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TRACE INEQUALITIES AND A QUESTION OF BOURIN

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Abstract

We give an affirmative answer to one of the questions posed by Bourin regarding a special type of inequality referred to as subadditivity inequalities in the case of the Hilbert–Schmidt and the trace norms. We formulate the solution for arbitrary commuting positive operators, and we conjecture that it is true for all unitarily invariant norms and all commuting positive operators. New related trace inequalities are also presented.

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1. Introduction

It is known that for a nonnegative concave function f on $[0, \infty)$, the subadditivity relation

$$f(a + b) \le f(a) + f(b)$$

holds for all $a, b \ge 0$.

Bourin and Uchiyama [6] have obtained a noncommutative version of this inequality for all positive operators on a finite-dimensional Hilbert space. This version says that if A and B are positive operators, then

$$|||f(A + B)||| \le |||f(A) + f(B)|||,$$

where $|||\cdot|||$ denotes any unitarily invariant norm. This result is due to Kosem [8] for the spectral norm.

Recently, Bourin [5] has generalised this inequality to normal operators.

If A and B are normal operators and $f : [0, \infty) \to [0, \infty)$ is a concave function, then

$$|||f(|A + B|)||| \le |||f(|A|) + f(|B|)|||$$

for every unitarily invariant norm $|||\cdot|||$, where $|X| = (X^*X)^{1/2}$.

An important special case of the above inequality is

$$|||\,|A + B|^p||| \leq |||\,|A|^p + |B|^p|||$$

for $0 < p \leq 1$ (see also [1, 3]).

This inequality prompted Bourin [4] to ask the following related questions.

QUESTION 1.1. Given $A, B \geq 0$ and $p, q > 0$, is it true that

$$|||A^{p+q} + B^{p+q}||| \leq |||(A^p + B^p)(A^q + B^q)|||?$$

Bourin also wondered whether stronger inequalities like

$$|||A^{p+q} + B^{p+q}||| \leq |||(A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2}|||$$

hold true.

Actually, we address and settle the above question affirmatively for the trace norm $\|\cdot\|_1$ and the Hilbert–Schmidt norm $\|\cdot\|_2$. We formulate the solution for arbitrary commuting positive operators, and we conjecture that the above question is true for all unitarily invariant norms and for all commuting positive operators.

CONJECTURE 1.2. Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then

$$|||A_1A_2 + B_1B_2||| \leq |||(A_1 + B_1)(A_2 + B_2)|||.$$

We also wonder whether a stronger inequality holds true.

CONJECTURE 1.3. Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then

$$|||A_1A_2 + B_1B_2||| \leq |||(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}|||.$$

Section 4 demonstrates some new related trace inequalities.

To prove our main results, we will use some useful trace inequalities which are presented in the following section.

2. Preliminary results

We start with some simple well-known facts that will be used in proving our main results.

(i) If A and B are any two operators, then $\operatorname{tr} AB = \operatorname{tr} BA$.

(ii) If $A, B \geq 0$, then

$$\operatorname{tr} AB \geq 0. \tag{2.1}$$

(iii) If $A \geq 0$ and C is any operator, then $CAC^* \geq 0$.

(iv) If $A, B \geq 0$ and $AB = BA$, then

$$AB \geq 0. \tag{2.2}$$

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Other results that will be needed are the following lemmas.

LEMMA 2.1. *Let A, B be any two operators such that the product AB is self-adjoint. Then, for every unitarily invariant norm $|||\cdot|||$,*

$$|||AB||| \leq |||\mathrm{Re}\, BA|||.$$

PROOF. See [7]. □

LEMMA 2.2. *Let $A, B \geq 0$. Then*

$$|||A^{1/2}BA^{1/2}||| \leq |||AB|||. \tag{2.3}$$

PROOF. The proof follows by using Lemma 2.1. □

LEMMA 2.3 (Weyl's Majorant Theorem). *If X is any operator and $m \in \mathbb{N}$, then*

$$|\mathrm{tr}\, X^m| \leq \mathrm{tr}\, |X|^m.$$

PROOF. See [2, p. 279]. □

Though we confine our discussion to operators on a finite-dimensional Hilbert space, by slight modifications our results can be extended to operators on an infinite-dimensional Hilbert space.

3. Main results

Now we use the facts of the previous section to prove our main results.

The following theorem gives the answer to the above question posed by Bourin for the case of the trace norm in a more general setting in terms of commuting positive operators.

THEOREM 3.1. *Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\|A_1A_2 + B_1B_2\|_1 \leq \|(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}\|_1.$$

PROOF. We have

$$\begin{aligned}
&\|(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}\|_1 \\
&\quad = \mathrm{tr}(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2} \\
&\quad = \mathrm{tr}(A_1 + B_1)(A_2 + B_2) \\
&\quad = \mathrm{tr}((A_1A_2 + B_1B_2) + (A_1B_2 + B_1A_2)) \\
&\quad \geq \mathrm{tr}(A_1A_2 + B_1B_2) \quad (\text{by } (2.1)) \\
&\quad = \|A_1A_2 + B_1B_2\|_1 \quad (\text{by } (2.2)).
\end{aligned}$$

This concludes the proof. □

Using (2.3), we have the following corollary.

COROLLARY 3.2. *Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\|A_1A_2 + B_1B_2\|_1 \leq \|(A_1 + B_1)(A_2 + B_2)\|_1.$$

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Corollary 3.3. *Let* $A, B \geq 0$ *and* $p, q > 0$. *Then*

$$\|A^{p+q} + B^{p+q}\|_1 \leq \|(A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2}\|_1.$$

Corollary 3.4. *Let* $A, B \geq 0$ *and* $p, q > 0$. *Then*

$$\|A^{p+q} + B^{p+q}\|_1 \leq \|(A^p + B^p)(A^q + B^q)\|_1.$$

The following theorem gives the answer to the same question posed by Bourin for the case of the Hilbert–Schmidt norm in a more general setting in terms of commuting positive operators.

Theorem 3.5. *Let* $A_1, A_2, B_1, B_2 \geq 0$ *with* $A_1A_2 = A_2A_1$ *and* $B_1B_2 = B_2B_1$. *Then*

$$\|A_1A_2 + B_1B_2\|_2 \leq \|(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}\|_2.$$

Proof. We have

$$\begin{aligned}
&\|(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}\|_2^2 \\
&\qquad = \mathrm{tr}((A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2})^2 \\
&\qquad = \mathrm{tr}(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)(A_2 + B_2)(A_1 + B_1)^{1/2} \\
&\qquad = \mathrm{tr}((A_1 + B_1)(A_2 + B_2))^2 \\
&\qquad = \mathrm{tr}((A_1A_2 + B_1B_2) + (A_1B_2 + B_1A_2))^2 \\
&\qquad = \mathrm{tr}(A_1A_2 + B_1B_2)^2 + 2\mathrm{tr}(A_1A_2 + B_1B_2)(A_1B_2 + B_1A_2) + \mathrm{tr}(A_1B_2 + B_1A_2)^2 \\
&\qquad = \|A_1A_2 + B_1B_2\|_2^2 + 2\mathrm{tr}(A_1A_2A_1B_2 + A_1A_2B_1A_2 + B_1B_2A_1B_2 + B_1B_2B_1A_2) \\
&\qquad\qquad + \mathrm{tr}(A_1B_2A_1B_2 + A_1B_2B_1A_2 + B_1A_2A_1B_2 + B_1A_2B_1A_2) \quad (\text{by } (2.2)) \\
&\qquad \geq \|A_1A_2 + B_1B_2\|_2^2 \quad (\text{by } (2.1)).
\end{aligned}$$

This concludes the proof. □

Using (2.3), we have the following corollary.

Corollary 3.6. *Let* $A_1, A_2, B_1, B_2 \geq 0$ *with* $A_1A_2 = A_2A_1$ *and* $B_1B_2 = B_2B_1$. *Then*

$$\|A_1A_2 + B_1B_2\|_2 \leq \|(A_1 + B_1)(A_2 + B_2)\|_2.$$

Corollary 3.7. *Let* $A, B \geq 0$ *and* $p, q > 0$. *Then*

$$\|A^{p+q} + B^{p+q}\|_2 \leq \|(A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2}\|_2.$$

Corollary 3.8. *Let* $A, B \geq 0$ *and* $p, q > 0$. *Then*

$$\|A^{p+q} + B^{p+q}\|_2 \leq \|(A^p + B^p)(A^q + B^q)\|_2.$$

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4. Some new trace inequalities

We start with the following two lemmas. The second lemma contains a generalisation of (2.1).

Lemma 4.1. *Let $A, B \geq 0$ and $n \in \mathbb{N}$. Then*

$$(AB)^n A \geq 0 \quad \textit{and} \quad B(AB)^n \geq 0.$$

Proof. We will prove the inequality $(AB)^n A \geq 0$ by induction. The proof of the second inequality is similar.

For $n = 1$, it is clear that $(AB)^1 A = ABA \geq 0$. Assume that the statement is true for all $k \leq n$, that is

$$(AB)^k A \geq 0$$

for all $k \leq n$. So, in particular,

$$(AB)^{n-1} A \geq 0. \tag{4.1}$$

Now let $k = n + 1$. By (4.1),

$$(AB)^{n+1} A = (AB)((AB)^{n-1} A)(AB)^* \geq 0. \tag{4.2}$$

This proves the inequality for $k = n + 1$, and completes the proof of the lemma. □

Lemma 4.2. *Let $A, B \geq 0$ and $n \in \mathbb{N}$. Then*

$$\operatorname{tr}(AB)^n \geq 0.$$

Proof. By Lemma 4.1, we have $(AB)^{n-1} A \geq 0$. This, together with (2.1), implies that $\operatorname{tr}(AB)^n = \operatorname{tr}(AB)^{n-1} AB \geq 0$. □

It should be mentioned here that Lemma 4.2 can also be proved using the spectral mapping theorem and the fact that the trace of an operator is the sum of its eigenvalues.

In view of (2.1) (the case $n = 1$ of Lemma 4.2), it is reasonable to conjecture that if $A, B, C \geq 0$, then $\operatorname{tr} ABC \geq 0$. However, this can be refuted by two-dimensional examples. If, in addition, we assume that $BC = CB$, then we have the following result, which generalises Lemma 4.2.

Proposition 4.3. *Let $A, B, C \geq 0$ with $BC = CB$ and $n \in \mathbb{N}$. Then*

$$\operatorname{tr}(AB)^n C \geq 0.$$

Proof. By Lemma 4.1, we have $(AB)^{n-1} A \geq 0$. This, together with (2.1) and (2.2), implies that $\operatorname{tr}(AB)^n C = \operatorname{tr}(AB)^{n-1} ABC \geq 0$. □

Now we will present some new trace inequalities related to the question posed by Bourin [4].

Theorem 4.4. *Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\operatorname{tr}(A_1B_2 + B_1A_2)^2 \leq \operatorname{tr}(A_1B_2 + B_1A_2)(A_2B_1 + B_2A_1).$$

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PROOF. We have

$$\begin{aligned}\operatorname{tr}(A_1B_2 + B_1A_2)^2 &= \operatorname{tr}(A_1B_2)^2 + \operatorname{tr}(B_1A_2)^2 + \operatorname{tr}A_1B_2B_1A_2 + \operatorname{tr}B_1A_2A_1B_2\\ &= \operatorname{tr}(A_1B_2)^2 + \operatorname{tr}(B_1A_2)^2 + 2\operatorname{tr}A_1A_2B_1B_2.\end{aligned}$$

Using (2.1) and Lemma 4.2, it can be easily shown that $\operatorname{tr}(A_1B_2 + B_1A_2)^2 \geq 0$. Now applying Lemma 2.3 to the operator $A_1B_2 + B_1A_2$ with $m = 2$ gives the result. □

COROLLARY 4.5. *Let $A, B \geq 0$ and $p, q > 0$. Then*

$$\operatorname{tr}(A^pB^q + B^pA^q)^2 \leq \operatorname{tr}(A^pB^q + B^pA^q)(A^qB^p + B^qA^p).$$

Using an argument similar to that used in the proof of Theorem 4.4, we can prove the following related trace inequality.

THEOREM 4.6. *Let $A_1, A_2, B_1, B_2 \geq 0$ with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\operatorname{tr}(A_1B_1 + B_2A_2)^2 \leq \operatorname{tr}(A_1B_1 + B_2A_2)(B_1A_1 + A_2B_2).$$

COROLLARY 4.7. *Let $A, B \geq 0$ and $p, q > 0$. Then*

$$\operatorname{tr}(A^pB^p + B^qA^q)^2 \leq \operatorname{tr}(A^pB^p + B^qA^q)(B^pA^p + A^qB^q).$$

References

[1] T. Ando and X. Zhan, 'Norm inequalities related to operator monotone functions', *Math. Ann.* **315** (1999), 771–780.

[2] R. Bhatia, *Matrix Analysis* (Springer, New York, 1997).

[3] R. Bhatia and F. Kittaneh, 'Norm inequalities for positive operators', *Lett. Math. Phys.* **43** (1998), 225–231.

[4] J. C. Bourin, 'Matrix subadditivity inequalities and block-matrices', *Internat. J. Math.* **20** (2009), 679–691.

[5] J. C. Bourin, 'A matrix subadditivity inequality for symmetric norms', *Proc. Amer. Math. Soc.* **138** (2010), 495–504.

[6] J. C. Bourin and M. Uchiyama, 'A matrix subadditivity inequality for $f(A + B)$ and $f(A) + f(B)$', *Linear Algebra Appl.* **423** (2007), 512–518.

[7] F. Kittaneh, 'A note on the arithmatic-geometric mean inequality for matrices', *Linear Algebra Appl.* **171** (1992), 1–8.

[8] T. Kosem, 'Inequalities between $\|f(A + B)\|$ and $\|f(A) + f(B)\|$', *Linear Algebra Appl.* **418** (2006), 153–160.

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