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DEFINABILITY PROBLEMS IN ELEMENTARY TOPOLOGY

MARIKO YASUGI

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Abstract

The elementary part of general topology is carried out in a system which is based on the arithmetically definable theory of the reals with definitions by definable induction (DDI), where a formal object is said to be definable if the quantifiers are restricted to the rationals, the names of the base members and the elements of the spaces.

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Introduction: our stand

The purpose and the guideline of our program to study "definability problems" in analysis were explained in our previous article Yasugi (1981a) and we shall not repeat them here. Let us say, however, a little more about our stand and method. First, our objective is to investigate the logical structure of mathematical thinking in various branches of analysis, independent of specific properties of given spaces. We therefore do not "construct" analysis as a counter-theory to classical mathematics, do not analyze "constructive analysis" nor do we claim "this is constructive analysis". Rather, we study classical texts of mathematics as they stand, formulate the mathematical theories there in a very "economical" formal system and observe the constructive (definable) aspects of them. We fully accept classical mathematics but we are convinced through our endeavors (see above) that mathematical thinking is quite "definable". That is why we use the term "definable" rather than "constructive". This does not mean that we are against the current trends of constructive mathematics; in fact we stand in with them. What we would like to emphasize is that we investigate mathematics from the purely foundational viewpoint.

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It is commonly held that the major obstacle in constructivizing mathematics is the axiom of choice. In practice, however, as Bishop has elaborated in a series of his works, common applications of the axiom of choice can be replaced by actual constructions of the objects which are claimed to exist. Another point that is usually brought into the discussion is the principle of excluded middle. The law of the excluded middle *per se*, however, does not obstruct mathematical constructions as long as comprehension is carefully controlled. Our intention is therefore to establish machinery by which one can project most parts of analysis into the definable world as soon as the spaces and other objects concerned are concretely given.

We employ "definable" logic with definitions by definable induction (DDI) as the basis of our machinery. (DDI was previously called ω-type inductive definitions.) Although the practice of mathematics in such a framework may appear somewhat unnatural and complicated, it cannot be helped in the present state of affairs. This is only the beginning of our program.

In this paper we demonstrate the definable nature of that part of point set topology which has a direct bearing on metric spaces. Here we only lay the foundations, hoping that the more sophisticated part of the theory will be developed in our formalism. In a sequel to this article, we treat metric spaces in the same framework (see Yasugi (1981b) for a summary).

The elementary theory of topology is carried out in a system which is a modest extension of Peano arithmetic and which is sound relative to given spaces. The topology of a space is determined by the base system whose members are named by indices. It is sufficient to talk about the indices instead of the base members (which are sets). Proof-theoretical arguments are much the same as in Yasugi (1981a), although in the present paper the notion of definability admits quantification over any atomic type while in the previous paper it was only allowed over the rationals. The reason is that here we do not "define" the space elements or indices.

The arithmetical part of our system is Takeuti's system FA or FA_1 (see Takeuti (1978)). Let us point out that in FA the content of any usual calculus text can be developed completely. Thus, for example, the convergence of an increasing sequence of reals which is bounded above can be easily demonstrated. The intermediate value theorem is also a natural consequence.

Various other approaches to constructive mathematics are listed in the references. Bishop's line of constructive analysis and its foundations (based on intuitionistic logic) are seen in Beeson (1979), Bishop (1967), Bridges (1979), Feferman (1979), Friedman (1977) and Myhill (1975). General topology as a part of intuitionistic mathematics is dealt with in Troelstra (1966, 1968). Demuth and Kučera have been engaged in constructive analysis along the lines of the Russian school, which places emphasis on algorithms. See, for example, Demuth and Kučera (1979). The difference of our approach from these should be clear from the introduction in Yasugi (1981a) and the discussion immediately above. Yasugi (1973) is a brief prelude to our present program.

Acquaintance with Takeuti (1978) and Yasugi (1981a) is assumed throughout.

1. Systems and axioms for one space

DEFINITION 1.1. Type. 1) τ_0, τ_1 and τ_2 are respectively atomic types of distinct sorts.

2) If $\sigma_1,\ldots,\sigma_n$ are types, $[\sigma_1,\ldots,\sigma_n]$ is a nonatomic type, which is also called a higher type.

DEFINITION 1.2. 1) For the language of basic logic and arithmetic, see Definition 1.2 of Yasugi (1981a). In particular 0 is of type τ_0 and N is of type $[\tau_0]$.

2) Symbols of a topological space are listed below.

$$\lambda_0,\, x_0,\, \Lambda,\, X,\, U,\, \mathrm{eq}_1,\, \mathrm{eq}_2.$$

Types of those symbols are respectively as follows.

$$\tau_1,\, \tau_2,\, [\tau_1],\, [\tau_2],\, [\tau_1,\, \tau_2],\, [\tau_1,\, \tau_1],\, [\tau_2,\, \tau_2].$$

3) Predicates for DDI, I_0, I_1, $I_2,\ldots$.

The type of I_i assumes the form $[\tau_0,\, \rho_1,\ldots,\rho_k,\, \sigma_1,\ldots,\sigma_l]$, where $\rho_1,\ldots,\rho_k$ are atomic types and $\sigma_1,\ldots,\sigma_l$ are arbitrary. The type of I_i is predetermined individually for each i.

DEFINITION 1.3. Definability, terms, formulas, abstracts, **min** and sequents are defined similarly to those in Definition 1.3 of Yasugi (1981a). Let us remark on a few points.

i) An object defined in our language is said to be definable if the only quantifiers it may contain are of atomic type (which are not necessarily of rationals).

ii) An expression of the form $\{\psi_1,\ldots,\psi_n\}F(\psi_1,\ldots,\psi_n)$, where $F(\psi_1,\ldots,\psi_n)$ is a definable formula and $\psi_1,\ldots,\psi_n$ are variables of type $\sigma_1,\ldots,\sigma_n$ respectively, is called an abstract of type $[\sigma_1,\ldots,\sigma_n]$.

iii) If Φ is a constant or a free variable of type $[\sigma_1,\ldots,\sigma_n]$ and if $J_1,\ldots,J_n$ are terms or abstracts of type $\sigma_1,\ldots,\sigma_n$ respectively, then $\Phi(J_1,\ldots,J_n)$ is an atomic formula.

iv) The objects of function types are not involved here.

DEFINITION 1.4. Substitution of an abstract for a free variable in an expression can be defined as usual; see for example Takeuti (1975) or Takeuti (1978).

We follow the notational conventions (adjusted to the present context) in Definition 1.2 of Yasugi (1981a). In particular, x, $y, \ldots,$ will be used for variables of type τ_2 and λ, $\mu, \ldots$ for those of type τ_1.

DEFINITION 1.5. Logical system $\mathfrak{L}$. The logical system $\mathfrak{L}$ is the predicate calculus of our language with the definable comprehension rule; namely the abstract of $\forall$ left of a higher type is definable. See Definition 1.4 in Chapter 1, Part II of Takeuti (1978) and Definition 1.4 of Yasugi (1981a) for the details.

DEFINITION 1.6. We define three sets of axioms in our language, $\mathcal{A}$, $\mathfrak{B}$, and $\mathcal{C}$.

1) $\mathcal{A}$ will stand for the set of axioms of arithmetic; see Definition 2.2 in Chapter 1 of Part II in Takeuti (1978) and 1) of Definition 1.5 in Yasugi (1981a).

2) $\mathfrak{B}$ will stand for the set of axioms of topology listed below.

$\mathfrak{B}1$. $\forall\lambda\Lambda(\lambda)$. $\forall x X(x)$.

$\mathfrak{B}2$. $\Lambda(\lambda_0)$. $X(x_0)$.

$\mathfrak{B}3$. Equivalence relations with respect to eq_1 and eq_2. Let us write $d = e$ for $\mathrm{eq}_i(d, e)$, $i = 1, 2$.

$$\forall x(x = x), \qquad \forall x(x = y \vdash y = x),$$
$$\forall x \forall y \forall z(x = y \wedge y = z \vdash x = z).$$

$\mathfrak{B}4$. $\forall x(X(x) \Leftrightarrow \exists\lambda U(\lambda, x))$.

$\mathfrak{B}5$. $\forall\lambda\exists x U(\lambda, x)$.

$\mathfrak{B}6$. $\forall\lambda\forall\mu\forall x\forall y(\lambda = \mu \wedge x = y \wedge U(\lambda, x) \vdash U(\mu, y))$.

$\mathfrak{B}7$.

$$\forall\lambda\forall\mu\forall x(U(\lambda, x) \wedge U(\mu, x)$$
$$\vdash \exists\nu(U(\nu, x) \wedge \forall y(U(\nu, y) \vdash U(\lambda, y) \wedge U(\mu, y)))).$$

These axioms suggest the following interpretations of the symbols. X is the set (called the space) upon which the theory of topology will be built. U represents a base for X whose members are indexed by the elements of Λ. λ_0 and x_0 are designated elements of Λ and X respectively. Notice that the axioms in $\mathfrak{B}$ are definable.

3) $\mathcal{C}$ will stand for the set of axioms of definitions by definable induction (which we abbreviate to DDI) given below.

Let $G_i(m, \varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_n, \Phi)$ be a definable formula which does not contain any of I_i, $I_{i+1}, \ldots,$ where m, $\varphi_1, \ldots, \varphi_k$, $\psi_1, \ldots, \psi_l$, Φ exhaust all the free

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variables G_i may contain, $\varphi_1,\ldots,\varphi_k$ are of atomic type, and Φ is of appropriate type. DDI is as follows.

$$\forall m \forall\varphi_1 \cdots \forall\varphi_k \forall\psi_1 \cdots \forall\psi_l \big(I_i(m,\varphi_1,\ldots,\varphi_k,\psi_1,\ldots,\psi_l) \\ \Leftrightarrow G_i(m,\varphi_1,\ldots,\varphi_k,\psi_1,\ldots,\psi_l, I_i[m])\big),$$

where $I_i[m]$ abbreviates

$$\{n\chi_1 \cdots \chi_k\}\big(n<m \wedge I_i(n,\chi_1,\ldots,\chi_k,\psi_1,\ldots,\psi_l)\big).$$

Although G_i should be specified in developing mathematics, the particular form of G_i is irrelevant in proof-theoretical arguments.

DEFINITION 1.7. Elementary theory of topology, $\mathfrak{T}$. A sequent $\Gamma \to \Delta$ of our language is said to be a theorem of $\mathfrak{T}$ if $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\Gamma \to \Delta$ is provable in the system $\mathfrak{L}$, where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ in the antecedent represent finite sequences of formulas from $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ respectively.

DEFINITION 1.8. Definable instantiations of the axioms in $\mathfrak{A}$ and $\mathfrak{C}$ can be defined as in Definition 1.7 of Yasugi (1981a). The universal quantifiers there can be of any higher types here.

Henceforth $\mathfrak{A}'$ will stand for the set of all definable instantiations of $\mathfrak{A}$ and $\mathfrak{A}^*$ will stand for a finite sequence from $\mathfrak{A}'$. Similarly for $\mathfrak{C}$.

DEFINITION 1.9. Logical system $\mathfrak{M}$. $\mathfrak{M}$ is obtained from $\mathfrak{L}$ by suppressing all variables of higher types.

DEFINITION 1.10. System $\mathfrak{P}$. $\mathfrak{P}$ is the system $\mathfrak{M}$ augmented by the following.

1°. Rule of inference: mathematical induction applied to the formulas of $\mathfrak{M}$.

2°. Initial sequents: formulas of $\mathfrak{A}'$ (with regard to any instantiations) except the instantiations of the mathematical induction.

3°. Initial sequents: formulas of $\mathfrak{C}'$.

2. Relative soundness

The following proposition is proved similarly to Theorem 16.5 of Takeuti (1975).

PROPOSITION 2.1. *Cut elimination holds in* $\mathfrak{L}$ (*see Definition* 1.5).

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THEOREM 1. *Let $\Gamma \to \Delta$ be a sequent which expresses an elementary theorem of the reals or of general topology (with one space). Then it is a theorem of* $\mathfrak{T}$, *namely* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \Gamma \to \Delta$ *is provable in* $\mathfrak{L}$, *hence without cuts.*

PROOF. For the theory of reals, see Part II of Takeuti (1978). The remaining sections of this article are devoted to the development of the elementary theory of topology in $\mathfrak{T}$.

Following the arguments in Section 2 of Yasugi (1981a) we obtain

PROPOSITION 2.2. *The consistency of* $\mathfrak{B}$ *relative to* $\mathfrak{P}$ *implies that of* $\{\mathcal{A}', \mathfrak{B}, \mathcal{C}'\}$ *relative to* $\mathfrak{M}$, *which in turn yields that of* $\{\mathcal{A}, \mathfrak{B}, \mathcal{C}\}$ *relative to* $\mathfrak{L}$.

THEOREM 2. $\mathfrak{P}$ *is consistent.*

PROOF. The proof of this theorem is a simplified version of that in Section 4 of Yasugi (1981a). Here one needs the well-ordered set $I_\infty = I_0 \cup I_1 \cup \cdots \cup I_i \cup \cdots$, where $I_i = \{(j, i); j \in I_*\} \cup \{\infty_i\}$ and $I_* = \omega \cup \tilde{\omega}$, in defining the rank of an occurrence of I_i in a formula. Thus, $|I_\infty| = (2\omega + 1)\omega = \omega^2$. The grade of a formula can be defined to be its norm, which is an element of ω^{ω^2}. The system Λ is therefore ω^{ω^2}. In assigning the elements of $\Pi(\Lambda)$ to the sequents in a proof, we need not consider comprehension, since there is none.

THEOREM 3 (relative soundness). *The theory* $\mathfrak{T}$ *is sound relative to* $\mathfrak{B}$. *In other words, the elementary theory of topology is sound relative to the theory of the given space.*

PROOF. By Proposition 2.2, and Theorems 1 and 2.

3. Basic properties of topology

DEFINITION 3.1. 1) ss$(X; A)$ will stand for the relation $\forall x \forall y(x = y \wedge A(x) \vdash A(y))$, where x and y are variables of type τ_2. ss$(X; A)$ is read "A is a subset of X." Subsets of Λ are defined similarly.

We use $A, B, C, \ldots$ to denote subsets of X, hence a restrictive expression such as ss$(X; A) \vdash F(A)$ will often be abbreviated to $F(A)$.

2) If $F(\psi)$ is a definable formula, then $\{\psi\}F(\psi)$ will be regarded as a set (of ψ satisfying F). Thus $\varphi \in \{\psi\}F(\psi)$ will express $F(\varphi)$. $\{\psi\}F(\psi)$ will also be written as $\{\psi; F(\psi)\}$ or simply F. $\cap$, $\cup$ and X-$*$ correspond to $\wedge$, $\vee$ and $\neg$ respectively. So, for example, $A \cup B$ is defined to be $\{\psi: A(\psi) \vee B(\psi)\}$.

We list some abbreviations. $A \subset B$: $\forall\psi(A(\psi) \vdash B(\psi))$,

$A = B$: $\forall\psi(A(\psi) \Leftrightarrow B(\psi))$, $A = \varphi$: $\forall\psi\neg A(\psi)$ and $\{\psi\}$: $\{\chi; \chi = \psi\}$.

Let F and G be definable. Then $\bigcup\{F(\psi); \psi \in G\}$: $\{\varphi\}\exists\psi(G(\psi) \wedge F(\psi, \varphi))$, where ψ is a variable of atomic type, and $\bigcap\{F(\psi); \psi \in G\}$: $\{\varphi\}\forall\psi(G(\psi) \vdash F(\psi, \varphi))$.

THEOREM 4. *The definability property and the subset property are both preserved under the set theoretical operations* $\cap$, $\cup$, X-$*$, $\bigcap$ *and* $\bigcup$, *when, for the last two, the defining formulas F and G are definable.*

From now on the statement in a proposition is to mean that it is provable in $\mathfrak{L}$ under the hypotheses of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, or to mean that it is a theorem of $\mathfrak{T}$ (see Definition 1.7). The proofs are formalizations of usual mathematical proofs (mostly taken out of Sections 8 and 9 of Royden (1968)); one has only to note that comprehension abstracts are definable. Although the arguments in this and subsequent sections are strictly formalizable, we shall state and prove propositions in a semi-formal manner so that it will assume the look of the usual text of topology. We shall deal with some exemplary cases, and what is not included in this paper can be formalized using a similar routine. We make these remarks here once and for all.

DEFINITION 3.2. opn(A) (A is an open subset of X):

$$\mathrm{ss}(X; A) \wedge \forall x \in A \exists\lambda(x \in U(\lambda) \subset A).$$

sbf(θ) (θ is a subfamily of the base u):

$$\forall\lambda(\theta(\lambda) = \varphi \vee \theta(\lambda) = U(\lambda)).$$

PROPOSITION 3.1. opn *complies with the usual notion of open sets. Let us list a few of its properties.*

1) opn(A), opn$(B) \to$ opn$(A \cap B)$.

2) sbf$(\theta) \to \forall\lambda$ss$(X; \theta(\lambda)) \wedge$ opn$(\bigcup\{\theta(\lambda); \lambda \in \Lambda\})$.

3) $\forall\psi(\Psi(\psi) \vdash$ opn$(\Phi(\psi)) \to$ opn$(\bigcup\{\Phi(\psi); \Psi(\psi)\})$.

PROOF OF 1).

$$x \in A \cap B \to \exists\lambda\exists\mu(x \in U(\lambda) \subset A \wedge x \in U(\mu) \subset B);$$
$$x \in U(\lambda) \cap U(\mu) \to \exists\nu(x \in U(\nu) \subset U(\lambda) \cap U(\mu)).$$

So,

$$x \in A \cap B \to \exists\nu(x \in U(\nu) \subset A \cap B).$$

Definition 3.3. $\mathrm{rl}(C, \lambda, x)$: $\mathrm{ss}(X; C) \wedge x \in C \cap U(\lambda)$.

$\mathrm{rlopn}(C, A)$: $\mathrm{ss}(X; A) \wedge \mathrm{ss}(X; C) \wedge A \subset C \wedge \forall x \in A \exists \lambda (x \in C \cap U(\lambda) \subset A)$.

Proposition 3.2. $\{A\}\mathrm{rlopn}(C, A)$ *can be regarded as the relative (to* C*) topology and* $\{\lambda, x\}\, \mathrm{rl}(C, \lambda, x)$ *can be regarded as a base for the relative topology.*

Definition 3.4. Sequence and convergence. Let S, S_1 and S_2 be variables of the appropriate type.

$\mathrm{sq}(\{n, x\}S(n, x))$($S$ is a sequence from X): $\forall n \exists x \forall y (x = y \Leftrightarrow S(n, y))$.

$\mathrm{cnv}(S, x)$(S converges to x):

$$\mathrm{sq}(S) \wedge \forall \lambda \big(x \in U(\lambda) \vdash \exists m \forall n \geqslant m \forall y \big(S(n, y) \vdash y \in U(\lambda)\big)\big).$$

$\mathrm{clst}(S, x)$(x is a cluster point of S):

$$\mathrm{sq}(S) \wedge \forall \lambda \big(x \in U(\lambda) \vdash \forall m \exists n \geqslant m \forall y \big(S(m, y) \vdash y \in U(\lambda)\big)\big).$$

$\mathrm{sbsq}(S_1, S_2)$(S_2 is a subsequence of S_1):

$$\mathrm{sq}(S_1) \wedge \mathrm{sq}(S_2) \wedge \forall n \forall x \big(S_2(n, x) \vdash \exists m S_1(m, x)\big)$$
$$\wedge \forall n \forall l \forall x \forall y \forall m \forall k \big(S_2(n, x) \wedge S_2(l, y) \wedge n < l$$
$$\wedge S_1(m, x) \wedge S_1(k, y) \vdash m < k\big).$$

Proposition 3.3. sq, cnv, clst *and* sbsq *comply with the usual mathematical notions suggested in parentheses. For example,*

1) $\mathrm{sq}(S) \leftrightarrow \forall n \exists ! x S(n, x) \wedge \forall n \forall x \forall y (x = y \wedge S(n, x) \vdash S(n, y))$, *where* $\exists ! x$ *expresses the unique existence of* x *with respect to* eq_2.

2) $\mathrm{cnv}(S, x) \to \mathrm{clst}(S, x)$.

Definition 3.5. $\mathrm{clsd}(A)$(A is closed): $\mathrm{opn}(X - A)$.

$\mathrm{cl}(A)$: $\{x; \forall \lambda (x \in U(\lambda) \vdash \neg(u(\lambda) \cap A = \varnothing))\}$.

$\mathrm{bd}(A)$: $\{x; \forall \lambda (x \in U(\lambda) \vdash \neg(U(\lambda) \cap A = \varnothing) \wedge \neg(U(\lambda) \cap (X - A) = \varnothing))\}$.

$\mathrm{int}(A)$: $\{x; \exists \lambda (x \in U(\lambda) \subset A)\}$.

$\mathrm{clst}(A)$: $\{x; \forall \lambda (x \in U(\lambda) \vdash \neg((U(\lambda) - \{x\}) \cap A = \varnothing))\}$.

Proposition 3.4. *The definability property and the subset property are closed with respect to* cl, bd, int *and* clst, *and these express respectively the closure, the boundary, the interior and the set of cluster points of* A.

1) $\mathrm{clsd}(A) \to \mathrm{ss}(X; A)$; $\mathrm{clsd}(\varphi)$; $\mathrm{clsd}(X)$.

2) $\mathrm{clsd}(\mathrm{cl}(A))$.

3) $\mathrm{clsd}(A) \leftrightarrow A \supset \mathrm{clst}(A)$.

4) $\mathrm{cl}(A) = A \cup \mathrm{clst}(A)$.

PROOF. We work 4) as an example.

$\mathrm{cl}(A) \supset A$ by definition and $\mathrm{clsd}(\mathrm{cl}(A))$ by 2). So by 3)

$$\mathrm{cl}(A) \supset \mathrm{clst}(\mathrm{cl}(A)) \supset \mathrm{clst}(A).$$

Thus $\supset$ follows.

$$x \in \mathrm{cl}(A) \wedge \neg x \in A \to \forall\lambda\big(x \in U(\lambda) \vdash \neg\big(U(\lambda) - \{x\}\big) \cap A = \varnothing\big)\big) \\ \to x \in \mathrm{clst}(A),$$

which implies $\subset$.

DEFINITION 3.6. $\mathrm{dcm}(A, B)$: $\mathrm{opn}(A) \wedge \mathrm{opn}(B) \wedge \neg A = \varnothing \wedge \neg B = \varnothing \wedge A \cap B = \varnothing \wedge A \cup B = X$.

$\mathrm{cnn}(X)$: $\forall A \forall B \mathrm{dcm}(A, B)$.

PROPOSITION 3.5. $\mathrm{cnn}(X)$, $\mathrm{clsd}(C)$, $\mathrm{opn}(C) \to C = X \vee C = \varnothing$.

PROOF. Put $A : C$ and $B : X - C$ in $\neg\mathrm{dcm}(A, B)$.

4. Maps between spaces

DEFINITION 4.1. Here we consider three spaces X, Y and Z, and thus add types, constants, variables and axioms for Y and Z. (See Definitions 1.1 through 1.8.) We shall not go into the details. We use λ, x, A, U for X, μ, y, B, V for Y and ν, z, C, W for Z, where $\lambda \in \Lambda$, $x \in X$, $A \subset X$ and $U \subset \Lambda \times X$ are assumed; and similarly for Y and Z.

Proposition 2.1 holds for the thus enlarged language.

THEOREM 1′. *Theorem 1 of Section 2 holds, where $\Gamma \to \Delta$ can be a theorem on maps between spaces.*

The rest of Section 2 goes through. In particular we have

THEOREM 3′. *The theory of continuous maps between spaces is sound relative to the definable theories of the given spaces.*

Now to the proof of Theorem 1′.

DEFINITION 4.2. For the notions and operations which are common to all spaces, such as opn, clsd and cl, we shall use the same symbolism. Thus, for

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example, opn(A) will mean that A is open in X, while opn(B) will mean that B is open in Y.

mp(f, A, Y)(f is a map from A to Y):

$$\mathrm{ss}(X; A) \wedge \forall x \in A \exists ! y f(x, y)$$
$$\wedge \forall x \forall u \forall y \forall v (x = u \wedge y = v \wedge f(x, y) \vdash f(u, v)).$$

img(f, A) (the image of A by f): $\{y; \exists x \in A f(x, y)\}$.

inv(f, B) (the inverse image of B by f): $\{x; \exists y \in B f(x, y)\}$.

cnt(f, A, Y) (f is a continuous map from A to Y):

$$\mathrm{mp}(f, A, Y) \wedge \forall \mu \, \mathrm{rlopn}(A, \mathrm{inv}(f, V(\mu))).$$

inj(f, A, Y) (f is injective):

$$\mathrm{mp}(f, A, Y) \wedge \forall x \in A \forall u \in A \forall y (f(x, y) \wedge f(u, y) \vdash x = u).$$

srj(f, A, Y) (f is surjective): $\mathrm{mp}(f, A, Y) \wedge \forall y \exists x \in A f(x, y)$.

cntinv(f, A, Y) (f has a continuous inverse):

$$\mathrm{inj}(f, A, Y) \wedge \forall \lambda \mathrm{opn}(f(U(\lambda))).$$

hmm(f, X, Y) (f is a homeomorphism between X and Y):

$$\mathrm{inj}(f, X, Y) \wedge \mathrm{srj}(f, X, Y) \wedge \mathrm{cnt}(f, X, Y) \wedge \mathrm{cntinv}(f, X, Y).$$

We can define the same concepts for maps between Y and Z and between X and Z.

cmpmp(h, f, g) (h is a composite map of f and g):

$$\mathrm{mp}(f, X, Y) \wedge \mathrm{mp}(g, Y, Z) \wedge \mathrm{mp}(h, X, Z)$$
$$\wedge \forall x \forall z (h(x, z) \Leftrightarrow \exists y (f(x, y) \wedge g(y, z))).$$

$f \upharpoonright A$ (the restriction of f to A): $\{(x, y); f(x, y) \wedge x \in A\}$.

We shall omit the letters X, Y and Z when ambiguity is unlikely.

PROPOSITION 4.1. 1) *When* mp(f, A, Y) *is assumed, the definability property and the subset property are preserved under* img, inv *and* $\upharpoonright$.

2) *The predicates and abstracts defined in Definition* 4.2 *serve as the usual mathematical notions.*

PROOF OF 2). Consider cnt(f), opn(B) $\to$ opn(inv(f, B)) as an example. $x \in \mathrm{inv}(f, B) \leftrightarrow \exists y \in B f(x, y)$: $y \in B$; $\mathrm{opn}(B) \to \exists \mu (y \in V(\mu) \subset B)$; $x \in \mathrm{inv}(f, B), y \in B, \mathrm{opn}(B) \to x \in \mathrm{inv}(f, V(\mu))$. Thus, for a μ as above,

$$\exists \lambda (x \in U(\lambda) \subset \mathrm{inv}(f, V(\mu)) \subset \mathrm{inv}(f, B)),$$

or opn(inv(f, B)).

PROPOSITION 4.2. 1) cnn(X), cnt(f, X, Y), srj(f) $\to$ cnn(Y); *see Definition* 3.6 *for* cnn.

2) $\mathrm{cnn}(X)$, $\mathrm{cnt}(f, X, R)$, $\exists x \exists u(f(x) < a < f(u)) \to \exists z(f(z) = a)$, *where* R *stands for the set of the (definable) reals,* a *stands for a real,* $<$ *and* $=$ *are relations of reals, and* x, u *and* w *are supposed to be some elements in* X. *The notations are slightly different when reals are involved. For instance,* $f(x)$ *stands for* $\{t\}f(x, t)$, t *denoting the rationals, and* $f(x) < a$ *abbreviates* $\exists t(\neg f(x, t) \wedge a(t))$. *See Takeuti* (1978) *for the details.*

Proof. 1) $\mathrm{dcm}(Y, B, D) \to \mathrm{dcm}(X, \mathrm{inv}(f, B), \mathrm{inv}(f, D))$.

2) Put $I = (-\infty, a)$ and $J = (a, \infty)$, $A = \mathrm{inv}(f, I)$, $B = \mathrm{inv}(f, J)$ and $C = A \cup B$. Thus $\neg C = X$ under the assumption, and so $\exists z \in X(\neg f(z) \in I \cup J)$. $f(z) = a$ for such a z.

5. Separation axioms

Henceforth we work with one space besides the real space, and hence shall return to the notations in Sections 1 and 2.

Definition 5.1. In the following $\mathbf{T}(i)$ expresses that a given space is a $\mathbf{T}_i$-space.

$\mathbf{T}(1)$: $\forall x \forall y(\neg x = y \vdash \exists \lambda(y \in U(\lambda) \wedge \neg x \in U(\lambda)))$.

$\mathbf{T}(2)$: $\forall x \forall y(\neg x = y \vdash \exists \lambda \exists \mu(x \in U(\lambda) \wedge y \in U(\mu) \wedge U(\lambda) \cap U(\mu) = \varnothing))$.

In the following, χ, ρ, π, θ, η are variables of appropriate type.

$\mathbf{T}(3; \chi, \rho)$:

$$\mathbf{T}(1) \wedge \forall x \forall E\big(\neg x \in E \wedge \mathrm{clsd}(E) \vdash \mathrm{opn}(\chi(x, E)) \wedge \mathrm{opn}(\rho(x, E)) \\ \wedge x \in \chi(x, E) \wedge E \subset \rho(x, E) \wedge \chi(x, E) \cap \rho(x, E) = \varnothing\big).$$

This reads "X is regular by χ and ρ". Similarly for the following two.

$\mathbf{T}(3\frac{1}{2}; \pi)$:

$$\forall x \forall E\big(\neg x \in E \wedge \mathrm{clsd}(E) \\ \vdash \mathrm{cnt}(\pi(x, E), X, R) \wedge \pi(x, E, x, 1) \wedge \forall y \in E \pi(x, E, y, 0)\big).$$

$\mathbf{T}(4; \theta, \eta)$:

$$\mathbf{T}(1) \wedge \forall D \forall E\big(\mathrm{clsd}(D) \wedge \mathrm{clsd}(E) \wedge D \cap E = \varnothing \\ \vdash \mathrm{opn}(\theta(D, E)) \wedge \mathrm{opn}(\eta(D, E)) \wedge D \subset \theta(D, E) \\ \wedge E \subset \eta(D, E) \wedge \theta(D, E) \cap \eta(D, E) = \varnothing\big).$$

Proposition 5.1. 1) $\mathbf{T}(1) \leftrightarrow \forall x \mathrm{clsd}(\{x\})$.

2) $\mathbf{T}_{i+1} \to \mathbf{T}_i$, $i = 1, 2, 3$.

Proof of $\mathbf{T}_4 \to \mathbf{T}_3$. Assume $\mathbf{T}(4; \theta, \eta)$ and define $\chi_0(\theta, \eta)$ to be $\{x, E\}\theta(\{x\}, E)$ and $\rho_0(\theta, \eta)$ to be $\{x, E\}\eta(\{x\}, E)$. Then $\mathbf{T}(3; \chi_0(\theta, \eta), \rho_0(\theta, \eta))$.

The following is the first case that requires the principle of DDI.

PROPOSITION 5.2 (Urysohn's lemma). *There is a definable* $f = f(\theta, \eta)$ *such that*

$$\begin{aligned}\mathbf{T}(4; \theta, \eta) \to \forall D \forall E\big(&\mathrm{clsd}(D) \wedge \mathrm{clsd}(E) \wedge D \cap E = \varnothing \\ &\vdash \mathrm{cnt}(f(\theta, \eta, D, E), X, R) \wedge 0 \leqslant f \leqslant 1 \\ &\wedge (f \equiv 0 \text{ on } D) \wedge (f \equiv 1 \text{ on } E)\big),\end{aligned}$$

where we have employed abbreviated notations whose meanings should be clear from the content.

PROOF. First notice that $\mathrm{cnt}(f, X, R)$ can be expressed as follows.

$$\mathrm{mp}(f, X, R) \wedge \forall r \forall s\big(r < s \vdash \mathrm{opn}(\{x; r < \{t\}f(x, t) < s\})\big),$$

where r, s, t stand for rationals and $<$ is taken to be the order relation of the reals.

Assume $\mathbf{T}(4; \theta, \eta)$, $\mathrm{clsd}(D)$, $\mathrm{clsd}(E)$ and $D \cap E = \varnothing$.

1°. Put B: $\theta(D, E)$, C: $\eta(D, E)$ and A: $X - E$. Then

$$D \subset B \subset \mathrm{cl}(B) \subset C \subset A. \tag{1}$$

2°. Let $\exp(a, b)$ express a^b, and let κ be an arithmetically definable enumeration of $\{(m, n); 0 < m \exp(2, -n) < 1\}$. Write $\delta(l) = m \exp(2, -n)$ if $\kappa(l) = (m, n)$. We are to construct a definable formula $G(l, x, D, E, \theta, \eta, \Phi)$ such that, under the assumption and with the scheme of DDI: $I(l, x, D, E, \theta, \eta) \leftrightarrow G(l, x, D, E, \theta, \eta, I[l])$, (2) below holds. Here $I[l]$ abbreviates $\{k, y\}(k < l \wedge I(k, y, D, E, \theta, \eta))$, and we write $I(l, x)$ for $I(l, x, D, E, \theta, \eta)$ and $I(l)$ for $\{x\}I(l, x)$.

$$\begin{aligned}\forall l\big(0 < \delta(l) < 1 \vdash \mathrm{opn}(I(l)) \wedge D \subset I(l) \subset A\big) \\ \wedge \forall l \forall k\big(0 < \delta(l) < \delta(k) < 1 \vdash \mathrm{cl}(I(l)) \subset I(k)\big).\end{aligned} \tag{2}$$

We shall give an informal account of defining $I(l)$ from $I(0), \ldots, I(l-1)$, which will explain how to construct G.

Put $I(0) : B$, or $\theta(D, E)$. Suppose $I(0), \ldots, I(l-1)$ have been defined so as to satisfy (2), and suppose l_1 and l_2 satisfy that $l_1, l_2 \leqslant l - 1$, $\delta(l_1) < \delta(l) < \delta(l_2)$ and $\delta(l_1)$ and $\delta(l_2)$ are each adjacent to $\delta(l)$. Then $\mathrm{cl}(I(l_1))$ and $X - I(l_2)$ are each closed and disjoint with one another (see (2) above). Thus one can apply the method in 1° to $\mathrm{cl}(I(l_1))$ (in the place of D) and $X - I(l_2)$ (in the place of E). Now if we put

$$I(l) : \theta\big(\mathrm{cl}(I(l_1)), X - I(l_2)\big),$$

then $\mathrm{opn}(I(l))$, $D \subset I(l) \subset A$ and $\mathrm{cl}(I(l)) \subset I(l_2)$. Thus by the hypotheses (2) holds for all $k \leqslant l$.

It is a matter of routine work to formulate the procedure above to give a precise form to G. We may assume that this G is G_0 and the corresponding I is I_0 in our language.

3°. Define f by $f(x, t)$: $t \leqslant 1 \wedge \forall l(x \in I(l) \vdash t < \delta(l))$. $\mathrm{mp}(f, X, R)$, $0 \leqslant f \leqslant 1$, "$f \equiv 0$ on D" and "$f \equiv 1$ on E" are easily proved.

For continuity, it suffices to establish $\mathrm{opn}(\{x;\ f(x) < s\})$ and $\mathrm{clsd}(\{x;\ f(x) \leqslant s\})$ for any rational s in $[0, 1]$. Let $J(r)$ denote $I(l)$ if $r = \delta(l)$. Then, by the definition of f $\{x;\ f(x) < s\} = \bigcup\{J(r);\ r < s\}$ and $\{x;\ f(x) \leqslant s\} = \bigcap\{\mathrm{cl}(J(r));\ r > s\}$.

PROPOSITION 5.3 (Tietzes's extension theorem). *There is a definable* $g \equiv g(\theta, \eta)$ *such that*

$$\mathbf{T}(4;\ \theta, \eta) \to \forall D \forall h\big(\mathrm{clsd}(D) \wedge \mathrm{cnt}(h, D, R) \\ \vdash \mathrm{cnt}(g(\theta, \eta, D, h), X, h) \wedge g \equiv h \text{ on } D\big).$$

PROOF. Assume $\mathbf{T}(4;\ \theta, \eta)$, $\mathrm{clsd}(D)$ and $\mathrm{cnt}(h, D, R)$. Using Urysohn's lemma, we can construct a sequence of maps $\{e(n)\}$ satisfying

$$\forall n\big(\mathrm{cnt}(e(n), X, R) \wedge |e(n)| < \exp(2, n-1)/\exp(3, n) \\ \wedge |e(0) - \Sigma\{e(i);\ i = 1, \ldots, n\}| < \exp(2, n)/\exp(3, n) \text{ on } D\big),$$

where $\Sigma\{e(i);\ i = 1, \ldots, n\}$ is the summation of $e(1), \ldots, e(n)$.

Formalizing the mathematical construction of $\{e(n)\}$ in a form $G_1(n, x, t, h, D, \theta, \eta, I[n])$ with G_1 definable, e can be regarded as a predicate constant I_1 to which we can apply DDI:

$$I_1(n, x, t, h, D, \theta, \eta) \leftrightarrow G_1(n, x, t, h, D, \theta, \eta, I_1[n]).$$

4°. Define $p(n, x) = \Sigma\{e(i, x);\ i = 1, \ldots, n\}$ and $q(x) = \limsup\{p(n, x);\ n = 1, 2, \ldots\}$. Then $q(x) = \lim\{p(n, x);\ n = 1, 2, \ldots\}$. Put $g \equiv q/(1 - |q|)$.

PROPOSITION 5.4. *There are definable* χ_1, ρ_1 *and* π_0 *such that* $\mathbf{T}(3\tfrac{1}{2};\ \pi) \to \mathbf{T}(3;\ \chi_1(\pi), \rho_1(\pi))$ *and* $\mathbf{T}(4;\ \theta, \eta) \to \mathbf{T}(3\tfrac{1}{2};\ \pi_0(\theta, \eta))$.

PROOF. By Urysohn's lemma where $E = \{x\}$.

6. Notions of compactness

There is no prospect of formulating the classical notion of compactness in our language. In metric spaces, however, various notions of compactness are all classically equivalent and a compact space is automatically separable. In fact it

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turns out that in a definably separable metric space, sequential compactness, Bolzano-Weierstrass property and countable compactness are mutually "definably interpretable". This fact is proven in a sequel to the present paper. It is most convenient to work on sequential compactness in the general setting.

DEFINITION 6.1. $\mathrm{sq}(A, S)$:

$$\mathrm{ss}(X; A) \wedge \mathrm{sq}(S) \wedge \forall n \forall x (S(n, x) \vdash x \in A).$$

$\mathrm{scmp}(A, \Phi)$ (A is sequentially compact by Φ):

$$\mathrm{ss}(X; A) \wedge \forall S(\mathrm{sq}(A, S) \vdash \mathrm{sbsq}(S, \Phi(S)) \wedge \exists x \in A\, \mathrm{cnv}(\Phi(S), x)).$$

Notice that sequential compactness can be formulated with the aid of a parameter if the concrete structure of X is not known.

PROPOSITION 6.1. 1) $\mathrm{scmp}(X; \Phi)$, $\mathrm{clsd}(A) \to \mathrm{scmp}(A, \Phi)$.

2) $\mathrm{scmp}(X, \Phi)$, $\mathrm{cnt}(f, X, Y)$, $\mathrm{srj}(f) \to \mathrm{scmp}(Y, \Psi(f, \Phi))$, *where* $\Psi(f, \Phi)(S)$ *is defined to be* $f(\Phi(\mathrm{inv}(f, S)))$.

DEFINITION 6.2. 1) $\mathrm{opnsq}(\alpha)$: $\forall i\, \mathrm{opn}(\alpha(i))$.

$\mathrm{opncv}(A, \alpha)$: $\mathrm{opnsq}(\alpha) \wedge \forall x \in A \exists i (x \in \alpha(i))$.

$\mathrm{fncv}(A, \alpha)$: $\mathrm{opnsq}(\alpha) \wedge \exists n \forall x \in A \exists i \leqslant n (x \in \alpha(i))$.

$\mathrm{ccmp}(A)$ (A is countably compact):

$$\mathrm{ss}(X; A) \wedge \forall \alpha (\mathrm{opncv}(A, \alpha) \vdash \mathrm{fncv}(A, \alpha)).$$

2) $\mathrm{clsq}(\beta)$: $\forall i\, \mathrm{clsd}(\beta(i))$.

$\mathrm{fip}(\beta)$: $\mathrm{clsq}(\beta) \wedge \forall n \exists x \in \bigcap \{\beta(i); i \leqslant n\}$.

FIP (X satisfies the finite intersection property):

$$\forall \beta (\mathrm{fip}(\beta) \vdash \exists x \in \bigcap \{\beta(i); i = 1, 2, \ldots\}).$$

BW (Bolzano-Weierstrass property): $\forall S(\mathrm{sq}(S) \vdash \exists x\, \mathrm{clst}(S, x))$.

PROPOSITION 6.2. 1) $\mathrm{ccmp}(X) \to \mathrm{FIP}$.

2) $\mathrm{FIP} \to \mathrm{ccmp}(X)$.

3) $\mathrm{ccmp}(X)$, $\mathrm{clsd}(A) \to \mathrm{ccmp}(A)$.

4) $\mathrm{ccmp}(X) \to \mathrm{BW}$.

5) $\mathrm{scmp}(X, \Phi) \to \mathrm{BW}$.

PROOF. We give the proof of 1). Assume $\mathrm{ccmp}(X)$, $\mathrm{fip}(\beta)$ and $\bigcap\{\beta(i); i = 1, 2, \ldots\} = \varnothing$. Define $\alpha^*(i)$ to be $X - \beta(i)$. Then $\mathrm{opncv}(X, \alpha^*)$. So, by $\mathrm{ccmp}(X)$ applied to α^*, $\mathrm{fncv}(X, \alpha^*)$, hence $\exists n(X = \bigcup\{\alpha^*(i); i \leqslant n\})$, or $\exists n(\bigcap\{\beta(i); i \leqslant n\} = \varnothing)$, contradicting $\mathrm{fip}(\beta)$. So $\exists x \in \bigcap\{\beta(i); i = 1, 2, \ldots\}$.

PROPOSITION 6.3. ccmp(X), cnt(f, X, Y), srj(f) $\to$ ccmp(Y).

PROOF. Assume the premise and suppose opncv(Y, γ). Define $\alpha^*(i)$ to be inv($f, \gamma(i)$). α^* is definable and opncv(X, α^*). So $\exists n(X = \bigcup\{\alpha^*(i): i \leq n\})$, hence $\exists n(Y = \bigcup\{\gamma(i); i \leq n\})$.

The notions and the consequences concerning the upper semicontinuous functions can be formulated in our system. This includes Dini's theorem.

DEFINITION 6.3. $K(A)$: opn(A) $\wedge$ ccmp(cl(A)).
LCMP(σ) (X is locally countably compact by σ): $\forall x(x \in \sigma(x) \wedge K(\sigma(x)))$.
CCB(τ): $\forall\lambda$ ccmp(cl($U(\lambda)$)) $\wedge$ $\forall x \exists! \lambda(\tau(\lambda, x) \wedge U(\lambda, x))$.

PROPOSITION 6.4. 1) *If we define* $\sigma_0(x)$ *to be* X *for every* x, *then* ccmp(X) $\to$ LCMP(σ_0).

2) *Under the assumption of* LCMP(σ), $\{A; K(A)\}$ *forms a base in the sense of* (1) *and* (2) *below.*

(1) $\forall x(x \in X \Leftrightarrow \exists B(x \in B \wedge K(B)))$.

(2) $\forall x \forall B \forall C(x \in B \cap C \wedge K(B) \wedge K(C) \vdash \exists D(x \in D \wedge K(D) \wedge D \subseteq B \cap C))$.

3) *Define* $K_0(\lambda)$: $K(U(\lambda))$. *Then* K_0 *serves as a base for* X.

4) CCB(τ) $\to$ LCMP($\{x, y\}\exists\lambda(\tau(\lambda, x) \wedge U(\lambda, y))$).

7. Countability axioms

In order to express the countability property, the axioms on the space (the axiom set $\mathfrak{B}$) must be presented in a manner suitable for that purpose. We leave the formulation of the first countability axiom to the reader.

DEFINITION 7.1. The theory $\mathfrak{T}$ where Λ is N will be called the theory of topology with the second countability axiom and will be denoted by $\mathfrak{T}_2$.

DEFINITION 7.2. spr(S) (X is separable by S): sq(S) $\wedge$ $X =$ cl(S).
dsg(T) (T is a set of designated elements of base members):

$$\mathrm{sq}(T) \wedge \forall n \forall y(T(n, y) \vdash y \in U(n)).$$

PROPOSITION 7.1. *The following are theorems of* $\mathfrak{T}_2$.
1) dsg(T) $\to$ spr(T).
2) **T**(2), ccmp(A) $\to$ clsd(A).

3) ccmp(X), cnt(f, X, Y), srj(f), inj(f), $\mathbf{T}(2; Y) \rightarrow$ hmm(f), *where* $\mathbf{T}(2; Y)$ *reads "Y is a* $\mathbf{T}_2$*-space."*

Proof of 2). Assume $\mathbf{T}(2)$ and ccmp(A). It suffices to show opn$(X - A)$. Let $y \in X - A$. Put

$$G(m, y)\colon \exists x \in A \exists n(x \in U(m) \wedge y \in U(n) \wedge U(m) \cap U(n) = \varnothing).$$

Then $A \subset \bigcup\{U(m); G(m, y)\}$ by $\mathbf{T}(2)$. So, ccmp(A) implies

$$\exists k\big(A \subset \bigcup\ \{U(m); G(m, y) \wedge m \leqslant k\}\big).$$

Also,

$$G(m, y) \rightarrow \exists n(y \in U(n) \wedge U(m) \cap U(n) = \varnothing).$$

If we define

$$\nu(m) = \mathbf{min}(n, y \in U(n) \wedge U(m) \cap U(n) = \varnothing),$$

then

$$\exists k\big(y \in \bigcap\ \{U(\nu(m)); m \leqslant k\} \subset X - A\big) \wedge \mathrm{opn}\big(\bigcap\ \{U(\nu(m)); m \leqslant k\}\big).$$

This implies opn$(X - A)$.

8. Product space

We shall work on the product of a sequence of topological spaces. Here we consider a language with the new letters Ξ and Θ, where Ξ represents the universe of elements and Θ represents the universe of indices.

Definition 8.1. 1) Let Θ, Ξ, Λ, X, Ω, ξ_0, ι_0, eq$_1$ and eq$_2$ be constant symbols. Types and intended interpretations of them should be figured out from the axioms given below; consult also some definitions in Section 1.

2) $\mathfrak{T}$ will stand for the axiom system on a sequence of topological spaces.

$\mathfrak{T}$1. Equivalence relations eq$_1$ and eq$_2$ on Θ and Ξ respectively. As before, we use $=$ for both.

$\mathfrak{T}$2. $\forall m \forall n \forall \lambda(\Lambda(m, \lambda) \wedge \Lambda(n, \lambda) \vdash m = n)$.

$\mathfrak{T}$3. $\forall \lambda(\Theta(\lambda) \Leftrightarrow \exists m \Lambda(m, \lambda))$; $\forall x(\Xi(x) \Leftrightarrow \exists m X(m, x))$.

$\mathfrak{T}$4. $\forall m \exists! \lambda(\iota_0(m, \lambda) \wedge \Lambda(m, \lambda))$; $\forall m \exists! x(\xi_0(m, x) \wedge X(m, x))$.

$\mathfrak{T}$5. $\forall m \forall \lambda \forall x(\Omega(m, \lambda, x) \vdash \Lambda(m, \lambda) \wedge X(m, x))$.
$\forall m \forall x(X(m, x) \vdash \exists \lambda(\Lambda(m, \lambda) \wedge \Omega(m, \lambda, x)))$.

$\mathfrak{T}$6. $\forall m \forall \lambda \forall \mu \forall x \forall y(\lambda = \mu \wedge x = y \wedge \Omega(m, \lambda, x) \vdash \Omega(m, \mu, y))$.

$\mathfrak{T}$7. $\forall m \forall \lambda(\iota_0(m, \lambda) \vdash \forall x(X(m, x) \Leftrightarrow \Omega(m, \lambda, x)))$.

$\mathfrak{D}8$.

$$\forall m \forall \lambda \forall \mu \forall x \big(\Lambda(m, \lambda) \wedge \Lambda(m, \mu) \wedge X(m, x) \wedge \Omega(m, \lambda, x) \wedge \Omega(m, \mu, x) \\ \vdash \exists \nu \big(\Lambda(m, \nu) \wedge \Omega(m, \nu, x) \wedge \forall y \big(X(m, y) \wedge \Omega(m, \nu, y) \\ \vdash \Omega(m, \lambda, y) \wedge \Omega(m, \mu, y) \big) \big) \big).$$

Proposition 8.1. *For every m regarding $\Lambda(m, \lambda)$ as $\lambda \in \Lambda$, $X(m, x)$ as $x \in X$, $\Omega(m, \lambda, x)$ as $x \in U(\lambda)$, the unique element of ι_0 as λ_0 and the unique element of ξ_0 as x_0, the axioms in $\mathfrak{B}$ (Definition 1.6) are provable uniformly in m. Thus the elementary theory of topology we have developed so far can be developed here uniformly in m.*

Note. We can write various notions in each space by placing m as a parameter. For example,

$$\mathrm{opn}(m, A): A \subset X(m) \wedge \forall x \in A \exists \lambda \big(\Lambda(m, \lambda) \wedge U(m, \lambda) \subset A \big).$$

Definition 8.2. Product space. Let ξ and ι be variables of appropriate type. The products of the index sets and the spaces are defined as follows.

$$\prod \Lambda(\iota): \forall m \forall \lambda \forall \mu \big(\lambda = \mu \wedge \iota(m, \lambda) \vdash \iota(m, \mu) \big) \wedge \exists n \forall m \geqslant n \\ \forall \lambda \big(\iota(m, \lambda) \vdash \lambda = \lambda_0 \big) \wedge \forall m \exists ! \lambda \big(\Lambda(m, \lambda) \wedge \iota(m, \lambda) \big).$$

$$\prod X(\xi): \forall m \forall x \forall y \big(x = y \wedge \xi(m, x) \vdash \xi(m, y) \big) \\ \wedge \forall m \exists ! x \big(X(m, x) \wedge \xi(m, x) \big).$$

We write $\iota \in \prod \Lambda$ and $\xi \in \prod X$ respectively for the relations above.

Proposition 8.2. *$\iota_0 \in \prod \Lambda$ and $\xi_0 \in \prod X$; thus the nonemptiness of the product space trivially holds (presuming that each space in the sequence is non-empty).*

Notation. We shall henceforth assume that $\iota \in \prod \Lambda$ and $\xi \in \prod X$, thus shall omit restrictive expressions such as "$\iota \in \prod \Lambda \vdash$ ".

Definition 8.3. $\xi_1 = \xi_2$: $\forall m \forall x \forall y (\xi_1(m, x) \wedge \xi_2(m, y) \vdash x = y)$.
$\xi_1 \equiv \xi_2$: $\forall m \forall x (\xi_1(m, x) \Leftrightarrow \xi_2(m, x))$.
Similarly for $\iota_1 = \iota_2$ and $\iota_1 \equiv \iota_2$.

Proposition 8.3. *$=$ and $\equiv$ are equivalence relations for their respective types. $\xi_1 = \xi_2$ if and only if $\xi_1 \equiv \xi_2$. Similarly for ι. We can thus regard either of $=$ and $\equiv$ as an equality relation for the product.*

Definition 8.4. $\Sigma(m, \iota, \xi)$:

$$\forall i \leqslant m \forall \lambda \forall x \big(\iota(i, \lambda) \wedge \xi(i, x) \vdash \Omega(i, \lambda, x) \big).$$

PROPOSITION 8.4. 1) $\Sigma(0, \iota_0, \xi)$.

2) *There is a definable* M *such that, abbreviating* $\{i, \nu\}M(m, n, \iota_1, \iota_2, \xi, i, \nu)$ *to* κ,

$$\forall m \forall n \forall \iota_1 \forall \iota_2 \forall \xi \big(\Sigma(m, \iota_1, \xi) \wedge \Sigma(n, \iota_2, \xi) \\ \vdash \exists l\big(\Sigma(l, \kappa, \xi) \wedge \forall \zeta\big(\Sigma(l, \kappa, \zeta) \vdash \Sigma(m, \iota_1, \zeta) \wedge \Sigma(n, \iota_2, \zeta)\big)\big)\big).$$

3)

$$\forall m \forall \iota_1 \forall \iota_2 \forall \xi_1 \forall \xi_2 \big(\iota_1 = \iota_2 \wedge \xi_1 = \xi_2 \wedge \Sigma(m, \iota_1, \xi_1) \vdash \Sigma(m, \iota_2, \xi_2)\big).$$

1) $\sim$ 3) *claim that*

$$\big\{\{\zeta\}\Sigma(n, \iota, \zeta);\ n = 1, 2, \ldots, \iota \in \textstyle\prod \Lambda\big\}$$

serves as a base for $\prod X$ *where the index set is* $N \times \prod X$.

PROOF OF 2). Put $l = \max(m, n)$ and $k = \min(m, n)$. Then define

$$M(m, n, \iota_1, \iota_2, \xi, i, \nu) \colon \big[i \leqslant k \wedge \exists \lambda \exists \mu \exists x\big(\iota_1(i, \lambda) \wedge \iota_2(i, \mu) \wedge \zeta(i, x) \\ \wedge \Omega(i, \nu, x) \wedge \forall y\big(\Omega(i, \nu, y)(\vdash \Omega(i, \lambda, y) \wedge \Omega(i, \mu, y)\big)\big)\big] \\ \vee\big[k < i \leqslant l \wedge \big(\iota_1(i, \nu) \vee \iota_2(i, \nu)\big)\big] \vee\big[k > l \wedge \iota_2(i, \nu)\big].$$

REMARK. If one wishes to develop the elementary theory of topology on the product space, one can do so by redoing the arguments in Sections 3 to 7 with the axiom system in Proposition 8.4 above. Since Proposition 8.4 can be established in $\mathfrak{L}$ under the assumption of $\mathfrak{D}$ as well as $\mathfrak{A}$ and $\mathfrak{C}$, it is sound relative to $\mathfrak{D}$. So, by reapplying Theorem 3 in Section 2, one can claim that the theory of the product space is sound relative to $\mathfrak{D}$.

PROPOSITION 8.5. 1) *If we define* $\pi(i, \xi, x)$ *to be* $\xi(i, x)$, *then* $\mathrm{mp}(\pi(i), \prod X, X(i))$.

2) *With a* $\rho \equiv \rho(i)$ *defined below* $\pi(i)$ *is continuous.*

$$\rho(i, j, \lambda, \mu, \xi) \colon (j = i \wedge \mu = \lambda) \vee (\neg j = i \wedge \iota_0(j, \mu)).$$

DEFINITION 8.5. For any two topological spaces X and Y, define $\mathrm{opn}(f, X, Y)$ to be $\mathrm{mp}(f, X, Y) \wedge \forall A(\mathrm{opn}(X; A) \vdash \mathrm{opn}(Y; f(A)))$.

PROPOSITION 8.6. $\mathrm{opn}(\pi(i), \prod X, X(i))$.

PROOF. Assume $\mathrm{opn}(A)$, $\zeta \in A$ and $\pi(i)(\xi) = x$. $\exists n \exists \iota(\xi \in \Sigma(n, \iota) \subset A)$ by definition. But

$$\Sigma(n, \iota, \xi) \leftrightarrow \forall j \leqslant n \forall \lambda \forall y\big(\iota(j, \lambda) \wedge \xi(j, y) \vdash \Omega(j, \lambda, y)\big).$$

So $\pi(i)(\Sigma(n, \iota)) \subset \pi(i)(A)$ and $x \in \pi(i)(\Sigma(n, \iota)) = \Omega(i, \lambda)$ where $\iota(i, \lambda)$.

Notice that, although we have proven an existential statement of a higher type, no comprehension has been used.

PROPOSITION 8.7. $\mathrm{cnt}(f, Y, \prod X) \leftrightarrow \forall i\, \mathrm{cnt}(\pi(i) \circ f, Y, X(i))$.

PROOF. ($\rightarrow$) Assume $\mathrm{cnt}(f, Y, \prod X)$. Suppose $U \subset X(i)$ and $\mathrm{opn}(U)$ in $X(i)$. By 2) of Proposition 8.5 U^*: $\mathrm{inv}(\pi(i), U)$ is open in $\prod X$, hence $\mathrm{inv}(f, U^*)$ is open in Y. So, $\mathrm{inv}(\pi(i) \circ f, U)$ is open in Y.

($\leftarrow$) Assume $\forall i\, \mathrm{cnt}(\pi(i) \circ f, Y, X(i))$. Suppose W is open in $\prod X$. Let W be $\Sigma(n, \iota)$. $\pi(i)(W)$ is open in $X(i)$ by Proposition 8.6. But $\pi(i)(W)$ is $X(i)$ if $i > n$ and it is $\Omega(i, \lambda)$ if $i \leqslant n$, where $\iota(i, \lambda)$ is satisfied. Thus $\mathrm{inv}(f, \mathrm{inv}(\pi(i), \pi(i)(W)))$ is open in Y and $W = \bigcap \{\mathrm{inv}(\pi(i), \pi(i)(W));\ i \leqslant n\}$. So $\mathrm{inv}(f, W) = \bigcap \{\mathrm{inv}(f, \mathrm{inv}(\pi(i), \pi(i)(W)));\ i \leqslant n\}$, which is open in Y.

PROPOSITION 8.8. *The product of a sequence of Hausdorff spaces is Hausdorff, where we assume that the Hausdorff property is defined uniformly in i (by parameters γ and δ); namely,*

$$\begin{aligned}\forall i \forall x \forall y (x, y \in X(i) \wedge \neg x = y \\ \vdash \exists \lambda \exists \mu (\gamma(i, x, y, \lambda) \wedge \delta(i, x, y, \mu) \wedge x \in \Omega(i, \lambda) \\ \wedge y \in \Omega(i, \mu) \wedge \Omega(i, \lambda) \cap \Omega(i, \mu) = \varnothing)).\end{aligned}$$

9. Product of sequentially compact spaces

PROPOSITION 9.1. *Let S be a parameter of appropriate type. Define $\sigma(X; n, \xi)$ to be $\forall i \forall x(\xi(i, x) \Leftrightarrow S(n, i, x))$. Under the assumption that $\forall n(\{i, x\} S(n, i, x) \in \prod X)$, $\{n, \xi\}\sigma(S, n, \xi)$ is a sequence from $\prod X$; that is: $\forall n \exists \xi \forall \eta(\xi = \eta \Leftrightarrow \sigma(S; n, \eta))$ is provable in our theory.*

Due to the specific form of Σ, we may regard S itself as a sequence from $\prod X$. Thus, we shall work on S hereafter.

PROPOSITION 9.2. *Suppose $X(i)$ is sequentially compact for each i (uniformly in i). Then there is a definable Φ^* such that, for any S a sequence from $\prod X$ in the sense above, $\Phi^*(S)$ is a subsequence of S which converges in $\prod X$.*

PROOF. Assume $\forall i\ \mathrm{scmp}(X(i), \Phi(i))$ and suppose S is a sequence from $\prod X$. Rewrite $S(n, 1, x)$ as $T(1, n, x)$. Then

$$\forall n \exists x \in X(1) \forall y \in X(1)(x = y \Leftrightarrow T(1, n, y)).$$

$\Phi(1, T(1))$ is a subsequence of $T(1)$ in $X(1)$ that converges, namely $\exists y \in X(1)\mathrm{cnv}(\Phi(1, T(1)), y)$. Let us write this as $\exists y Q(1, y)$. $\Phi(1, T(1))$ determines a sequence of natural numbers $\{j\}M(1, j)$ so that $\Phi(1, T(1)) \equiv \{j\}T(1, M(1, j))$. Then $\{j, i, x\}S(M(1, j), i, x)$ is a subsequence of S in $\prod X$.

Similarly, we can define $M(l, j)$, $T(l)$ and $Q(l, y)$ for $l = 1, 2, \ldots,$ which satisfy the following.

$$\Phi\big(l, \Phi\big(l-1, \ldots, \Phi\big(1, \{n, x\}S(n, 1, x)\big) \cdots\big)\big)$$
$$\equiv \{j\}T\big(l, M(l, j)\big) \equiv \{j, x\}S\big(M(l, j), l, x\big);$$
$$Q(l, y) \leftrightarrow \mathrm{cnv}\big(\{j\}T\big(l, M(l, j)\big), y\big).$$

So, $\exists! y \in X(l)Q(l, y)$, $\{j\}M(l+1, j)$ is a subsequence of $\{j\}M(l, j)$ and $\{j\}T(l+1, M(l+1, j))$ is a subsequence of $\{j\}T(l, M(l, j))$ in $\prod X$. Now define $\Phi^*(n, x)$ to be $T(n, M(n, n), x)$. $\{n\}M(n, n)$ is an increasing sequence of natural numbers, and Φ^* is a subsequence of S. Redefine $Q(n, x)$ as $\xi^*(n, x)$. Then $\xi^* \in \prod X$. It is a routine work to establish $\mathrm{cnv}(\Phi^*, \xi^*)$ in $\prod X$.

The definability of the inductive construction of Φ^* from S should be clear from the discussion given above.

Note. In order that σ be a sequence from $\prod X$, one should assume a parameter S which represents the nth entry of σ for each n, that is: $\forall n(\{i, x\}S(n, i, x) \in \prod X) \wedge \forall n \forall \xi(\sigma(n, \xi) \Leftrightarrow \forall i \forall x(\xi(i, x) \Leftrightarrow S(n, i, x)))$ should hold. It is therefore sufficient to deal with the sequences from $\prod X$ in the form of Proposition 9.1, and hence we may regard Proposition 9.2 as claiming that the product of a sequence of sequentially compact spaces is again sequentially compact.

10. One-point compactification

Definition 10.1. Let $\mathfrak{L}\mathfrak{C}$ be the theory $\mathfrak{T}_2$ augmented by the axioms $\mathbf{T}(2)$ and

$$\forall n \mathrm{ccmp}(\mathrm{cl}(U(n))) \wedge \neg\mathrm{ccmp}(X).$$

(See Definitions 5.1, 6.2 and 7.1.) The propositions in this section are meant to be provable in $\mathfrak{L}\mathfrak{C}$.

Definition 10.2. 1) $s(N) \equiv \{s(n); n \in N\}$, where s is a new symbol which designates a new constant $s(n)$ corresponding to each n. Let Λ be the set $N \cup s(N)$, and λ will be used as a variable ranging over the elements of Λ. The type of λ is regarded as atomic and the quantification over λ is understood to be definable. $\lambda_1 = \lambda_2$ is defined to be: $(\lambda_1, \lambda_2 \in N \wedge \lambda_1 = \lambda_2) \vee (\lambda_1 = s(n_1) \wedge \lambda_2 = s(n_2) \wedge n_1 = n_2)$.

2) Let X^* be the set $X \cup \{\omega\}$, where ω is a new symbol with the axiom $\forall x \in X(x \neq \omega)$. Henceforth $x, y, \ldots$, will be used as variables on the elements of X^*.

3) $W(k) \equiv \bigcup\{U(i); i \leqslant k\}$, where we assume $W(0) = \varnothing$.

$$V(\lambda) \equiv \{x; (\lambda \in N \wedge x \in U(\lambda)) \vee \exists k \in N(\lambda = s(k) \wedge x \in X^*\text{-cl}(W(k)))\}.$$

4) $C^*(A, k)$: $X - \text{cl}(W(k)) \subset A - \{\omega\}$; $D^*(A, n)$: $U(n) \subset A - \{\omega\}$.

Proposition 10.1. 1) $\forall k \text{ccmp}(\text{cl}(W(k)))$.

2) $\{V(\lambda); \lambda \in \Lambda\}$ *forms a countable base for* X^*.

3) *Suppose* $A \subset X$. *Then* A *is open with regard to* V *if and only if it is open with regard to* U.

4) *Suppose* $\omega \in A$. *Then* A *is open with regard to* V *if and only if* $\exists k C^*(A, k)$ *and*

$$A - \{\omega\} = [\bigcup \{X - \text{cl}(W(k)); C^*(A, k)\}] \cup [\bigcup \{U(n); D^*(A, n)\}].$$

Proposition 10.2 (one-point compactification). (X^*, Λ, V) *is a countably compact Hausdorff space in which* X *is dense.*

The proof is straightforward from Proposition 10.1.

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References

M. Beeson (1979), 'Continuity in intuitionistic set theory', *Logic colloquium* 78, Studies in logic and foundations of mathematics 97, edited by M. Boffa, D. Van Dalen, K. McAloon, pp. 1–52 (North-Holland Publ. Co., Amsterdam).

E. Bishop (1967), *Foundations of constructive analysis* (McGraw-Hill Book Co., New York).

D. S. Bridges (1979), *Constructive functional analysis* (Pitman, London).

O. Demuth and A. Kučera (1979), 'Remarks on constructive mathematical analysis', *Logic colloquium* 78, Studies in logic and the foundations of mathematics 97, edited by M. Boffa, D. Van Dalen, K. McAloon, pp. 81–129 (North-Holland Publ. Co., Amsterdam).

S. Feferman (1979), 'Constructive theories of functions and classes', *Logic colloquium* 78, Studies in logic and the foundations of mathematics 97, edited by M. Boffa, D. Van Dalen, K. McAloon, pp. 159–224 (North-Holland Publ. Co., Amsterdam).

H. Friedman (1977), 'Set-theoretic foundations for constructive analysis', *Ann. Math.* **105**, 1–28.

J. Myhill (1975), 'Constructive set theory', *J. Symbolic Logic* **40**, 347–383.

H. L. Royden (1968), *Real analysis*, second edition (Collier-Macmillan Limited, London).

G. Takeuti (1975), *Proof theory* (North-Holland Publ. Co., Amsterdam).

G. Takeuti (1978), *Two applications of logic to mathematics* (Iwanami Shoten and Princeton Univ. Press, Tokyo).

A. S. Troelstra (1966), *Intuitionistic general topology* (Thesis, Amsterdam).

A. S. Troelstra (1968), 'One point compactifications of intuitionistic locally compact spaces', *Fund. Math.* **62**, 75–93.

M. Yasugi (1973), 'Arithmetically definable analysis', *Proc. Res. Inst. Math. Sci.* **180**, 39–51.

M. Yasugi (1981a), 'The Hahn-Banach theorem and a restricted inductive definition', *Lecture Notes in Math.* 891, pp. 359–394 (Springer-Verlag, Berlin).

M. Yasugi (1981b), 'Definability problems in metric spaces; a summary', *Proc. Res. Inst. Math. Sci.* **441**, 66–82.

The Institute of Information Science
University of Tsukuba
Sakuramura, Ibaraki
Japan 305