n) = f_n \upharpoonright \rho(n)$.
4. For all $n \in \omega$, $\rho(n) < \Phi(f_n) \leq \Sigma(f)$.

Let $f_0 = f$ and note that $\rho(0) = \Gamma(f) < \Phi(f) = \Phi(f_0)$. Suppose f_n has been constructed satisfying the above four properties. Since $f_n \sqsubseteq f$, Claim 1 implies $\Sigma(f_n) = \Sigma(f)$ and $\Gamma(f_n) = \Gamma(f) \leq \rho(n)$. By condition (4), $\rho(n) < \Phi(f_n)$ and therefore by the main property of Γ, $\Lambda(f_n, \rho(n)) = \Lambda(f_n, \Gamma(f_n)) = \Sigma(f_n) = \Sigma(f)$. Since $\rho(n+1) < \Sigma(f)$, there exists an $h \sqsubseteq f_n$ with $\mathfrak{C}_h = \mathfrak{C}_{f_n}$, $h \upharpoonright \rho(n) = f_n \upharpoonright \rho(n)$ and $\rho(n+1) < \Phi(h) \leq \Sigma(h) = \Sigma(f)$ by Claim 1. Since $\Phi(h) \in \mathfrak{C}_h$ (is a closure point of h) by the assumptions, $\Phi(h)$ is a limit ordinal and therefore, $\Phi(h) \leq \sup(h \upharpoonright \Phi(h)) < h(\Phi(h))$. Since $\Phi(h) \leq \Sigma(f)$, $h \upharpoonright \Sigma(f)$ is a continuity point for Φ relative to C_2. Define $f_{n+1} \in [C_2]^{\omega_1}_*$ by

$$f_{n+1}(\alpha) = \begin{cases} h(\alpha) & \alpha < \Sigma(f) \\ f(\alpha) & \Sigma(f) \leq \alpha \end{cases}.$$

Note that f_{n+1} is indeed an increasing and discontinuous function since $\Sigma(f) \in \mathfrak{C}_f$ and $\mathfrak{C}_h = \mathfrak{C}_f$ imply that $\sup(f_{n+1} \upharpoonright \Sigma(f)) = \sup(h \upharpoonright \Sigma(f)) = \Sigma(f) < f(\Sigma(f)) = f_{n+1}(\Sigma(f))$. Since $h \upharpoonright \Sigma(f)$ is a continuity point for Φ relative to C_2, $\Phi(f_{n+1}) = \Phi(h)$. This function f_{n+1} satisfies all the required properties relative to the previous f_n.

By DC, there is a sequence $\langle f_n : n \in \omega \rangle$ with all the required properties. Define $f_\omega \in [C_2]^{\omega_1}_*$ by $f_\omega(\alpha) = \sup\{f_n(\alpha) : n < \omega\}$. Note that for all n, $f_\omega \sqsubseteq f_n$, $\mathfrak{C}_{f_\omega} = \mathfrak{C}_f$ and $f_\omega \upharpoonright \rho(n) = f_n \upharpoonright \rho(n)$. (Note that $\Sigma(f) \in \mathfrak{C}_{f_\omega}$ since $\Sigma(f) \in \mathfrak{C}_{f_n}$ for all $n \in \omega$.) Since Φ satisfies subsequence monotonicity on C_2, $f_\omega \sqsubseteq f_n$ implies that $\rho(n) < \Phi(f_n) \leq \Phi(f_\omega)$. Thus, $\Sigma(f) \leq \Phi(f_\omega)$. Since $f_\omega \sqsubseteq f$ and $f_\omega \upharpoonright \Gamma(f) = f \upharpoonright \Gamma(f)$, Claim 1 implies that $\Sigma(f) = \Sigma(f_\omega)$. By definition, $\Phi(f_\omega) \leq \Sigma(f_\omega)$. Hence, $\Sigma(f_\omega) = \Phi(f_\omega)$. Since $f_\omega \in [C_2]^{\omega_1}_*$, $P(f_\omega) = 0$ and C_2 is homogeneous for P, one must have that C_2 is homogeneous for P taking value 0. This completes the argument for Claim 2.

For each $f \in [C_2]^{\omega_1}_*$, $\Phi(f) = \Sigma(f) = \Lambda(f, \Gamma(f))$. This implies the club C_2 and the function Γ satisfy condition (1) and (3). For each $f \in [C_2]^{\omega_1}_*$, let $\delta_f = \mathrm{ot}(\{\alpha : \Gamma(f) \leq \alpha < \Phi(f)\})$. Let ϵ_f be the least ordinal so that there exists an additively indecomposable ordinal ν_f with $\delta_f = \epsilon_f + \nu_f$. Let $\Gamma^\Phi(f) = \Gamma(f) + \epsilon_f$. Now, Γ^Φ satisfies all the desired properties. □

Definition 6.16. Let $n \in \omega$, $C \subseteq \omega_1$ be a club and $\Gamma_0, ..., \Gamma_{n-1} : [C]^{\omega_1}_* \to \omega_1$ be a sequence of functions. $\langle C, \Gamma_0, ..., \Gamma_{n-1} \rangle$ is a good sequence if and only if the following holds.

1. For all $f \in [C]^{\omega_1}_*$ and for all $k < n - 1$, $\Gamma_{k+1}(f) < \Gamma_k(f)$.
2. For each $i < n - 1$, $\mathrm{ot}(\{\alpha : \Gamma_{i+1}(f) \leq \alpha < \Gamma_i(f)\})$ is an additively indecomposable ordinals or the ordinal 1.

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3. For all $j < n-1$, for all $f \in [C]^{\omega_1}_*$, for all $g \in [C]^{\omega_1}_*$, if
   - $g \sqsubseteq f$ and $\mathfrak{C}_g = \mathfrak{C}_f$,
   - for all $j < k < n-1$, $\sup(g \upharpoonright \Gamma_k(f)) = \sup(f \upharpoonright \Gamma_k(f))$,
   - and $g \upharpoonright \Gamma_{n-1}(f) = f \upharpoonright \Gamma_{n-1}(f)$,

   then $\Gamma_j(f) = \Gamma_j(g)$.

Lemma 6.17. *Assume* DC*, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and that the almost everywhere short length club uniformization principle holds at ω_1. If $\langle D, \Gamma_0, ..., \Gamma_{n-1}\rangle$ is a good sequence so that Γ_{n-1} is not $\mu^{\omega_1}_{\omega_1}$-almost everywhere the constant 0 function, then there is a $C \subseteq D$ and a function Γ_n so that $\langle C, \Gamma_0, ..., \Gamma_{n-1}, \Gamma_n\rangle$ is a good sequence.*

Proof. It will be shown by induction on the length $n \geq 1$ of the good sequence. Suppose $\langle D, \Gamma_0\rangle$ is a good sequence so that Γ_0 is not $\mu^{\omega_1}_{\omega_1}$-almost everywhere constantly 0. Depending on whether for $\mu^{\omega_1}_{\omega_1}$-almost all f, $\Gamma_0(f)$ is a successor ordinal, a limit ordinal which is not a closure point of f or a closure point of f, let $C \subseteq D$ and $\Gamma_1 = \Gamma^{\Gamma_0}$ be given by Lemma 6.10, Lemma 6.11 or Lemma 6.15. $\langle C, \Gamma_0, \Gamma_1\rangle$ is the desired extension.

Now, suppose that $n > 1$ and that any length $n-1$ good sequence where the last function is not $\mu^{\omega_1}_{\omega_1}$-almost everywhere constantly 0 can be extended. Let $\langle D, \Gamma_0, ..., \Gamma_{n-1}\rangle$ be a good sequence of length n with Γ_{n-1} not $\mu^{\omega_1}_{\omega_1}$-almost everywhere constantly 0. The restriction $\langle D, \Gamma_1, ..., \Gamma_{n-1}\rangle$ is a length $n-1$ good sequence. Applying the induction hypothesis to this sequence, there is a $D_0 \subseteq D$ and a function Γ so that $\langle D_0, \Gamma_1, , ..., \Gamma_{n-1}, \Gamma\rangle$ is a length n good sequence and by applying Fact 6.9 to Γ, one may also assume that for all $f \in [D_0]^{\omega_1}_*$ and $\alpha < \omega_1$, if $\Gamma(f) < f(\alpha)$, then $f \upharpoonright \alpha$ is a continuity point for Γ relative to D_0.

(Case 1) For all $f \in [D_0]^{\omega_1}_*$, $\mathrm{ot}(\{\alpha : \Gamma(f) \leq \alpha < \Gamma_{n-1}(f)\}) = 1$.

Setting $C = D_0$ and $\Gamma_n = \Gamma$, $\langle C, \Gamma_0, ..., \Gamma_{n-1}, \Gamma_n\rangle$ is the desired extension.

(Case 2) For all $f \in [D_0]^{\omega_1}_*$, $\mathrm{ot}\{(\alpha : \Gamma(f) \leq \alpha < \Gamma_{n-1}(f)\})$ is an indecomposable ordinal.

For all β such that $\Gamma(f) \leq \beta < \Gamma_{n-1}(f)$, let

$$T^\beta_f = \{g \sqsubseteq f : \mathfrak{C}_g = \mathfrak{C}_f \wedge g \upharpoonright \beta = f \upharpoonright \beta \wedge (\forall 0 < i \leq n-1)(\sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(f)))\}.$$

Let $\mathfrak{E}^\beta_f = \{\Gamma_0(g) : g \in T^\beta_f\}$. Let $\Lambda(f, \beta) = \sup(\mathfrak{E}^\beta_f)$ which is an ordinal less than ω_1 by Fact 6.14 since $T^f_\beta \subseteq A^{\Gamma_0,0}_f$ and hence $\mathfrak{E}^\beta_f \subseteq \mathfrak{B}^{\Gamma_0,0}_f$ (recall that $A^{\Gamma_0,0}_f$ and $\mathfrak{B}^{\Gamma_0,0}_f$ were defined in Definition 6.13). Note that if $\Gamma(f) \leq \beta_0 \leq \beta_1 < \Gamma_{n-1}(f)$, then $\Lambda(f, \beta_1) \leq \Lambda(f, \beta_0)$. Let $Z_f = \{\Lambda(f, \beta) : \Gamma(f) \leq \beta < \Gamma_{n-1}(f)\}$. Define $\Gamma_n : [D_0]^{\omega_1}_* \to \omega_1$ by $\Gamma_n(f) = \min\{\beta : \Gamma(f) \leq \beta < \Gamma_{n-1}(f) \wedge \min(Z_f) = \Lambda(f, \beta)\}$. Define $\Sigma : [D_0]^{\omega_1}_* \to \omega_1$ by $\Sigma(f) = \Lambda(f, \Gamma_n(f))$. The main property is that for all β with $\Gamma_n(f) \leq \beta < \Gamma_{n-1}(f)$, $\Lambda(f, \beta) = \Lambda(f, \Gamma_n(f)) = \Sigma(f)$. Applying Fact 6.12 to Σ, Γ_n and Γ_0, there is a club $D_1 \subseteq D_0$ so that Σ, Γ_n and Γ_0 are subsequence monotonic on D_1.

Claim 1: For all $f \in [D_1]^{\omega_1}_*$, if $g \in [D_1]^{\omega_1}_*$ has the property that $g \sqsubseteq f$, $\mathfrak{C}_g = \mathfrak{C}_f$, for all $1 \leq i \leq n-1$, $\sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(f))$ and $f \upharpoonright \Gamma_n(f) = g \upharpoonright \Gamma_n(f)$, then $\Sigma(f) = \Sigma(g)$ and $\Gamma_n(f) = \Gamma_n(g)$.

To see Claim 1: Since $g \sqsubseteq f$, subsequence monotonicity implies that $\Gamma_n(f) \leq \Gamma_n(g)$. Since $\Gamma(f) \leq \Gamma_n(f)$ and $\langle D_0, \Gamma_1, ..., \Gamma_{n-1}, \Gamma\rangle$ is a good sequence, one has by Definition 6.16 condition (3) that for all $1 \leq i \leq n-1$, $\Gamma_i(f) = \Gamma_i(g)$. Hence, one has that $T^{\Gamma_n(g)}_g \subseteq T^{\Gamma_n(f)}_f$ and hence $\Sigma(g) \leq \Sigma(f)$. However, by subsequence monotonocity, one has $\Sigma(f) \leq \Sigma(g)$. Hence, $\Sigma(f) = \Sigma(g)$. Suppose for sake of contradiction that $\Gamma_n(f) < \Gamma_n(g)$. Since Γ satisfies the fine continuity property of Fact 6.9 relative to D_0, $\Gamma(g) = \Gamma(f)$. Now, since $\Gamma(g) = \Gamma(f) < \Gamma_n(f) < \Gamma_n(g)$, the definition of $\Gamma_n(g)$ implies there is an h with $\mathfrak{C}_h = \mathfrak{C}_g$, $h \upharpoonright \Gamma_n(f) = g \upharpoonright \Gamma_n(f) = f \upharpoonright \Gamma_n(f)$, for all $1 \leq i \leq n-1$, $\sup(h \upharpoonright \Gamma_i(f)) = \sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(h))$ and $\Gamma_0(h) > \Sigma(g) = \Sigma(f)$. However, $h \in T^{\Gamma_n(f)}_f$ and thus $\Lambda(f, \Gamma(f)) \geq \Gamma_0(h) > \Sigma(f) = \Lambda(f, \Gamma(f))$ which is a contradiction. This shows that one must have $\Gamma_n(f) = \Gamma_n(g)$. The proof of Claim 1 is complete.

Define $P : [D_1]^{\omega_1}_* \to 2$ by $P(f) = 0$ if and only if $\Gamma_0(f) = \Sigma(f)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $D_2 \subseteq D_1$ which is homogeneous for P.

Claim 2: D_2 is homogeneous for P taking value 0.

First, if there is an $f \in [D_2]^{\omega_1}_*$ such that $\Sigma(f)$ is a successor, then the supremum in the definition of $\Sigma(f) = \Lambda(f, \Gamma_n(f))$ is obtained. That is, there is a $g \sqsubseteq f$ with $\mathfrak{C}_g = \mathfrak{C}_f$, $g \upharpoonright \Gamma_n(f) = f \upharpoonright \Gamma_n(f)$, for all $1 \leq i \leq n-1$, $\sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(f))$ and $\Gamma_0(g) = \Sigma(f)$. By Claim 1, $\Sigma(g) = \Sigma(f)$. Thus, $P(g) = 0$ and since $g \in [D_2]^{\omega_1}_*$, D_2 must be homogeneous for C taking value 0.

Thus, assume that for all $f \in [D_2]^{\omega_1}_*$, $\Sigma(f)$ is a limit ordinal. Pick any $f \in [D_2]^{\omega_1}_*$. Let $\rho : \omega \to \Gamma_{n-1}(f)$ be an increasing cofinal sequence through $\Gamma_{n-1}(f)$ with $\rho(0) = \Gamma_n(f)$. Let $\tau : \omega \to \Sigma(f)$ be an increasing sequence through $\Sigma(f)$. One will construct a sequence $\langle f_k : k \in \omega\rangle$ with the following properties:

1. $f_0 = f$. For all $k \in \omega$, $f_k \in [D_2]^{\omega_1}_*$, $f_{k+1} \sqsubseteq f_k$ and $\mathfrak{C}_{f_{k+1}} = \mathfrak{C}_{f_k} = \mathfrak{C}_f$.
2. For all $k \in \omega$, for all $\alpha \geq \Gamma_{n-1}(f)$, $f_k(\alpha) = f(\alpha)$.
3. For all $k \in \omega$, $f_{k+1} \upharpoonright \rho(k) = f_k \upharpoonright \rho(k)$.
4. For all $k \in \omega$, $\tau(k) < \Gamma_0(f_{k+1}) \leq \Sigma(f)$.

Let $f_0 = f$. Suppose f_k has been constructed satisfying the above properties. Claim 1 implies that $\Sigma(f_k) = \Sigma(f)$ and $\Gamma_n(f_k) = \Gamma_n(f)$. The main property above and the fact that $\rho(k) \geq \rho(0) = \Gamma_n(f_k)$ give that $\Lambda(f_k, \rho(k)) = \Lambda(f_k, \Gamma_n(f_k)) = \Sigma(f_k) = \Sigma(f)$. Thus, there is an $h \sqsubseteq f_k$ with $\mathfrak{C}_h = \mathfrak{C}_{f_k}$, $h \upharpoonright \rho(k) = f_k \upharpoonright \rho(k)$, for all $1 \leq i \leq n-1$, $\sup(h \upharpoonright \Gamma_i(f_k)) = \sup(f_k \upharpoonright \Gamma_i(f_k))$ and $\tau(k) < \Gamma_0(h) \leq \Sigma(f_k)$. Now, define f_{k+1} by

$$f_{k+1}(\alpha) = \begin{cases} h(\alpha) & \alpha < \Gamma_{n-1}(f) \\ f(\alpha) & \Gamma_{n-1}(f) \leq \alpha \end{cases}.$$

Since $\langle D_0, \Gamma_1, ..., \Gamma_{n-1}, \Gamma\rangle$ is a good sequence and $h \upharpoonright \Gamma(f) = f \upharpoonright \Gamma(f)$, one has that $\Gamma_i(h) = \Gamma_i(f)$ for all $1 \leq i \leq n-1$. (In particular, $\Gamma_{n-1}(h) = \Gamma_{n-1}(f)$.) Since $f_{k+1} \upharpoonright \Gamma_{n-1}(h) = h \upharpoonright \Gamma_{n-1}(h)$, for all $i < n-1$, $\sup(f_{k+1} \upharpoonright \Gamma_i(h)) = \sup(h \upharpoonright \Gamma_i(h))$ and $\langle D, \Gamma_0, ..., \Gamma_{n-1}\rangle$ is a good sequence, one has that $\Gamma_0(f_{k+1}) = \Gamma_0(h) > \tau(k)$. Thus, f_{k+1} satisfies the required properties.

By DC, there is a sequence $\langle f_k : k \in \omega\rangle$ with the desired properties. Define $f_\omega \in [D_2]^{\omega_1}_*$ by $f_\omega(\alpha) = \sup\{f_k(\alpha) : k < \omega\}$. Since $f_\omega \sqsubseteq f_{k+1}$, the subsequence monotonicity of Γ_0 implies that $\tau(k) < \Gamma_0(f_{k+1}) \leq \Gamma_0(f_\omega)$. Hence, $\Sigma(f) \leq \Gamma_0(f_\omega)$. Claim 1 implies that $\Sigma(f) = \Sigma(f_\omega)$. Therefore, $\Sigma(f_\omega) \leq \Gamma_0(f_\omega)$. Since $\Gamma_0(f_\omega) \leq \Sigma(f_\omega)$ by definition, one has shown that $\Gamma_0(f_\omega) = \Sigma(f_\omega)$. Since $P(f_\omega) = 0$ and $f_\omega \in [D_2]^{\omega_1}_*$, one must have that D_2 is homogeneous for P taking value 0. This completes the proof of Claim 2.

Claim 2 implies that $\langle C, \Gamma_0, ..., \Gamma_{n-1}, \Gamma_n\rangle$ is a good sequence where $C = D_2$. This completes the argument. □

Theorem 6.18. *Assume* DC, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ *and that the almost everywhere short length club uniformization principle holds at* ω_1. *Let* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. *There is a club* $C \subseteq \omega_1$ *and finitely many functions* $\Upsilon_0, ..., \Upsilon_{n-1}$ *so that for all* $f \in [C]^{\omega_1}_*$, *for all* $g \in [C]^{\omega_1}_*$, *if* $\mathfrak{C}_g = \mathfrak{C}_f$ *and for all* $i < n$, $\sup(g \upharpoonright \Upsilon_i(f)) = \sup(f \upharpoonright \Upsilon_i(f))$, *then* $\Phi(f) = \Phi(g)$.

Proof. Fix $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. Let $\mathcal{T}$ consists of good sequences $\langle C, \Gamma_0, ..., \Gamma_{n-1}\rangle$ with $\Gamma_0(f) = \Phi(f)$ for all $f \in C$. Define an ordering on $\prec$ on $\mathcal{T}$ by $\langle D, \Psi_0, ..., \Psi_{m-1}\rangle \prec \langle C, \Gamma_0, ..., \Gamma_{n-1}\rangle$ if and only if $n < m$, $D \subseteq C$ and for all $f \in [D]^{\omega_1}_*$, for all $i < n$, $\Gamma_i(f) = \Psi_i(f)$.

Claim 1: There is a $\langle C, \Gamma_0, ..., \Gamma_{n-1}\rangle \in \mathcal{T}$ so that Γ_{n-1} is $\mu^{\omega_1}_{\omega_1}$-almost everywhere constantly 0.

To see Claim 1: Suppose not. Then Lemma 6.17 implies that $(\mathcal{T}, \prec)$ is a tree with no dead branches. DC implies there is an infinite $\prec$-descending sequence $\langle\langle C_j, \Gamma^j_0, ..., \Gamma^j_j\rangle : j \in \omega\rangle$. Let $C = \bigcap_{j<\omega} C_j$. Pick $f \in [C]^{\omega_1}_*$. Then $\langle\Gamma^i_i(f) : i \in \omega\rangle$ is an infinite descending sequence of ordinals. Contradiction. This completes the proof of Claim 1.

Let $\langle C, \Gamma_0, ..., \Gamma_{n-1}\rangle$ be a good sequence so that for all $f \in [C]^{\omega_1}_*$, $\Gamma_0(f) = \Phi(f)$ and $\Gamma_{n-1}(f) = 0$. Now, suppose $f, g \in [C]^{\omega_1}_*$ with the property that $\mathfrak{C}_g = \mathfrak{C}_f$ and for all $0 < i < n$, $\sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(f))$. Let $h : \omega_1 \to \omega_1$ be defined by $h(0) = \min\{f[\omega_1] \cup g[\omega_1]\}$. Suppose $h \upharpoonright \alpha$ has

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been defined. Let $h(\alpha)$ be the least element of $f[\omega_1] \cup g[\omega_1]$ greater than $\sup(h \upharpoonright \alpha)$. Note that $h \in [C]^{\omega_1}_*$, that is, is increasing, discontinuous, and has uniform cofinality ω. For each $0 < i < n-1$, let $K_i = \{\alpha : \Gamma_{i+1}(f) \leq \alpha < \Gamma_i(f)\}$ and $\nu_i = \mathrm{ot}(K_i)$ which is either an additively indecomposable ordinal or 1. Therefore, for each $0 < i < n-1$, $\mathrm{ot}(\{f(\alpha) : \alpha \in K_i\}) = \mathrm{ot}(\{g(\alpha) : \alpha \in K_i\}) = \mathrm{ot}(\{f(\alpha), g(\alpha) : \alpha \in K_i\}) = \nu_i$ since ν_i is an additively indecomposable ordinal or 1. Hence, for each $0 < i < n$, $\sup(f \upharpoonright \Gamma_i(f)) = \sup(g \upharpoonright \Gamma_i(f)) = \sup(h \upharpoonright \Gamma_i(f))$, $\mathfrak{C}_f = \mathfrak{C}_g = \mathfrak{C}_h$, $f \sqsubseteq h$ and $g \sqsubseteq h$.

Claim 2: For all $k < n$, $\Gamma_{n-1-k}(h) = \Gamma_{n-1-k}(f) = \Gamma_{n-1-k}(g)$.

To see Claim 2: This will be shown by induction on k. If $k = 0$, then $\Gamma_{n-1}(h) = \Gamma_{n-1}(f) = \Gamma_{n-1}(h) = 0$. Now, suppose $k < n$ and for all $j < k$, it has been shown that $\Gamma_{n-1-j}(h) = \Gamma_{n-1-j}(f) = \Gamma_{n-1-j}(g)$. Since it was shown above that $f \sqsubseteq h$, $\mathfrak{C}_f = \mathfrak{C}_h$, $\sup(f \upharpoonright \Gamma_{n-1-j}(h)) = \sup(f \upharpoonright \Gamma_{n-1-j}(f)) = \sup(h \upharpoonright \Gamma_{n-1-j}(f)) = \sup(h \upharpoonright \Gamma_{n-1-j}(h))$ for each $j < k$, Definition 6.16 condition (3) for the pair (f, h) at Γ_{n-1-k} implies that $\Gamma_{n-1-k}(f) = \Gamma_{n-1-k}(h)$. The same argument for the pair (g, h) implies $\Gamma_{n-1-k}(g) = \Gamma_{n-1-k}(h)$. This concludes the proof of Claim 2.

Applying Claim 2 for $k = n-1$, one has that $\Phi(f) = \Gamma_0(f) = \Gamma_0(h) = \Gamma_0(g) = \Phi(g)$. For each $i < n-2$, let $\Upsilon_i = \Gamma_{i+1}$. Then C and the function $\Upsilon_0, \ldots, \Upsilon_{n-2}$ are the desired objects. (This just removes $\Gamma_0 = \Phi$ which is redundant.) □

Next, one will show that the continuity property expressed in Theorem 6.18 holds under the axiom of determinacy. The following is a consequence of the Moschovakis coding lemma.

Fact 6.19. Assume AD. $\mathscr{P}(\omega_1) = (\mathscr{P}(\omega_1))^{L(\mathbb{R})}$.

The next fact asserts that every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is equal to a function in $L(\mathbb{R})\mu^{\omega_1}_{\omega_1}$-almost everywhere.

Fact 6.20. ([2]) Assume AD. Suppose $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. Then there is a club $C \subseteq \omega_1$ so that $\Phi \upharpoonright [C]^{\omega_1}_* \in L(\mathbb{R})$.

It is not known if AD implies $\mathsf{DC}_{\mathbb{R}}$; however, Kechris showed that $L(\mathbb{R})$ satisfies DC.

Fact 6.21. ([11]) Assume AD. Then $L(\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$.

Theorem 6.22. *Assume* AD. *Let* $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$. *There is a club* $C \subseteq \omega_1$ *and finitely many function* $\Gamma_0, \ldots, \Gamma_{n-1}$ *so that for all* $f \in [C]^{\omega_1}_*$, *for all* $g \in [C]^{\omega_1}_*$, *if* $\mathfrak{C}_g = \mathfrak{C}_f$ *and for all* $i < n$, $\sup(g \upharpoonright \Gamma_i(f)) = \sup(f \upharpoonright \Gamma_i(f))$, *then* $\Phi(f) = \Phi(g)$.

Proof. By Fact 6.20, there is a club $C_0 \subseteq \omega_1$ so that $\Phi \upharpoonright [C_0]^{\omega_1}_* \in L(\mathbb{R})$. By Fact 6.21, $L(\mathbb{R})$ satisfies AD and DC. AD implies $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ by Fact 2.16. The almost everywhere short length club uniformization for ω_1 holds by Fact 6.5. Fact 6.19 implies $\omega_1 = (\omega_1)^{L(\mathbb{R})}$. Theorem 6.18 applied inside $L(\mathbb{R})$ for $\Phi \upharpoonright [C_0]^{\omega_1}_*$ will provide a club $C_1 \subseteq C$ and functions $\Gamma_0, \ldots, \Gamma_{n-1}$ which satisfies the required property in $L(\mathbb{R})$. Fact 6.19 will imply that these objects continue to have the desired property in the original universe satisfying determinacy. □

Competing interest. The authors have no competing interest to declare.

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