NORMAL AND QUASINORMAL WEIGHTED COMPOSITION OPERATORS

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Introduction. In their paper [**1**], Campbell and Jamison attempted to give necessary and sufficient conditions for a weighted composition operator on an L^2 space to be normal, and to be quasinormal. Those conditions, specifically Theorems I and II of that paper, are not valid (see [**2**] for precise comments on the other results in that paper). In this paper we present a counterexample to those theorems and state and prove characterizations of quasinormality (Theorem 1 below) and normality (Theorem 2 and Corollary 3 below). We also discuss additional examples and information concerning normal weighted composition operators which contribute to the further understanding of this class.

In what follows, (X, Σ, μ) will be a complete σ-finite measure space. $T: X \to X$ will be a measurable transformation of X onto itself with the properties that the measure $\mu \circ T^{-1}$ is absolutely continuous with respect to μ, and $\mu \circ T^{-1}$ is finite. We set $h = d\mu \circ T^{-1}/d\mu$. By $T^{-1}\Sigma$ we mean the relative completion of the σ-algebra generated by $\{T^{-1}A : A \in \Sigma\}$. With the space X and the measure μ fixed, if $\Gamma \subseteq \Sigma$ is a σ-algebra we write $L^2(\Gamma)$ as the usual equivalence classes of Γ measurable functions whose modulus squared is integrable over X.

We denote by $E: L^2(\Sigma) \to L^2(T^{-1}\Sigma)$ the so-called conditional expectation operator with respect to the σ-algebra $T^{-1}\Sigma$. More generally, $E(f)$ may be defined for bounded measurable functions f or non-negative measurable functions f; for details on the properties of E see [**1**], [**3**], [**4**].

Given a Σ-measurable function $\phi: X \to \mathbb{C}$, the *weighted composition operator* (w.c.o.) induced by T with weight ϕ is defined by

$$W_{T,\phi} f(x) = \phi(x) f(Tx), \qquad f \in L^2(\Sigma).$$

Usually ϕ and T are understood and we just write W. The operator norm of W is $\|W\| = \|hE(|\phi|^2) \circ T^{-1}\|_\infty^{1/2}$ (see [**1**] for a discussion of $E(\cdot) \circ T^{-1}$ when T is not invertible). All of our w.c.o.'s will be bounded. The *support* of a measurable function g is $\bigcup_{n=1}^{\infty} \{x : |g(x)| > 1/n\}$; we shall let $\operatorname{supp} g$ denote the support of g. Equalities and inequalities between measurable functions are interpreted in the almost everywhere sense, and equality between sets is interpreted up to a set of measure 0.

We use the following non-standard notation. Whenever Γ is a sub-σ-algebra of Σ and A is any Σ-measurable set, by $\Gamma \cap A$ we mean $\{B \cap A : B \in \Gamma\}$. The statement $\Gamma \cap A = \Sigma \cap A$ means that for each Σ-measurable set $C \subseteq A$, there exists a set $B \in \Gamma$ so that $B \cap A = C$.

In our statements and proofs of the theorems below, we assume $\phi \geq 0$. The results can be easily extended to the case of a complex-valued ϕ.

The following example illustrates that the characterizations of normal and quasinormal w.c.o.'s given in [**1**] are false.

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EXAMPLE 1. Let $X = \mathbb{Z} \cup \{a\}$ where $a \notin \mathbb{Z}$, $\Sigma = 2^X$, and $\mu(x) = 1$ for each $x \in X$. Define $T: X \to X$ by $T(n) = n + 1$ for $n \in \mathbb{Z}$, and $T(a) = 0$. Define ϕ by $\phi(n) = 1$ for $n \in \mathbb{Z}$ and $\phi(a) = 0$. A straightforward computation shows that W is normal, but that (i) ϕ is not $T^{-1}\Sigma$ measurable, (ii) $T^{-1}(\operatorname{supp} \phi) \neq \operatorname{supp} \phi$, and (iii) $hE(\phi^2) \circ T^{-1} \neq h \circ TE(\phi^2)$. Properties (i) and (ii) invalidate Theorem 1 of **[1]** and property (iii) invalidates Theorem 2 of **[1]**.

The correct characterization of quasinormality is given by the restriction of the condition of Theorem 2 in **[1]** to the support of ϕ.

THEOREM 1. *W is quasinormal if and only if $h \circ TE(\phi^2) = hE(\phi^2) \circ T^{-1}$ on the support of ϕ.*

Proof. Compute $WW^*Wf = \phi h \circ TE(\phi^2) f \circ T$ and $W^*WWf = hE(\phi^2) \circ T^{-1} \phi f \circ T$. Then W is quasinormal if and only if

$$\phi h \circ TE(\phi^2) f \circ T = hE(\phi^2) \circ T^{-1} \phi f \circ T, \quad \text{for all } f \in L^2.$$

This last condition is equivalent to the one in the statement of the theorem. □

REMARK. Theorem 1 may be proved using the polar decomposition approach attempted in **[1]** provided one observes a factor of ϕ in each term of VM and MV, where $M = |W^*W|^{1/2}$ and V is the partial isometry which gives the unique, canonical polar form $VM = W$.

We may now establish the correct characterization of normality in many ways, and the most useful would be one which required the fewest and easiest calculations involving h, ϕ, E, and T. The following theorem gives a minimal set of conditions which are necessary and sufficient for the normality of W. Since $\operatorname{supp} \phi \in \Sigma$, by our previous convention condition (ii) in the following theorem means that for each $B \in \Sigma$, $B \subseteq \operatorname{supp} \phi$, there exists $C \in \Sigma$ so that $T^{-1}C \cap \operatorname{supp} \phi = B$.

THEOREM 2. *W is normal if and only if*
(i) *$\phi E(\phi) h \circ T = hE(\phi^2) \circ T^{-1}$, and*
(ii) *$T^{-1}\Sigma \cap \operatorname{supp} \phi = \Sigma \cap \operatorname{supp} \phi$.*

Proof. Suppose (i) and (ii) are true. Since $h \circ T > 0$, it follows from (i) that $\operatorname{supp} hE(\phi^2) \circ T^{-1} = \operatorname{supp} \phi E(\phi)$. Moreover for any non-negative f we have $\operatorname{supp} fE(f) = \operatorname{supp} f$, so that $\operatorname{supp} hE(\phi^2) \circ T^{-1} = \operatorname{supp} \phi$. Now let B be a $T^{-1}\Sigma$-measurable set with finite measure. From (i) and the support condition just established,

$$\phi E(\phi \chi_B) h \circ T = \phi E(\phi) h \circ T \chi_B = hE(\phi^2) \circ T^{-1} \chi_B, \tag{1}$$

where χ_B denotes the indicator function of B. From (ii) it follows that (1) holds for any Σ-measurable subset B of $\operatorname{supp} \phi$, as long as B has finite measure. Again, since $\operatorname{supp} \phi = \operatorname{supp} hE(\phi^2) \circ T^{-1}$, (1) holds for *all* Σ-measurable sets B of finite measure. Consequently $WW^*\chi_B = W^*W\chi_B$ for all such B, implying that W is normal.

Assume that W is normal. Then $WW^*f = W^*Wf$ for all $f \in L^2(\Sigma)$. This is equivalent to

$$\phi h \circ TE(\phi f) = hE(\phi^2) \circ T^{-1} f, \quad \text{for all } f \in L^2(\Sigma), \tag{2}$$

which implies that

$$\phi h \circ TE(\phi) f = hE(\phi^2) \circ T^{-1} f, \quad \text{for all } f \in L^2(T^{-1}\Sigma). \tag{2$'$}$$

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From (2)′ it follows easily that (i) is true. Combining (i) and equation (2) we have

$$\phi E(\phi f) = \phi E(\phi)f, \quad f \in L^2(\Sigma). \tag{3}$$

We now consider the special case where $\phi > 0$. Then by (3) $E(\phi f) = E(\phi)f$ for all $f \in L^2(\Sigma)$. In particular for $f = \chi_B$ with $\mu(B)$ finite we obtain

$$\int_X \phi\chi_B \, d\mu = \int_X E(\phi\chi_B) \, d\mu = \int_X E(\phi)\chi_B \, d\mu.$$

Therefore

$$\int_B \phi \, d\mu = \int_B E\phi \, d\mu \quad \text{for all} \quad B \in \Sigma, \qquad \mu(B) < \infty.$$

Consequently $E(\phi) = \phi$. It now follows from equation (3) that $E(f) = f$ for all L^2 functions f, or equivalently that $T^{-1}\Sigma = \Sigma$.

We now drop the restriction that $\phi > 0$. Since W is normal, $\overline{\text{ran } W}$ (the closure of the range of W) is a reducing subspace for W. By (3) we have that $WW^*f = \phi h \circ TE(\phi f) = \phi h \circ TE(\phi)f$, so that $\overline{\text{ran } W} = L^2(\text{supp } \phi E(\phi), \Sigma, \mu) = L^2(\text{supp } \phi, \Sigma, \mu)$. Thus $L^2(\text{supp } \phi, \Sigma, \mu)$ is a reducing subspace for W on which W is normal. On the other hand $\overline{\text{ran } W} = \overline{\text{ran } W^*} = L^2(\text{supp } hE(\phi^2) \circ T^{-1})$, so that $\text{supp } \phi = \text{supp } hE(\phi^2) \circ T^{-1}$. Consequently $\phi(Tx) = 0$ implies that $h(Tx)E(\phi^2)(x) = 0$ which implies that $\phi(x) = 0$. Therefore T maps the support of ϕ into itself, and we see that W is a normal w.c.o. on $L^2(\text{supp } \phi, \Sigma, \mu)$. For this space the weight is non-zero, so if we define $\Sigma_1 := \Sigma \cap \text{supp } \phi$ and T_1 as the restriction of T to $\text{supp } \phi$, the preceding paragraph implies that $T_1^{-1}\Sigma_1 = \Sigma_1$, which is precisely condition (ii). □

REMARK 1. In the preceding proof it is shown that if $\phi E(\phi)h \circ T = hE(\phi^2) \circ T^{-1}$ (in particular if W is normal), then $\text{supp } \phi = \text{supp } hE(\phi^2) \circ T^{-1}$. Moreover, if $\text{supp } hE(\phi^2) \circ T^{-1} \subseteq \text{supp } \phi$, then T maps $\text{supp } \phi$ into itself so that W is a weighted composition operator on the invariant subspace $L^2(\text{supp } \phi, \Sigma, \mu)$.

REMARK 2. If $\phi > 0$, then, by Theorem 1, W is quasinormal if and only if $h \circ TE(\phi^2) = hE(\phi^2) \circ T^{-1}$ and, by Theorem 2, W is normal if and only if $T^{-1}\Sigma = \Sigma$ and $\phi^2 h \circ T = h\phi^2 \circ T^{-1}$. In particular if $\phi \equiv 1$, W is quasinormal exactly when $h \circ T = h$ and normal exactly when $h \circ T = h$ and $T^{-1}\Sigma = \Sigma$. These last two results are found in [**5**] and are two of the earliest results connecting operator theoretic properties of W and measure-theoretic properties of T.

In [**4**] Lambert proves that W is hyponormal if and only if (i) $\text{supp } \phi \subseteq \text{supp } hE(\phi^2) \circ T^{-1}$, and (ii) $h \circ TE\left(\dfrac{\phi^2}{hE(\phi^2) \circ T^{-1}}\right) \leq 1$ (where the fraction is interpreted as 0 off of $\text{supp } hE(\phi^2) \circ T^{-1}$). It is reasonable to conjecture that equality in (ii) will imply normality. However equality holds in the following example in which W is not even quasinormal.

EXAMPLE 2. Let $\phi \equiv 1$, $X = \{n\}_{n=0}^{\infty} \cup \{a_k\}_{k=1}^{\infty} \cup \{b_k\}_{k=1}^{\infty}$, Σ the σ-algebra of all subsets of X. Define $T: X \to X$ by $T(a_{k+1}) = a_k$ and $T(b_{k+1}) = b_k$ for $k \geq 1$, $T(a_1) = T(b_1) = 0$, and $T(n) = n + 1$ for all $n \geq 0$. Define the measure by $\mu(a_k) = 2^{k-1}$ and

$\mu(b_k)=2^{2k-1}$ for $k\geq 1$, and $\mu(n)=3^{-n}$ for $n\geq 0$. Direct computation shows that $h\circ TE\left(\frac{\phi^2}{hE(\phi^2)\circ T^{-1}}\right)=1$ but that $h(a_1)\neq h(Ta_1)$ so that W is not quasinormal. □

We can, however, generalize Lambert's result to the normal case, and the following corollary gives conditions for normality analogous to his conditions for hyponormality.

Corollary 3. *W is normal if and only if the following conditions hold.*
(i) $\operatorname{supp}\phi=\operatorname{supp} hE(\phi^2)\circ T^{-1}$,
(ii) $T^{-1}\Sigma\cap\operatorname{supp}\phi=\Sigma\cap\operatorname{supp}\phi$,
and
(iii) $h\circ TE\left(\frac{\phi^2}{hE(\phi^2)\circ T^{-1}}\right)=\chi_{\operatorname{supp} E(\phi)}$,
where the fraction is interpreted to be 0 *off of* $\operatorname{supp} hE(\phi^2)\circ T^{-1}$.

Proof. Assume that W is normal so that (i) and (ii) are true by Remark 1 and Theorem 2. Therefore

$$h\circ TE\left(\frac{\phi^2}{hE(\phi^2)\circ T^{-1}}\right)=h\circ TE\left(\frac{\phi^2}{\phi E(\phi)h\circ T}\right)=E\left(\frac{\phi}{E(\phi)}\chi_{\operatorname{supp} E(\phi)}\right),$$

where the last equality follows since $\operatorname{supp}\phi\subseteq\operatorname{supp}E(\phi)$. Finally this last expression equals $\chi_{\operatorname{supp} E(\phi)}$.

Now assume that (i), (ii), and (iii) are true. By Theorem 2 we need only show that $\phi E(\phi)h\circ T=hE(\phi^2)\circ T^{-1}$. It follows from (iii) that

$$E\left(\phi\frac{\phi E(\phi)h\circ T}{hE(\phi^2)\circ T^{-1}}f\circ T\right)=E(\phi f\circ T),\quad\text{for all}\quad f\in L^2(\Sigma),$$

or $E(k\phi f\circ T)=0$ for all $f\in L^2(\Sigma)$, where $k=\frac{\phi E(\phi)h\circ T}{hE(\phi^2)\circ T^{-1}}-1$. In particular, if $C\in\Sigma$ has finite measure and $f=\chi_C$ we may conclude that

$$\int_{T^{-1}(C)\cap\operatorname{supp}\phi} k\phi\,d\mu=0.$$

Condition (ii) now implies that $k\phi=0$. This, together with condition (i), implies that $\phi E(\phi)h\circ T=hE(\phi^2)\circ T^{-1}$, and the conditions of Theorem 2 are satisfied. □

It is an open question whether conditions analogous to those in Corollary 3 exist in the quasinormal case.

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