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HYPERDECIDABILITY OF PSEUDOVARIETIES OF ORTHOGROUPS

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Abstract. Let **W** denote the intersection with the pseudovariety of completely regular semigroups of the Mal'cev product of the pseudovariety of bands with a pseudovariety **V** of completely regular semigroups. It is shown that the (pseudo)-word problem for **W** is reduced to that for **V** in such a way that decidability is preserved in the case in which terms involving only multiplication and weak inversion are considered. It is also shown that, if **V** is a hyperdecidable (respectively canonically reducible) pseudovariety of groups, then so is **W**.

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1. Introduction. Motivated by the Krohn-Rhodes complexity problem [**24**], the search for uniform algorithms for computing semidirect products of pseudovarieties has led to substantial research in the theory of finite semigroups. Even though there is no universal solution, since the semidirect product of decidable pseudovarieties is not necessarily decidable [**1**], under suitable assumptions on the factors, the semidirect product might be decidable. The notions of hyperdecidability [**3**] and σ-reducibility [**8**] have been devised in connection with this question and provide key links with the known proofs of the Rhodes type II conjecture [**12**, **33**, **21**] (cf. [**3**, **7**, **8**]). The paper [**8**] also brings forth a crucial role played by word problems for relatively free semigroups with extra operations, the unary operation of taking the weak inverse (in the subsemigroup generated by the argument) being of special interest.

Word problems, on the other hand, have long been central to the theory of varieties of completely regular semigroups. The free completely regular semigroup has been described as a relatively free unary semigroup [**17**, **35**] and its word problem has been solved [**23**]. The successes in the study of varieties of completely regular semigroups and of hyperdecidability and canonical reducibility for pseudovarieties of groups prompted the authors to study word problems, hyperdecidability, and canonical reducibility for pseudovarieties of completely regular semigroups. In the present paper, pseudovarieties of orthogroups (orthodox completely regular semigroups) are considered, while in future papers we plan to deal with the non-orthodox case.

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2. Preliminaries. This paper assumes some familiarity with the theory of finite semigroups. The reader is referred to [**27**, **2**] for general background and motivation. A good knowledge of the basic literature on the free band ([**18**], or see [**22**, Section IV.4]) is also recommended, although not required. In this section we gather the essential basic notions and notation for the rest of the paper.

For an element s of a finite semigroup S, we denote by s^ω the unique element of S that is an idempotent power s^n of s with n positive. The inverse of ss^ω (also naturally denoted $s^{\omega+1}$) in the maximal subgroup containing s^ω is denoted by $s^{\omega-1}$. Hence $s^{\omega-1}$ is a *weak inverse* of s in the sense that $s^{\omega-1}ss^{\omega-1} = s^{\omega-1}$ and in fact it is the only weak inverse of s in the subsemigroup generated by s. The operations $s \mapsto s^\omega$ and $s \mapsto s^{\omega-1}$ (henceforth called simply *weak inversion*) are examples of unary "implicit operations" on finite semigroups.

More generally, for a set A, an *A-ary* (or *$|A|$-ary*) *implicit operation* on a class $\mathscr{C}$ of semigroups is a family $(\pi_S)_{S\in\mathscr{C}}$ of functions $\pi_S : S^A \to S$ such that, for any homomorphism $\varphi : S \to T$ between elements of the class $\varphi \circ \pi_S = \pi_T \circ \varphi^A$, where the mapping $\varphi^A : S^A \to T^A$ is φ on each component. In particular, the basic operation defining a semigroup is an example of a (binary) implicit operation on the class of all semigroups. For each $a \in A$, the projection on the a-component of S^A into S also defines an implicit operation on the class of all semigroups which we usually identify with a.

By *pseudovarieties* we mean classes of finite semigroups which are closed under taking homomorphic images, subsemigroups, and finite direct products. For a pseudovariety $\mathbf{V}$, the set of all A-ary implicit operations on $\mathbf{V}$ is denoted $\overline{\Omega}_A\mathbf{V}$. If π is an n-ary implicit operation on $\mathbf{V}$ and $\rho_1, \ldots, \rho_n \in \overline{\Omega}_A\mathbf{V}$, then the composite operation $\pi(\rho_1, \ldots, \rho_n)$ given by $(\pi(\rho_1, \ldots, \rho_n))_S(f) = \pi_S(\rho_{1S}(f), \ldots, \rho_{nS}(f))$, for $f \in S^A$, is again an A-ary implicit operation on $\mathbf{V}$. Considering in particular for π the basic semigroup operation, we see that $\overline{\Omega}_A\mathbf{V}$ is itself a semigroup on which implicit operations on $\mathbf{V}$ have a natural interpretation. Under the initial topology for the homomorphisms into semigroups from $\mathbf{V}$, which are themselves viewed as discrete topological spaces, the set $\overline{\Omega}_A\mathbf{V}$ becomes a compact zero-dimensional space. With respect to this space the interpretation of any implicit operation on $\mathbf{V}$ is continuous and such that (continuous) homomorphisms into members of $\mathbf{V}$ suffice to separate distinct points; i.e., $\overline{\Omega}_A\mathbf{V}$ is *residually* in $\mathbf{V}$ [**2**]. In general, a compact semigroup which, as a topological semigroup, is residually in $\mathbf{V}$, is said to be a *pro-*$\mathbf{V}$ *semigroup*. The topological semigroup $\overline{\Omega}_A\mathbf{V}$ is then characterized as the *free* pro-$\mathbf{V}$ semigroup on the set A in the sense that, for any mapping $\varphi : A \to S$ into a pro-$\mathbf{V}$ semigroup, there is a unique continuous homomorphism $\bar{\varphi} : \overline{\Omega}_A\mathbf{V} \to S$ whose restriction to A is φ [**11**]. Pro-$\mathbf{S}$ semigroups are also called *profinite semigroups*, where $\mathbf{S}$ is the pseudovariety of all finite semigroups. The unique continuous homomorphism $\overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{V}$ that fixes the members of A is denoted $p_\mathbf{V}$.

By a *pseudoidentity* for finite semigroups we mean a formal equality $u = v$ between two implicit operations of the same arity on the class $\mathbf{S}$. The pseudoidentity $u = v$ is said to *hold* in a finite semigroup S if $u_S = v_S$. We write $\mathscr{C} \models u = v$ for a class $\mathscr{C}$ of finite semigroups if the pseudoidentity $u = v$ holds in every member of $\mathscr{C}$. For a set Σ of pseudoidentities, we denote by $[\![\Sigma]\!]$ the class of all finite semigroups in which all pseudoidentities from Σ hold. By a well-known theorem of Reiterman [**32**], the classes of this form are precisely the pseudovarieties of finite semigroups.

The following pseudovarieties will be used later in this paper. See for instance [**2**] for their significance.

Ab	=	$[\![x^\omega = 1,\ xy = yx]\!]$	*(Abelian groups).*
B	=	$[\![x^2 = x]\!]$	*(bands).*
CR	=	$[\![x^{\omega+1} = x]\!]$	*(completely regular).*
DA	=	$[\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega,\ x^{\omega+1} = x^\omega]\!]$	*(regular $\mathscr{D}$-classes are aperiodic subsemigroups).*
DG	=	$[\![(xy)^\omega = (yx)^\omega]\!]$	*(regular $\mathscr{D}$-classes are groups).*
DO	=	$[\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!]$	*(regular $\mathscr{D}$-classes are orthodox subsemigroups).*
DS	=	$[\![((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega]\!]$	*(regular $\mathscr{D}$-classes are subsemigroups).*
G	=	$[\![x^\omega = 1]\!]$	*(groups).*
J	=	$[\![(xy)^\omega = (yx)^\omega,\ x^{\omega+1} = x^\omega]\!]$	*($\mathscr{J}$-trivial).*
OCR	=	$[\![x^{\omega+1} = x,\ (x^\omega y^\omega)^2 = x^\omega y^\omega]\!]$	*(orthogroups).*
Sl	=	$[\![x^2 = x,\ xy = yx]\!]$	*(semilattices).*

Let σ be a set of implicit operations over finite semigroups containing the basic semigroup operation '$\cdot$'. Such a set is called an *implicit signature*. We view σ as an algebraic type for which every profinite semigroup has a natural structure as a σ-algebra, namely interpreting each implicit operation from σ as described above. A σ-algebra is said to be a σ-*semigroup* if it is a semigroup under the interpretation of the binary operation '$\cdot$'. We denote by $\Omega_A^\sigma \mathbf{V}$ the σ-subsemigroup of $\overline{\Omega}_A \mathbf{V}$ generated by a given set A. It is well known that $\Omega_A^\sigma \mathbf{S}$ is the free σ-semigroup on the set A.

An important example of an implicit signature is that of the signature $\kappa = \{\cdot,\ ^{\omega-1}\}$, also called the *canonical signature*, consisting of the basic semigroup operation and weak inversion. The reason for the word "canonical" in this context is to emphasize that its use pervades most of semigroup theory, in one form or another, from inverse semigroups and completely regular semigroups to finite semigroups and, more generally, epigroups and compact semigroups. There is in its choice no technical connotation in the sense of category theory. For example, the κ-semigroups $\Omega_A^\kappa \mathbf{G}$ and $\Omega_A^\kappa \mathbf{CR}$ are respectively the free group and the free completely regular semigroup on the set A since these algebras are residually finite [**25**,**26**]. By κ-terms in general, which we shall also call *weak terms*, we mean elements of a suitable free κ-semigroup $\Omega_A^\kappa \mathbf{S}$. We shall refrain however from using ambiguous terminology like "free weak semigroup" and "weak reducibility" (which already appears in [**8**] with a different meaning), preferring to retain the more formal but more precise reference to the signature κ.

For a pseudovariety $\mathbf{H}$ of groups, let $\overline{\mathbf{H}}$ denote the pseudovariety consisting of all finite semigroups all of whose subgroups lie in $\mathbf{H}$. Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups. Their *Mal'cev product* $\mathbf{V} ⓜ \mathbf{W}$ consists of all divisors of semigroups S such that there is a homomorphism $\varphi : S \to T$ into $T \in \mathbf{W}$ for which $\varphi^{-1}(e) \in \mathbf{W}$, for every idempotent $e \in T$.

By a *graph* we mean what is sometimes called in the literature a directed multigraph. More formally, a graph Υ is the union of two disjoint sets $V = V(\Upsilon)$ and $E = E(\Upsilon)$, respectively of *vertices* and *edges*, together with two functions $\alpha, \omega : E \to V$ describing respectively the *beginning* and the *end* vertices of each edge. An *undirected path* in the graph Υ is a path from one specified vertex to another in the graph obtained from Υ by forgetting directions of edges, which could be described as the union of Υ with the dual graph obtained by exchanging the functions α and ω. In such an undirected path, we say that an edge in the path appears *in the*

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opposite direction of the path if it comes from the dual of Υ and otherwise that it appears *in the direction of the path*.

3. The Rhodes and Birget expansions. In his proof of the "fundamental lemma of complexity", J. Rhodes introduced what came to be known as the *Rhodes expansions* (cf. [**15**, Chapter XII by B. Tilson]). By iterating left and right Rhodes expansions (which are idempotent functors on the category of semigroups on a fixed generating set), Birget [**13**] obtained another expansion which we will call the *Birget expansion* and which is used later in the paper. We review in this subsection the necessary definitions and properties of these expansions.

Let S be a semigroup and denote by $>_{\mathscr{R}}$ and $\geq_{\mathscr{R}}$ respectively the strict and the non-strict Green $\mathscr{R}$-orderings of S. Let $\bar{S}^{\mathscr{R}}$ denote the semigroup of all finite $\geq_{\mathscr{R}}$-chains $(s_1 \geq_{\mathscr{R}} s_2 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_n)$ of S under the following operation:

$$\begin{aligned}(s_1 \geq_{\mathscr{R}} s_2 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_m)&(t_1 \geq_{\mathscr{R}} t_2 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} t_n)\\ &= (s_1 \geq_{\mathscr{R}} s_2 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_m \geq_{\mathscr{R}} s_m t_1 \geq_{\mathscr{R}} s_m t_2 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_m t_n).\end{aligned}$$

Consider the reduction operation Red on $\bar{S}^{\mathscr{R}}$ that replaces, in a $\geq_{\mathscr{R}}$-chain, each maximal consecutive section in which all elements are $\mathscr{R}$-equivalent by the rightmost element. Then the set $\hat{S}^{\mathscr{R}}$ of all reduced (i.e., strict) $\geq_{\mathscr{R}}$-chains is a semigroup under the operation $s \cdot t = \operatorname{Red}(st)$, where the operation on the right hand side is the one defined above for the semigroup $\bar{S}^{\mathscr{R}}$.

Assume next that S is an *A-generated semigroup*; i.e., a semigroup S endowed with a function $\iota : A \to S$ whose image generates S. We define an associated mapping $\nu : A \to \hat{S}^{\mathscr{R}}$ by letting νa be the singleton chain (a), for each $a \in A$. Then the image of ν generates an A-generated subsemigroup, which we denote by $\hat{S}^{\mathscr{R}}_A$, and which we call the *right Rhodes expansion* of the A-generated semigroup S. Note that the mapping sending each finite reduced $\geq_{\mathscr{R}}$-chain to its lowest element defines an onto homomorphism $\eta^{\mathscr{R}}_S : \hat{S}^{\mathscr{R}}_A \to S$ that maps (a) to a. Usually, for an A-generated semigroup S, we shall use indiscriminately a symbol a to denote an element of A and the corresponding generator of S. This is justified whenever the associated mapping $\iota : A \to S$ is injective which is often the case.

Let S be an A-generated semigroup. We say that S *has a content function* c if there is a monoid homomorphism $c : S^1 \to \mathscr{P}(A)$ into the semilattice of all subsets of A under union such that $c(\iota a) = \{a\}$, for every $a \in A$. In particular, there is at most one content function on the A-generated semigroup S. As no confusion should result from it, we shall adopt the convention that all content functions, irrespective of the semigroup, are denoted by c. In case S has a content function, for each $s \in S$ and each $X \subseteq c(s)$, we let $0_X(s)$ denote the set of all $s_0 \in S^1$ such that there is a factorization $s = s_0 a s_1$ with $a \in X$ and X is the disjoint union of $c(s_0) \cap X$ and $\{a\}$; we also denote by $\bar{0}_X(s)$ the set of all such $a \in X$. In case $X = c(s)$, we write simply $0(s)$ and $\bar{0}(s)$ respectively for $0_X(s)$ and $\bar{0}_X(s)$. The sets $1_X(s)$, $\bar{1}_X(s)$, $1(s)$, and $\bar{1}(s)$ are defined dually. Note that $0(s)$ and $\bar{0}(s)$ are always nonempty sets but each may have more than one element. We say that S *has* 0 (respectively $\bar{0}$, 1, $\bar{1}$) *function* if $0(s)$ (respectively $\bar{0}(s)$, $1(s)$, $\bar{1}(s)$) is a singleton for every $s \in S$. For instance, the free semigroup A^+ and the free band on A have 0, $\bar{0}$, 1, $\bar{1}$ functions. The following generalization is essentially part of the proof of [**31**, Theorem 4.4] and was probably well known in 1990.

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PROPOSITION 3.1. *Let S be an A-generated semigroup with a content function. Then $\hat{S}_A^{\mathscr{R}}$ has content, 0, and $\bar{0}$ functions.*

Proof. Composing the mapping $\eta_S^{\mathscr{R}}$ with a content function for S, we obtain a content function for $\hat{S}_A^{\mathscr{R}}$.

Suppose next that $s_0, s_0' \in 0(s)$ for a given $\geq_{\mathscr{R}}$-chain $s \in \hat{S}_A^{\mathscr{R}}$. By definition of $0(s)$, there are factorizations $s = s_0 a s_1 = s_0' b s_1'$ such that $a, b \in A$, $c(s_0) \neq c(s) = c(s_0) \cup \{a\}$, and $c(s_0') \neq c(s) = c(s_0') \cup \{b\}$. We must show that $s_0 = s_0'$ and $a = b$. Since the content function is a homomorphism, we have the relations $s \leq_{\mathscr{R}} s_0 a <_{\mathscr{R}} s_0$ and $s \leq_{\mathscr{R}} s_0' b <_{\mathscr{R}} s_0'$. By [**15**, Proposition XII.12.1] (or [**13**, Fact 2.8]), the elements s_0, $s_0 a$, s_0', and $s_0' b$ are all comparable under the relation $\leq_{\mathscr{R}}$. But, in the presence of a content function, if the elements are $\leq_{\mathscr{R}}$-comparable, then their contents are $\subseteq$-comparable. It follows that $c(s_0) = c(s_0')$ and $a = b$. Without loss of generality, we may assume that $s_0 a \leq_{\mathscr{R}} s_0' a$. Then, since $s_0 a <_{\mathscr{R}} s_0$ and $s_0' a <_{\mathscr{R}} s_0'$, from the definition of the Rhodes expansion and [**15**, Proposition XII.12.1] it follows easily that $s_0 = s_0'$. □

The *left Rhodes expansion* $\hat{S}_A^{\mathscr{L}}$ of the A-generated semigroup S is defined dually to the right Rhodes expansion and naturally enjoys dual properties.

The *Birget expansion* $\hat{S}_A^{+}$ of the A-generated semigroup S is defined to be the projective limit of the successive alternating right and left Rhodes expansions $\hat{S}_A^{\mathscr{R}}$, $\widehat{(\hat{S}_A^{\mathscr{R}})}_A^{\mathscr{L}}$, and so on. Birget [**13**] showed that $\hat{S}_A^{+}$ coincides with its own right and left Rhodes expansions. We thus have the following result.

COROLLARY 3.2. *Let S be an A-generated semigroup with a content function. Then $\hat{S}_A^{+}$ has content, 0, $\bar{0}$, 1, and $\bar{1}$ functions.* □

Another important property of the Birget expansion is that its value on a finite semigroup is again a finite semigroup which can be effectively computed from the given semigroup [**13**].

We say that a pseudovariety $\mathbf{V}$ is *closed under right Rhodes expansions* if, for every A-generated semigroup $S \in \mathbf{V}$, $\hat{S}_A^{\mathscr{R}} \in \mathbf{V}$. Similar definitions may be given for the left Rhodes expansion and the Birget expansion. Note that a pseudovariety is closed under Birget expansions if and only if it is closed under both left and right Rhodes expansions.

For example, by [**31**, Lemma 4.3] and [**15**, Property XII.(9.4)] respectively, the pseudovarieties $\mathbf{CR}$ and $\overline{\mathbf{H}}$ are closed under Birget expansions for any pseudovariety $\mathbf{H}$ of groups. Of course, the intersection of a nonempty family of pseudovarieties closed under Birget expansions is still closed under Birget expansions. It is an easy exercise to verify that $\mathbf{DS}$ is also closed under Birget expansions. Further examples may be derived from the following result.

PROPOSITION 3.3. *For any pseudovariety $\mathbf{V}$ of semigroups, $\mathbf{B} ⓜ \mathbf{V}$ is closed under Birget expansions.*

Proof. Let S be an A-generated semigroup in $\mathbf{B} ⓜ \mathbf{V}$. Then S is a homomorphic image of some finite A-generated semigroup S' such that there is a homomorphism $\varphi : S' \to T$ with $T \in \mathbf{V}$ and $\varphi^{-1}e \in \mathbf{B}$, for every idempotent e in T. Since $\hat{S}_A^{\mathscr{R}}$ is then a homomorphic image of $\widehat{S'}_A^{\mathscr{R}}$, we may as well assume that $S' = S$. Let $\eta_S^{\mathscr{R}} : \hat{S}_A^{\mathscr{R}} \to S$ be the natural projection. Then $\varphi \circ \eta_S^{\mathscr{R}} : \hat{S}_A^{\mathscr{R}} \to T$ is a homomorphism and, for an

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arbitrary idempotent $e \in T$ and $x = (s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n)$ such that $\varphi\eta_S^{\mathscr{R}}(x) = e$, we have $\varphi s_n = e$, and so s_n is idempotent by the assumption on φ, which implies that

$$x^2 = \mathrm{Red}\,(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n \geq_{\mathscr{R}} s_n s_1 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_n s_{n-1} \geq_{\mathscr{R}} s_n^2) = x.$$

Hence $\hat{S}_A^{\mathscr{R}} \in \mathbf{B}ⓜ\mathbf{V}$, which shows that $\mathbf{B}ⓜ\mathbf{V}$ is closed under right Rhodes expansions. Similarly, $\mathbf{B}ⓜ\mathbf{V}$ is closed under left Rhodes expansions and therefore it is closed under Birget expansions. □

Note that the converse of Proposition 3.3, namely "the closure under Mal'cev product on the left by **B**" for a pseudovariety closed under Birget expansions, fails for instance for **CR** since $\mathbf{B}ⓜ\mathbf{CR} \supseteq \mathbf{B}ⓜ\mathbf{G} \not\subseteq \mathbf{CR}$. See [**14**].

For the following corollary, we recall that, as a particular case of [**28**, Theorem 4.1], if **V** is a pseudovariety of semigroups, then

$$\mathbf{B}ⓜ\mathbf{V} = [\![u^2 = u : \mathbf{V} \models u^2 = u]\!], \tag{1}$$

where u stands for an arbitrary implicit operation.

Corollary 3.4. *The pseudovarieties* **OCR**, **DA**, *and* **DO** *are closed under Birget expansions.*

Proof. For **OCR**, it suffices to observe that $\mathbf{OCR} = \mathbf{CR} \cap (\mathbf{B}ⓜ\mathbf{G})$. See [**30**]. Hence **OCR** is closed under Birget expansions, by Proposition 3.3, and since that property is preserved by intersection.

For **DA**, we have the decomposition $\mathbf{DA} = \mathbf{B}ⓜ\mathbf{J}$: by definition of **DA** and **J**, an implicit operation is idempotent in one of the pseudovarieties if and only if it is idempotent in both and so the equality follows from (1).

Similarly, we have $\mathbf{DO} = \mathbf{DS} \cap (\mathbf{B}ⓜ\mathbf{DG})$: by [**6**, Theorem 4.11], an implicit operation is idempotent in **DO** if and only if it is idempotent in **DG** and so again the equality follows from (1). □

The significance of closure under Birget expansions comes from the following simple observation.

Proposition 3.5. *If a pseudovariety* **V** *is closed under right Rhodes expansions and* **V** *contains* **Sl**, *then* $\overline{\Omega}_A\mathbf{V}$ *has content,* 0 *and* $\overline{0}$ *functions.*

Proof. The content function is just the canonical projection $\overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{Sl}$.

Suppose next that $u \in \overline{\Omega}_A\mathbf{V}$, $a, a' \in A$, and $u_0, u_0', u_1, u_1' \in (\overline{\Omega}_A\mathbf{V})^1$ are such that

$$u = u_0 a u_1 = u_0' a' u_1' \text{ with } a \notin c(u_0),\ a' \notin c(u_0'),\ \text{and } c(u_0 a) = c(u) = c(u_0' a'). \tag{2}$$

Suppose further that $u_0 \neq u_0'$. Then there is a semigroup S in **V** and there is an onto homomorphism $\varphi : \overline{\Omega}_A\mathbf{V} \to S$ such that $\varphi u_0 \neq \varphi u_0'$. Notice that, since $\mathbf{Sl} \subseteq \mathbf{V}$ and $\overline{\Omega}_A\mathbf{Sl}$ is the free (finite) semilattice on A, the subsemigroup S' of $S \times \overline{\Omega}_A\mathbf{Sl}$ generated by $\{(\varphi a, c(a)) : a \in A\}$ is such that the projection of $\overline{\Omega}_A\mathbf{V}$ onto S' separates u_0 and u_0' and has a content function. By Proposition 3.1, $\widehat{S'}_A^{\mathscr{R}}$ also has these properties and has a 0-function. We may therefore assume that $S = \widehat{S'}_A^{\mathscr{R}}$.

Since S has a content function, the conditions (2) are preserved under the application of φ in the sense that $\varphi u = (\varphi u_0) a (\varphi u_1) = (\varphi u_0') a' (\varphi u_1')$, $a \notin c(\varphi u_0)$,

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$a' \notin c(\varphi u_0')$, and $c((\varphi u_0)a) = c(\varphi u) = c((\varphi u_0')a')$. Since S has a 0 function, we reach the contradiction $\varphi u_0 = 0(\varphi u) = \varphi u_0'$. Hence $u_0 = u_0'$ and the equality $a = a'$ follows from (2), since $\{a\} = c(u) \setminus c(u_0)$. □

COROLLARY 3.6. *If a pseudovariety* $\mathbf{V}$ *is closed under Birget expansions and* $\mathbf{V}$ *contains* $\mathbf{Sl}$*, then* $\overline{\Omega}_A\mathbf{V}$ *has content,* 0*,* $\bar{0}$*,* 1*, and* $\bar{1}$ *functions.* □

We do not know whether the converses of Proposition 3.5 and Corollary 3.6 are also valid.

4. Pseudovarieties of the form $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V}$. For a pseudovariety $\mathbf{V}$, denote by $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V}$ the intersection $(\mathbf{B} \textcircled{m} \mathbf{V}) \cap \mathbf{CR}$. The aim of this section is to show that if $\mathbf{V}$ is a hyperdecidable pseudovariety of groups then so is $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V}$.

4.1. The word problem. The following result provides a relatively easy theoretical solution of the word problem for pseudovarieties of the form $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V}$.

THEOREM 4.1. *Let* $u, v \in \overline{\Omega}_A\mathbf{S}$ *and* $\mathbf{V} \subseteq \mathbf{CR}$*. Then* $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V} \models u = v$ *if and only if the following conditions hold:*

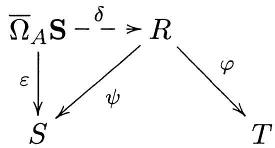
(i) $c(u) = c(v)$;
(ii) $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V} \models 0(u) = 0(v)$;
(iii) $\mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V} \models 1(u) = 1(v)$;
(iv) $\mathbf{V} \models u = v$.

Proof. Let $\mathbf{W} = \mathbf{B} \textcircled{m}_{\mathrm{CR}} \mathbf{V}$.

Assuming that the pseudoidentity $u = v$ is valid in $\mathbf{W}$, we obtain (i) and (iv), since $\mathbf{W}$ contains $\mathbf{Sl}$ and $\mathbf{V}$, respectively. By Corollary 3.6 and Proposition 3.3, since $\mathbf{CR}$ is closed under Birget expansions, we also obtain (ii) and (iii).

Conversely, suppose that conditions (i)–(iv) hold. We also view elements of $\overline{\Omega}_A\mathbf{S}$ as being members of $\overline{\Omega}_A\mathbf{W}$ by restriction. Since $\mathbf{W} \subseteq \mathbf{DS}$, by (i) and [**2**, Theorem 8.1.7] we deduce that $u \mathscr{J} v$. Since $\mathbf{W} \subseteq \mathbf{CR}$, letting $x = \bar{0}(u)$ ($= \bar{0}(v)$ by (i) and (ii)), we have $u \mathscr{R} 0(u)x = 0(v)x \mathscr{R} v$ and, dually, $u \mathscr{L} v$. Hence $u \mathscr{H} v$.

Let S be an element of $\mathbf{W}$. Then there are semigroups R and T and there are onto homomorphisms $\varphi : R \to T$ and $\psi : R \to S$ such that $T \in \mathbf{V}$ and $\varphi^{-1}e \in \mathbf{B}$ for every idempotent $e \in T$. Consider an evaluation homomorphism $\varepsilon : \overline{\Omega}_A\mathbf{S} \to S$. Since ψ is onto, ε lifts to a homomorphism δ such that $\psi \circ \delta = \varepsilon$ as depicted in the following diagram.



Since $\mathbf{V} \models u = v$, the equality $\varphi\delta u = \varphi\delta v$ holds in T. Now the element $e = \varphi\big((\delta u)^{\omega-1}\,\delta v\big) = (\varphi\delta u)^{\omega}$ is an idempotent. Since $\varphi^{-1}e \in \mathbf{B}$, it follows that $(\delta u)^{\omega-1}\delta v$ is idempotent and therefore so is its image under ψ; that is $(\varepsilon u)^{\omega-1}\varepsilon v$. Hence $\varepsilon u = \varepsilon v$, since the two elements lie in the same group, by the preceding paragraph. This shows that $\mathbf{W} \models u = v$, as desired. □

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Note that, if $\mathbf{V} \subseteq \mathbf{CR}$ and a "solution" of the word problem (not necessarily in an algorithmic sense) is known for $\mathbf{V}$, then an inductive "solution" (on the content) of the word problem for $\mathbf{B} \textcircled{m}_{\mathbf{CR}} \mathbf{V}$ follows from Theorem 4.1. If we restrict attention to implicit operations which are given by weak terms, then we get a solution of the corresponding word problem by applying Theorem 4.1. In order to explain how the solution works, we just need to describe how to compute the operations 0 and 1 on weak terms. By the "usual way of representing" a weak term $w \in \Omega_A^\kappa \mathbf{S}$ we mean the sequence $\rho(w)$ of symbols that describe it defined recursively as follows:

- for $a \in A$, $\rho(a)$ is the singleton sequence $\langle a \rangle$,
- for $w = u \cdot v$, $\rho(w) = \rho(u)\, \rho(v)$,
- for $w = (u)^{\omega-1}$, $\rho(w) = \langle\, (\,\rangle\, \rho(u)\, \langle\,)^{\omega-1}\, \rangle$,

where we are representing by juxtaposition the usual operation of concatenation of sequences. Note that a sequence in the alphabet $A \cup \{\,(\,,\ )^{\omega-1}\}$ represents a κ-term if and only if it has no consecutive entries and $(\quad)^{\omega-1}$ and the subsequence obtained by erasing the entries from A is a Dyck word (i.e., *parentheses match*). The following lemma is easily proved.

LEMMA 4.2. *Let w be an element of the free κ-semigroup $\Omega_A^\kappa \mathbf{S}$. Then*
(a) $0(w)$ *is the κ-term obtained from w by taking in $\rho(w)$ the shortest prefix which contains among its entries all but one member of $c(w)$ and deleting unmatched parentheses;*
(b) $1(w)$ *is obtained dually.*
In particular, $\Omega_A^\kappa \mathbf{S}$ is closed under the operations 0 *and* 1. □

This immediately yields the following result [**29**, Theorem 1] which extends the case of $\mathbf{V} = \mathbf{G}$ found in [**16**].

COROLLARY 4.3. *Let $\mathbf{V} \subseteq \mathbf{CR}$. The word problem for the relatively free κ-semigroup $\Omega_A^\kappa \mathbf{2}(\mathbf{B} \textcircled{m}_{\mathbf{CR}} \mathbf{V})$ is decidable if it is decidable for $\Omega_A^\kappa \mathbf{V}$.* □

4.2. Some geometric constructions. Let S be a finite A-generated semigroup with content, 0 and 1 functions. We denote by φ and $\bar{\varphi}$ respectively the homomorphisms $A^+ \to S$ and $\overline{\Omega}_A \mathbf{S} \to S$ determined by the choice of generators.

Let $s \in S$. In a factorization $s = 0(s)\, s_1\, 1(s)$ with $s_1 \in S$, we say that 0 *and* 1 *do not overlap* while, in a factorization $s = s_2\, 1_a(0(s))\, s_3$ with $s_2\, 1_a(0(s)) = 0(s)$, $a = \bar{1}(s)$, $b = \bar{0}(s)$, $1_a(0(s)) = 0_b(1(s))$, and $0_b(1(s))\, s_3 = 1(s)$, we say that 0 *and* 1 *overlap*. Note that for any choice of $u \in \overline{\Omega}_A \mathbf{S}$ such that $\bar{\varphi} u = s$, a factorization of u yields a factorization of s satisfying one of the two conditions and every such factorization of s can be obtained in this way. On such factorizations, we may further

- take s_1 to be of one of the forms $s_1 = \bar{0}(s)\, s_4\, \bar{1}(s)$, with $s_4 \in S^1$, or $s_1 = \bar{0}(s) = \bar{1}(s)$,
- take s_2 of the form $s_2 = s_5\, \bar{1}(s)$,
- and take s_3 of the form $s_3 = \bar{0}(s)\, s_6$. (See Figure 1.)

In general, for each $s \in S$, there are several ways of factorizing it in the above forms but, as a problem involving just the computation of products, contents, 0 and 1 function values, we may effectively find all such factorizations.

We define recursively a set of S-edge-labeled chains $\Gamma(s)$ associated with $s \in S^1$ as follows. A graph γ in $\Gamma(s)$ is generically represented by $\overset{\gamma}{\rightsquigarrow}$. At the basis of

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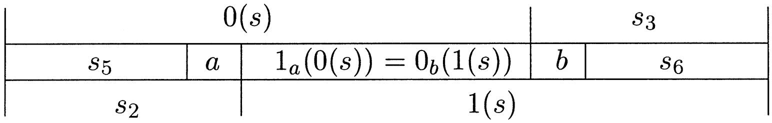
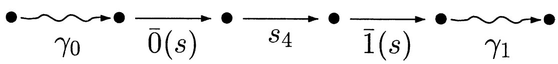


Figure 1: A possible overlapping factorization of s.

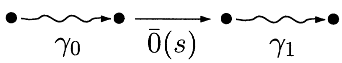
this recursive definition, take $\Gamma(1) = \{\bullet\}$, the trivial graph reduced to one vertex. For $s \neq 1$, each element of $\Gamma(s)$ is associated with an expression for s, according to three cases of overlap between the prefix $0(s)\bar{0}(s)$ and the suffix $\bar{1}(s)1(s)$ in the sense above, as follows:

(a) to the factorization $s = 0(s)\,\bar{0}(s)\,s_4\,\bar{1}(s)\,1(s)$ associate any of the graphs



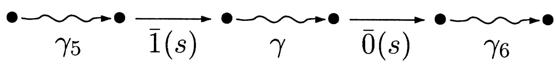
where $\overset{\gamma_0}{\rightsquigarrow}$ (respectively $\overset{\gamma_1}{\rightsquigarrow}$) stands for an element of $\Gamma(0(s))$ (respectively $\Gamma(1(s))$);

(b) to the factorization $s = 0(s)\,\bar{0}(s)\,1(s)$ associate any of the graphs



with γ_0 and γ_1 as in the preceding case;

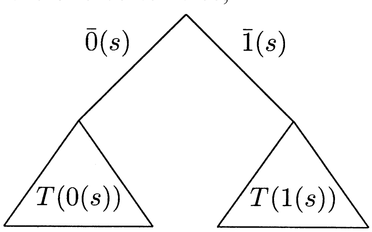
(c) to the factorization $s = s_5\,\bar{1}(s)\,1_a(0(s))\,\bar{0}(s)\,s_6$ associate any of the graphs



where $\bullet\overset{\gamma_5}{\rightsquigarrow}\bullet\xrightarrow{\bar{1}(s)}\bullet\overset{\gamma}{\rightsquigarrow}\bullet$ is an element of $\Gamma(0(s))$, the distinguished arrow being the last labeled with $\bar{1}(s)$, and $\bullet\overset{\gamma}{\rightsquigarrow}\bullet\xrightarrow{\bar{0}(s)}\bullet\overset{\gamma_6}{\rightsquigarrow}\bullet$ is an element of $\Gamma(1(s))$, the distinguished arrow being the first labeled with $\bar{0}(s)$.

The letters $\bar{0}(s)$ and $\bar{1}(s)$ are called the *top markers* of s. Each element s of S^1 determines a labeled tree $T(s)$ called the *tree of markers*, which is defined recursively as follows:

- $T(1)$ is the one-vertex tree;
- $T(s) =$



By the well-known solution of the word problem for bands [**22**], the set of all such trees of markers constitutes a free band on the set A under the multiplication $T(s)T(t) = T(st)$, and so the function $s \mapsto T(s)$ may be viewed as a homomorphism $S \to \overline{\Omega}_A\mathbf{B}$.

For the remainder of this subsection, we fix a pseudovariety $\mathbf{H}$ of groups and we let $\mathbf{W} = \mathbf{B}\,ⓜ_{\mathrm{CR}}\mathbf{H}$.

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Let Υ be a finite graph and let $f : \Upsilon \to S^1$ be a labeling in which edges are labeled by elements of S. The labeling f is *consistent* if, for every edge x in Υ, $f(\alpha x)f(x) = f(\omega x)$. Recall that the labeling f is said to be *inevitable with respect to a pseudovariety* **W** if there is a lifting of f to a labeling $g : \Upsilon \to (\overline{\Omega}_A\mathbf{S})^1$ such that $\bar{\varphi} \circ g = f$ and $h = p_{\mathbf{W}} \circ g : \Upsilon \to (\overline{\Omega}_A\mathbf{W})^1$ is consistent. The pseudovariety **W** is said to be *hyperdecidable* if there is an algorithm to test **W**-inevitability of labelings of finite graphs by finite semigroups. For the source and motivation for these notions, see [**3**].

Sometimes only the labels of edges are considered. A function $f : E(\Upsilon) \to S$ is said to be an *edge-labeling* of the graph Υ by S. By the *label* of an undirected path p in Υ we mean the product, in order, of either the label $f(e)$ of each edge e if it appears in the direction of the path, or $(f(e))^{\omega-1}$ otherwise. We say that the edge-labeling f *commutes* if the label of every undirected path in Υ depends only on where it starts and where it ends. If S is a group, the edge-labeling $f : E(\Upsilon) \to S$ commutes if and only if the label of every circuit is the identity element of S. A pseudovariety **H** of groups is hyperdecidable if and only if there is an algorithm to test whether an edge-labeling of a finite graph by an A-generated finite semigroup may be lifted to an edge-labeling by $\overline{\Omega}_A\mathbf{S}$ whose composite with $p_{\mathbf{H}}$ commutes, in which case we also say that the initial edge-labeling is **H**-*inevitable* [**3**].

Let σ be an implicit signature. A pseudovariety **W** is said to be σ-*reducible* [**8**, **9**] if every **W**-inevitable labeling of a finite graph by a finite A-generated semigroup has a lifting to a labeling by $\Omega^\sigma_A\mathbf{S}$ whose composite with $p_{\mathbf{W}}$ is consistent. Again, in the case of a pseudovariety **W** of groups, this condition is equivalent to the following: every **W**-inevitable edge-labeling of a finite graph by a finite A-generated semigroup has a lifting to an edge-labeling by $\Omega^\sigma_A\mathbf{S}$ whose composite with $p_{\mathbf{W}}$ commutes. A κ-reducible pseudovariety is also said to be *canonically reducible*. By [**12**, **8**], the pseudovariety of all finite groups is canonically reducible. By [**8**], a canonically reducible pseudovariety **W** such that the word problem for relatively free κ-semigroups $\Omega^\kappa_A\mathbf{W}$ is decidable is hyperdecidable.

Let S be a finite semigroup with a content, 0 and 1 functions and let $s \in S$. In the following, we shall construct from the graphs in the sets $\Gamma(s)$ certain S-labeled graphs for which we shall be interested in testing **H**-inevitability. In these graphs, some edges, corresponding to markers, are to be labeled with generators, and we want to lift such labels to the same generators of $\overline{\Omega}_A\mathbf{S}$. But, it may happen that a generator has in S a nontrivial expression in terms of generators. This difficulty may be overcome in an elementary way by ensuring from the start, before constructing the Birget expansion, that $\bar{\varphi}|_{A^+} : A^+ \to S$ recognizes each of the languages $\{a\}$ with $a \in A$, which can be easily achieved by replacing S by an effectively constructible semigroup that S divides.

If a labeled graph $\gamma \in \Gamma(s)$ is relabeled by elements of $\overline{\Omega}_A\mathbf{S}$ such that, for each edge $\xrightarrow{t}$, the new label belongs to $\bar{\varphi}^{-1}t$, then the product of the new labels in the order they appear in the chain γ produces some $u \in \overline{\Omega}_A\mathbf{S}$ such that $\bar{\varphi}u = s$. Moreover, the successive 0, 1, $\bar{0}$, $\bar{1}$ factorization of u, projected via $\bar{\varphi}$, defines precisely the graph γ.

Since $\mathbf{B} \subseteq \mathbf{W}$, a necessary condition for **W**-inevitability of f is that it be **B**-inevitable. Since $\overline{\Omega}_A\mathbf{B}$ is finite and effectively constructible, this condition may be effectively tested. From here on, we assume that f is **B**-inevitable.

For each $x \in \Upsilon$, let $\gamma_x \in \Gamma(f(x))$. We build up an edge-labeled graph $\Gamma(\gamma_x; x \in \Upsilon)$ as follows. For each edge x of Υ, we first build a "triangle" $\triangle_x$ from the chains $\gamma_{\alpha x}$, γ_x, and $\gamma_{\omega x}$ by identifying endpoints, as in the following picture.

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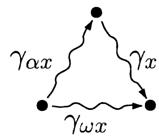
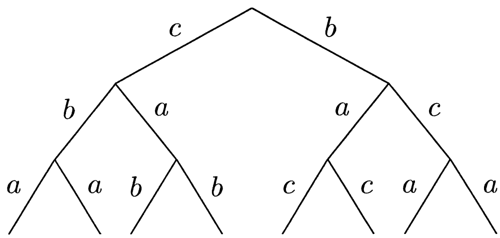


Each edge in the triangle $\triangle_x$ corresponds to either a vertex (αx or ωx) or the edge x of the graph Υ. Denote by $\check{\Gamma}(\gamma_x; x \in \Upsilon)$ the underlying unlabeled graph of the labeled graph $\Gamma(\gamma_x; x \in \Upsilon)$ and by $f' : \check{\Gamma}(\gamma_x; x \in \Upsilon) \to S$ the label itself.

Since a vertex may be an endpoint of several edges, or even both endpoints of the same edge, the triangles $\triangle_x$ $(x \in E(\Upsilon))$ share their edges corresponding to vertices and the "triangle" $\triangle_x$ will be somewhat degenerated in case $\alpha x = \omega x$.

We define a gluing procedure which is meant to deal with consistency of the labeling f at the edge x; it comes from comparing the trees of markers for the labeled chain $\gamma_{\omega x}$ and the concatenated chain $\overset{\gamma_{\alpha x}}{\leadsto}\overset{\gamma_x}{\leadsto}$, which we denote by $\gamma_{\alpha x}\gamma_x$, for each edge x of Υ. Observe that the markers in the concatenated chain $\gamma_{\alpha x}\gamma_x$ are already distinguished as markers in one of the chains $\gamma_{\alpha x}$ or γ_x, although some of the markers in each of $\gamma_{\alpha x}$ and γ_x may not be markers in $\gamma_{\alpha x}\gamma_x$. Since, by the hypothesis that f is **B**-inevitable, the trees corresponding to $\gamma_{\alpha x}\gamma_x$ and $\gamma_{\omega x}$ are equal, we may identify edges which, in the "triangulated" graph previously constructed, belong to edges in the sides of the same triangle that correspond to the same markers in these trees.

EXAMPLE 4.4. Suppose that a, b, c, d are distinct elements of A. Then, from the factorizations of elements of S associated with the words $ab^4cbcabc^3a$ and abc^4bca^2, we obtain the same tree of markers



the respective graphs, (where the label words are to be viewed as their values in S under $\bar{\varphi}$), being

$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{b^2}\xrightarrow{b}\xrightarrow{c}\xrightarrow{bca}\xrightarrow{b}\xrightarrow{c}\xrightarrow{c}\xrightarrow{c}\xrightarrow{a}$$

$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{c}\xrightarrow{c^3}\xrightarrow{b}\xrightarrow{c}\xrightarrow{a}\xrightarrow{a}\;.$$

Observe that $1(0(ab^4cbcabc^3a)) = b^4$ and the factorization is

$$\bar{0}(1(0(ab^4cbcabc^3a)))\, b^2\, \bar{1}(1(0(ab^4cbcabc^3a))) = b \cdot b^2 \cdot b,$$

as shown. Consider the S-labeled graph

$$1 \bullet \xrightarrow{\quad abc^4bca^2 \quad} \bullet\; ab^4cbcabc^3a.$$

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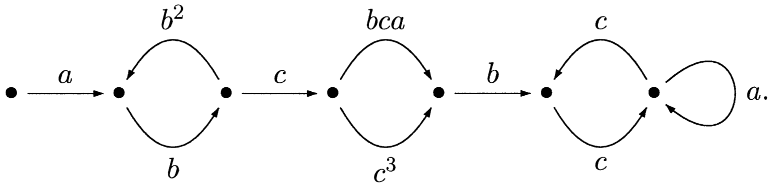
The corresponding glued graph is the following:

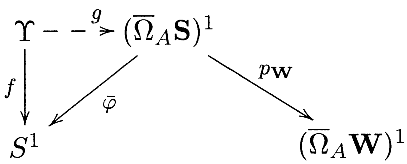


Notice that $1(0(abc^4bca^2)) = b$, so that the next factorization is overlapping with $\bar{0}(0(1(abc^4bca^2)))$ and $\bar{1}(0(1(abc^4bca^2)))$ being identical (of value b). Hence these markers, as well as the markers $\bar{0}(0(1(ab^4cbcabc^3a)))$ and $\bar{1}(0(1(ab^4cbcabc^3a)))$, are all glued together as shown. Similarly, the final two glued sections of the graph can be explained.

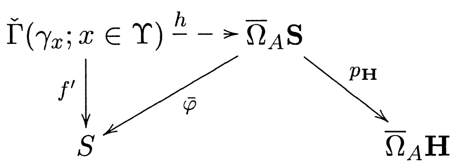
Each edge in the graph $\Gamma(\gamma_x;\, x \in \Upsilon)$ is either obtained from gluing markers, and thus may come from markers in edges of several of the triangles Δ_x (x edge of Υ), or else it can be traced to a specific edge in a unique chain γ_x ($x \in \Upsilon$).

PROPOSITION 4.5. *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $\mathbf{W} = \mathbf{B} \circledm_{\mathrm{CR}} \mathbf{H}$. *Then, with the above notation and hypotheses, the following are equivalent.*

(i) *There is a labeling* $g : \Upsilon \to (\overline{\Omega}_A \mathbf{S})^1$ *such that* $\bar{\varphi} \circ g = f$, $p_{\mathbf{W}} \circ g : \Upsilon \to (\overline{\Omega}_A \mathbf{W})^1$ *is consistent, and, for each* $x \in \Upsilon$, *the successive* $0, 1, \bar{0}, \bar{1}$ *factorization of* $g(x)$ *defines the chain* γ_x.



(ii) *There is an edge-labeling* h *of* $\check{\Gamma}(\gamma_x;\, x \in \Upsilon)$ *by elements of* $\overline{\Omega}_A \mathbf{S}$ *such that* $\bar{\varphi} \circ h$ *is the corresponding edge-labeling* f' *and* $p_{\mathbf{H}} \circ h$ *commutes.*



Proof. (i)⇒(ii). Consider a labeling g as in (i). As was observed above, each edge of $\Gamma(\gamma_x;\, x \in \Upsilon)$ comes from an edge of a chain γ_x and in a unique way if it is a non-marking edge. Since we are assuming that the values in S of the generators do not admit any other expressions in terms of the generators, it is only the non-marking edges that are to be relabeled. We relabel such an edge, coming from a specific edge of γ_x by the corresponding factor of $g(x)$. This clearly defines a relabeling h of $\Gamma(\gamma_x;\, x \in \Upsilon)$ which projects under $\bar{\varphi}$ to the original labeling. We must show that $p_{\mathbf{H}} \circ h$ commutes. For this purpose, it is convenient to localize the problem by further labeling the vertices of $\Gamma(\gamma_x;\, x \in \Upsilon)$ and showing that the extended labeling $\hat{h}$ is such that $p_{\mathbf{H}} \circ \hat{h}$ is consistent.

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Note that h may also be viewed as a labeling of each individual chain γ_x. To label the vertices of $\Gamma(\gamma_x; x \in \Upsilon)$, we first label each vertex of the chains γ_x by taking the product of the labels of the path from the initial vertex to the vertex in question, the empty product being taken to be 1. If x is an edge, we then further multiply on the left the labels of all the vertices of γ_x by $g(\alpha x)$.

We claim that, if the vertices p from γ_y and q from γ_z are identified in $\Gamma(\gamma_x; x \in \Upsilon)$, then their labels are equal in $\mathbf{H}$. Without loss of generality, we may assume that y and z are vertices in chains making up the sides of the same triangle $\triangle_x$, for the global identification of vertices is just the transitive closure of this local identification. This observation brings the claim down to showing that in the basic gluing process of two chains labeled in $(\overline{\Omega}_A\mathbf{S})^1$, with initial vertices labeled 1, and with the same tree of markers, if two vertices q and r are identified, then their labels are equal in $\mathbf{H}$. We may use the additional hypothesis, which results in our case from consistency of the labeling $p_{\mathbf{W}} \circ g$, that the final vertices of the two chains have matching labels in $\mathbf{W}$, say π_q and π_r. We next prove this localized claim by induction on the size of the content, the case of empty content being trivial.

Since the two vertices q and r are identified, they must be either both beginnings or both endings of edges corresponding to the same marker when markers in the two chains are glued together. Clearly the beginnings of two marker edges which are identified have labels which are equal in $\mathbf{H}$ if and only if the same happens with their endings, the labels for the latter being obtained from those for the former by multiplying by the same element of A. Note also that every vertex in a chain γ_x is either a beginning or an ending of a marker edge. This is not true for chains of the form $\gamma_{\alpha x}\gamma_x$, but the remaining vertices do not get identified in the gluing process within the triangle $\triangle_x$. Moreover, for such concatenated chains, replacing maximal paths whose intermediate vertices are not ends of marker edges in the concatenated chain by single edges whose label is the product of the labels in the path, we obtain a 0, 1, $\bar{0}$, $\bar{1}$ factorization of the product $g(\alpha x)g(x)$. Hence it suffices to show that the claim holds if q and r appear both in the 0 or both in the 1 portions of their chains. Now, by the solution of the word problem for $\mathbf{W}$ given by Theorem 4.1, $\mathbf{W}$ satisfies the pseudoidentities $0(\pi_q) = 0(\pi_r)$ and $1(\pi_q) = 1(\pi_r)$, which reduces the claim to a smaller content case, for which the induction hypothesis applies. This establishes the claim.

In view of the claim, we label each vertex v of $\Gamma(\gamma_x; x \in \Upsilon)$ with the label of any of the vertices which are glued to produce v. Since each chain was labeled consistently over $\mathbf{S}$, from the claim it follows that $\hat{h}$ is consistent over $\mathbf{H}$. Hence the edge-labeling h commutes over $\mathbf{H}$.

(ii)$\Rightarrow$(i). We shall construct a labeling g as in (i) from a labeling h as in (ii).

Let $x \in \Upsilon$. If γ_x is a trivial chain, then let $g(x) = 1$. Otherwise, retain the labels of the marker edges of γ_x and replace the label of each non-marker edge by its label under h; we let $g(x)$ be the product of the resulting labels. From the hypothesis that $\bar{\varphi} \circ h$ is the edge-labeling of $\Gamma(\gamma_x; x \in \Upsilon)$ constructed above and the choice of γ_x, it follows that $\bar{\varphi} \circ g = f$ and the 0, 1, $\bar{0}$, $\bar{1}$ factorization of each $g(x)$ defines the chain γ_x. Hence it remains to verify that $p_{\mathbf{W}} \circ g$ is consistent.

Let x be an edge of Υ. We claim that $\mathbf{W}$ satisfies the pseudoidentity $g(\alpha x)g(x) = g(\omega x)$. This corresponds to the gluing of chains associated with the triangle $\triangle_x$. Again, we prove by induction on the content a simplified version of the claim that we now formulate. Suppose γ and δ are two chains associated with 0, 1, $\bar{0}$, $\bar{1}$ factorizations of the elements u and v of $\overline{\Omega}_A\mathbf{S}$, respectively. Suppose further that u and v have the same tree of markers and that the edge-labeled graph $\gamma \amalg \delta$ resulting

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from gluing the two chains by identification of corresponding marker edges commutes in **H**. Then we claim that $\mathbf{W} \models u = v$. Here, if $c(u)$ (which is equal to $c(v)$ by the assumption on the trees of markers) is a singleton set, then the claim is obvious. Assume the claim holds for smaller content cases.

Now, by Theorem 4.1, since certainly $\mathbf{H} \models u = v$, it suffices to show that **W** satisfies the pseudoidentities $0(u) = 0(v)$ and $1(u) = 1(v)$. Since the two pseudoidentities may be treated dually, we only consider $0(u) = 0(v)$. Consider the maximal subchains of γ and δ that end at the beginning vertices of the marker edges corresponding to $\bar{0}(u)$ $(= \bar{0}(v))$. Denote these subchains respectively by $0(\gamma)$ and $0(\delta)$ and note that they also have the same tree of markers. Note also that there is a natural homomorphism from the graph $0(\gamma) \amalg 0(\delta)$ into $\gamma \amalg \delta$ that respects labels: an element of say $0(\gamma)$ is also an element of γ which determines an element of $\gamma \amalg \delta$; if two elements of $0(\gamma) \cup 0(\delta)$ are identified in $0(\gamma) \amalg 0(\delta)$, then they are elements of markers and the identification comes from the gluing of markers at corresponding positions, and so they are also identified in $\gamma \amalg \delta$. It follows that the edge-labeled graph $0(\gamma) \amalg 0(\delta)$ commutes over **H**, since cycles in this graph map to cycles in the graph $\gamma \amalg \delta$ under the natural homomorphism. Hence $\mathbf{W} \models 0(u) = 0(v)$ by the induction hypothesis. This proves the claim and completes the proof of the proposition. □

We may now establish one of the main results of this paper.

THEOREM 4.6. *Let* **H** *be a hyperdecidable pseudovariety of groups. Then* $\mathbf{B} \circledm_{\mathrm{CR}} \mathbf{H}$ *is also hyperdecidable.*

Proof. Let $f : \Upsilon \to S^1$ be a labeling of a finite graph by a finite semigroup S. Let A be a finite set such that S is A-generated say via the mapping $\varphi : A \to S$. As argued above, we may assume that each generator has no nontrivial expression in terms of the generators and that S has content, 0 and 1 functions. Moreover, since each label $f(x)$ has a finite number of possible overlap patterns in the successive 0, 1, $\bar{0}$, $\bar{1}$ factorization, it suffices to show that it is decidable whether there is a labeling g as in condition (i) of Proposition 4.5 for a given choice of chains $\gamma_x \in \Gamma(f(x))$ $(x \in \Upsilon)$. But, by the proposition, the existence of such a labeling is equivalent to **H**-inevitability of the effectively constructible edge-labeled graph $\Gamma(\gamma_x; x \in \Upsilon)$. Since **H** is assumed to be hyperdecidable, the result follows. □

Examples of (non-locally finite) hyperdecidable pseudovarieties of groups are the pseudovariety **G** of all finite groups [**12**, **3**] and the pseudovariety $\mathbf{G}_p$ of all finite p-groups [**34**]. Hence as a particular case of Theorem 4.6, we have the following result.

COROLLARY 4.7. *The pseudovariety* **OCR** *of all finite orthogroups is hyperdecidable.* □

4.3. Canonical reducibility. Our next aim is to show that the word "hyperdecidable" may be replaced by "canonically reducible" in the statement of Theorem 4.6. We start with a result complementary to Lemma 4.2.

LEMMA 4.8 *Each* $w \in \overline{\Omega}_A\mathbf{S}$ *has a unique* $0, \bar{0}, 1, \bar{1}$ *factorization.*

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Proof. It suffices to show that, if $w = 0(w)\bar{0}(w)w' = 0(w)\bar{0}(w)w''$ with $w', w'' \in (\overline{\Omega}_A\mathbf{S})^1$, then $w' = w''$. The result then follows from Proposition 3.5 and duality. To establish the claim, the shortest proof seems to be to consider the double semidirect product equality $\mathbf{S} = \mathbf{Sl} ** \mathbf{S}$ and to represent $\overline{\Omega}_A\mathbf{S}$ as a closed subsemigroup of a double semidirect product of a free profinite semilattice F, freely generated by the infinite profinite set $(\overline{\Omega}_A\mathbf{S})^1 \times A \times (\overline{\Omega}_A\mathbf{S})^1$, by $\overline{\Omega}_A\mathbf{S}$. We only sketch here the argument, leaving the details to the reader together with the reference [**10**, Theorem 5.1] for the representation in question. The two factorizations of w give rise to two representations of the image of w in $F{**}\overline{\Omega}_A\mathbf{S}$. Looking at the first components in these two representations, we find respectively the free profinite semilattice generators $(0(w), \bar{0}(w), w')$ and $(0(w), \bar{0}(w), w'')$, which can be found nowhere else in either of these representations. This implies that these free generators must be the same and so $w' = w''$. □

LEMMA 4.9. *Let w be an element of the free κ-semigroup $\Omega^\kappa_A\mathbf{S}$.*
(a) *If $w = 0(w)\bar{0}(w)w'\bar{1}(w)1(w)$ with $w' \in (\overline{\Omega}_A\mathbf{S})^1$, then $w' \in (\Omega^\kappa_A\mathbf{S})^1$.*
(b) *If $w = w'\bar{1}(w)1_a(0(w))\bar{0}(w)w''$ with $w', w'' \in (\overline{\Omega}_A\mathbf{S})^1$ and $a = \bar{1}(w)$, then $w', w'' \in (\Omega^\kappa_A\mathbf{S})^1$.*
Hence $\Omega^\kappa_A\mathbf{S}$ has 0 *and* 1 *functions.*

Proof. In view of Lemma 4.8, it suffices to show that the $0, \bar{0}, 1, \bar{1}$ factorization can be made in $(\Omega^\kappa_A\mathbf{S})^1$. Now, we already observed in Lemma 4.2 that $\Omega^\kappa_A\mathbf{S}$ is closed under the operations 0 and 1. Moreover, say to compute $0(w)$, Lemma 4.2 indicates how to fetch $\bar{0}(w)$ from a factor term $u^{\omega-1}$ of w, should it be found in there. Such a factor $u^{\omega-1}$ should then be replaced by the $0, \bar{0}$ factorization of u followed by $u^{\omega-1}u^{\omega-1}$. The result then follows by induction on the depth of application of the unary operation of weak inversion. □

We may now prove the second main result in this paper.

THEOREM 4.10. *Let* **H** *be a canonically reducible pseudovariety of groups. Then* $\mathbf{B}ⓜ_{\mathrm{CR}}\mathbf{H}$ *is also canonically reducible.*

Proof. Let $\mathbf{W} = \mathbf{B}ⓜ_{\mathrm{CR}}\mathbf{H}$. Let $f : \Upsilon \to S^1$ be a labeling of a finite graph by a finite semigroup S and suppose that it is **W**-inevitable. Let A be a finite set such that S is A-generated and let $\varphi : A \to S$ be a mapping describing S as an A-generated semigroup. Then there is a lifting of f to a labeling $g : \Upsilon \to (\overline{\Omega}_A\mathbf{S})^1$ such that $\bar{\varphi} \circ g = f$ and $p_{\mathbf{W}} \circ g$ is consistent.

For each $x \in \Upsilon$, let the chain γ_x be defined by the successive $0, \bar{0}, 1, \bar{1}$ factorization of $g(x)$ as in subsection 4.2. Then, by Proposition 4.5, there is an edge-labeling h of the glued graph $\Gamma(\gamma_x; x \in \Upsilon)$ by elements of $\overline{\Omega}_A\mathbf{S}$ such that $\bar{\varphi} \circ h$ is the natural edge-labeling f' of this graph by S and $p_{\mathbf{H}} \circ h$ commutes. In particular, the edge labeling f' is **H**-inevitable and so, since **H** is canonically reducible, there is an edge-labeling ℓ of Υ by $\Omega^\kappa_A\mathbf{S}$ such that $\bar{\varphi} \circ \ell = f'$ and $p_{\mathbf{H}} \circ \ell$ commutes. In view of Lemmas 4.2 and 4.9, the argument given in subsection 4.2 for the proof of (ii)⇒(i) in Proposition 4.5 shows that, multiplying the labels in each chain γ_x as attributed by ℓ (after gluing) gives rise to a labeling $\check{\ell}$ of Υ by $\Omega^\kappa_A\mathbf{S}$ that also lifts f and such that $p_{\mathbf{W}} \circ \check{\ell}$ is consistent. This shows that **W** is canonically reducible. □

In particular, we have the following result.

COROLLARY 4.11. *The pseudovariety* **OCR** *of all finite orthogroups is canonically reducible.* □

Note that the crucial additional ingredient in Theorem 4.10, besides Proposition 4.5, is the fact that $\Omega_A^\kappa\mathbf{S}$ admits 0, $\bar{0}$, 1, $\bar{1}$ factorizations and so, Theorem 4.10 remains valid for other implicit signatures that share this property. At present, to the best of our knowledge, the only non-locally finite canonically reducible pseudovariety of groups that is known is the pseudovariety **G** of all finite groups. The pseudovariety $\mathbf{G}_p$ of all finite p-groups is known not to be canonically reducible [**8**].

5. Final comments. We conclude with a few remarks.

The arguments in Section 4 may be adapted to show that $\mathbf{V}\,\textcircled{m}_{\mathrm{CR}}\mathbf{H}$ is hyperdecidable (respectively canonically reducible) for every pseudovariety **V** of bands and every hyperdecidable (respectively canonically reducible) pseudovariety **H** of groups.

By tracing the kinds of graphs that come up when trying to establish that $\mathbf{B}\,\textcircled{m}_{\mathrm{CR}}\mathbf{H}$ has computable pointlikes [**19**, **20**, **4**], one can use the arguments in Section 4 to show that, if the pseudovariety of groups **H** is such that **H**-inevitability of labelings of graphs whose underlying undirected graphs are circuits, then $\mathbf{B}\,\textcircled{m}_{\mathrm{CR}}\mathbf{H}$ has computable pointlikes.

The first author has shown recently that the pseudovariety $\mathbf{G}_p$ is σ-reducible for a suitable enlarged (infinite) signature σ [**5**]. We have not checked whether $\Omega_A^\sigma\mathbf{S}$ admits 0, $\bar{0}$, 1, $\bar{1}$ factorizations and thus, while Theorem 4.6 applies to show that $\mathbf{B}\,\textcircled{m}_{\mathrm{CR}}\mathbf{G}_p$ is hyperdecidable, we do not know whether the argument in the proof of Theorem 4.10 also applies to show that $\mathbf{B}\,\textcircled{m}_{\mathrm{CR}}\mathbf{G}_p$ is still σ-reducible and for that matter whether Theorem 4.1 yields from a solution of the word problem for $\Omega_A^\sigma\mathbf{G}_p$ [**5**] a solution of the word problem for $\Omega_A^\sigma(\mathbf{B}\,\textcircled{m}_{\mathrm{CR}}\mathbf{G}_p)$.

From the results of Section 3 it follows that the functions 0, $\bar{0}$, 1, and $\bar{1}$ are continuous on the free profinite semigroup. This does not however help in establishing the uniqueness of 0, $\bar{0}$, 1, $\bar{1}$ factorizations (Lemma 4.8) but just the existence of such factorizations. That the factors involved stay within the σ-subsemigroup $\Omega_A^\sigma\mathbf{S}$ is very much dependent on the signature σ (cf. Lemma 4.9). For instance, for the signature σ consisting of multiplication together with the unary operation given by ω-power, this property fails.

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