CHARACTERIZATION OF CERTAIN BINARY RELATIONS ON CONNECTED ORDERED SPACES

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1. Introduction. In an earlier paper **(2)** reflexive transitive binary relations were considered on a connected ordered space. These relations were topologically restricted and their minimal sets were either an end point of the space or empty. It was shown that these relations could be characterized as one of the two orders of the space. Viewing the situation somewhat differently as suggested by I. S. Krule, one could say that this class of relations was characterized in terms of the identity function on the space. In this case the relations are considered in their natural setting, the product of the space with itself.

Pursuing this viewpoint one might ask if a more general class of transitive binary relations could be characterized in terms of suitably chosen continuous functions on subsets of the space to itself.

This paper considers the class of all transitive binary relations that are monotone, closed above, closed below, and continuous, and shows that these relations can be characterized in this manner.

2. Notation and definitions. Throughout this paper it is assumed that X is a set consisting of more than a single element. It is further assumed that X is a connected topological space and R is a relation on X such that (X, R) is an ordered set and the order topology induced by R is the topology on X. The usual meaning regarding lower and upper bounds, infima, and suprema with respect to R of subsets of X will be used. Topological closure will be denoted by * throughout this paper.

If L (the dual of L is denoted by σL) is any relation on the topological space X, then the following terminology and notation will be used:

(1) if $x \in X$ then $L(x) = \{y \in X \mid (y, x) \in L\}$;

(2) if $A \subset X$ then $L(A) = \bigcup\{L(a) \mid a \in A\}$;

(3) L is continuous provided $L(A^*) \subset L(A)^*$ for each $A \subset X$;

(4) L is monotone provided $L(x)$ is connected for each $x \in X$;

(5) if $k \in X$, then k will be called L-minimal provided the following condition is satisfied: whenever $x \in L(k)$ and $x \in X$ then $k \in L(x)$ (the set of L-minimal elements is denoted by K_L);

(6) L is closed above provided $\sigma L(x)$ is closed for each $x \in X$;

(7) L is closed below provided $L(x)$ is closed for each $x \in X$;

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(8) if $B \subset X$, then B will be called an L-ideal provided $B \neq \emptyset$ and $L(B) \subset B$;

(9) if $x \in X$, then $L_x = \{y \in X \mid x \cup L(x) = y \cup L(y)\}$.

If R is a relation on X such that (X, R) is an ordered set, then the following additional terminology and notation will be used:

(10) if $x \in X$, then x will be called R-minimal provided $x \in R(y)$ for each $y \in X$;

(11) if $x \in X$, then x will be called R-maximal provided $x \in \sigma R(y)$ for each $y \in X$;

(12) if $x, y \in X$ then $x < y$ $(x > y)$ means $x \in R(y) - y$ $(x \in \sigma R(y) - y)$.

Throughout this paper $^\circ$ will denote the interior of a set. The pair

$$[p, q] = \{x \in X \mid p < x < q\}$$

where p and q are elements of the space. The symbols $\rightarrow$ and $\leftarrow$ indicate convergence.

It is well known that if L is a transitive, closed-below relation on the T_1-space X and if A is a compact L-ideal, then $A \cap K_L \neq \emptyset$.

The notion of nets will be used frequently and the reader is referred to **(1)** for the definitions and the general theory. In addition to nets, a notion of convergence of sets will be used. If Γ is a directed set and if $\{A_\alpha \mid \alpha \in \Gamma\}$ is a family of subsets of the space X, then the two following definitions of subsets of X are made:

(1) $\limsup A_\alpha$ is defined as follows: $x \in \limsup A_\alpha$ if and only if for each open set U about x there exists a cofinal subset $\Gamma_u \subset \Gamma$ such that $U \cap A_\alpha \neq \emptyset$ for each $\alpha \in \Gamma_u$;

(2) $\liminf A_\alpha$ is defined as follows: $x \in \liminf A_\alpha$ if and only if for every open set U about x there exists a residual subset $\Gamma_u \subset \Gamma$ such that $U \cap A_\alpha \neq \emptyset$ for each $\alpha \in \Gamma_u$.

The following lemma will be used frequently without reference. It is contained in more general form in **(4)**.

LEMMA. *Let L be a relation on the Hausdorff space X. Then L is continuous if and only if for each net $\{x_\alpha \mid \alpha \in \Gamma\}$ converging to x, $L(x) \subset \liminf L(x_\alpha)$.*

For additional information and facts pertaining to relations similar to those discussed in this paper, the reader is referred to **(2; 3;** and **5)**.

3. Preliminary lemmas. This section is a collection of lemmas, establishing certain properties of monotone, continuous, closed above and below, transitive relations in terms of nets, continuous functions, and connected subsets of the space. These lemmas are proved in general and should be most helpful in studying one specific relation of this type.

Throughout this section it is assumed that (X, R) is a connected ordered space and L is a monotone, continuous, closed above and below, transitive relation on X.

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LEMMA 1. *Either* $L_x = \{x\}$ *for each* $x \in K_L$ *or* $\emptyset \neq L(x) = K_L \neq \{x\}$ *for each* $x \in K_L$.

Proof. Suppose $L_x \neq \{x\}$ for each $x \in K_L$ and consider the case where $\emptyset \neq K_L \neq X$. Let a be an element of K_L such that $L_a \neq \{a\}$. By definition $L(a) \subset K_L$ and for each $x \in L(a)$, $L(x) = L(a)$. Without loss of generality suppose $\sup L(a) = p$ and p is not R-maximal. Assume there exists a net $\{x_\alpha \mid \alpha \in \Gamma\}$ in $\sigma R(p) - p$ converging to p such that $x_\alpha \in K_L$. Continuity of L implies that $L(p) \subset \liminf L(x_\alpha)$. It follows that for some $\alpha \in \Gamma$, $L(x_\alpha) \cap L(a) \neq \emptyset$, a contradiction. Therefore there exists $z > p$ such that $[p, z] \cap K_L = \emptyset$. Let $N = \{x \in K_L \mid z \leqslant x\}$ and if $N \neq \emptyset$, let $q = \inf N$ and let $H = \{y \in X \mid y \notin K_L \text{ and } p < y \leqslant q\}$. The continuity of L and the connectedness of H imply $L(a) \subset L(y)$ for each $y \in H$. Now establish that $N = \emptyset$ by showing that the existence of $q = \inf N$ leads to a contradiction.

(1) Suppose $q \in H$ and let $\{y_\alpha \mid \alpha \in \Gamma\}$ be a net in K_L converging to q. The continuity of L implies $L(q) \subset \liminf L(y_\alpha)$ and hence for some $\alpha \in \Gamma$, $L(y_\alpha) \cap L(a) \neq \emptyset$, a contradiction.

(2) Suppose $q \notin H$ and let $\{y_\alpha \mid \alpha \in \Gamma\}$ be a net in H converging to q. Now $\{y_\alpha\} \subset \sigma L(p)$ and since L is closed above, $q \in \sigma L(p)$, a contradiction. Hence $N = \emptyset$.

By a similar argument it can easily be shown that there does not exist any element of K_L less than $L(a)$. Therefore $\{a\} \neq K_L = L(a) \neq \emptyset$. For $K_L = \emptyset$ the lemma is obvious and in the case where $K_L = X$ the proof just presented will suffice.

LEMMA 2. *The minimal set* K_L *is closed.*

Proof. Consider only the case where $L_x = \{x\}$ for each $x \in K_L$. Let $y \in K_L^* - K_L$ and let $\{y_\alpha \mid \alpha \in \Gamma\}$ be a net in K_L converging to y. For $y \notin K_L$ it follows that $\emptyset \neq L(y) \neq \{y\}$. Let $t \in L(y)^\circ - y$ and let U and V be open connected sets such that $y \in V$, $t \in U \subset L(y)$, and $U \cap V = \emptyset$. Since $L_{y_\alpha} = \{y_\alpha\}$ it follows that either $L(y_\alpha) = \{y_\alpha\}$ or $L(y_\alpha) = \emptyset$. Therefore, there exists $\alpha \in \Gamma$ such that $L(y_\beta) \subset V$ for $\beta > \alpha$, which implies $L(y_\beta) \cap U \neq \emptyset$. Hence $t \notin \liminf L(y_\alpha)$, a contradiction.

LEMMA 3. *If* $x \in X - K_L$, *then* $L_x^\circ = \emptyset$.

Proof. Suppose $x \in X - K_L$ such that $L_x^\circ \neq \emptyset$. Let $y \in L_x^\circ$, where $y \neq x$. Now $y \cup L(y) = x \cup L(x)$ implies $x \in L(y) = L(x)$ and it follows that $L_x \subset L(x)$. Clearly $L_x \neq L(x)$, since $x \notin K_L$. Let $A = L(x) - L_x$. Now $A^* \cap L_x \neq \emptyset$ since L is monotone and L_x is closed. The continuity of L implies

$$L(x) \subset L(L_x) \subset L(A^*) \subset L(A)^* \subset A^*,$$

since clearly $L(A) \subset A$. But $A \subset L(x)$ and $L(x)$ is closed; therefore $A^* \subset L(x)$ and hence $L(x) = A^*$, a contradiction since $y \in L_x^\circ \subset L(x)$ and

$$A^* \cap L_x^\circ = \emptyset.$$

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LEMMA 4. *If* $x \in X - K_L$ *and* $L(x) \cap (R(x) - x) \neq \emptyset$, *then* $L(x) \subset R(x)$.

Proof. Suppose $x \in X - K_L$ such that $L(x) \cap (R(x) - x) \neq \emptyset$ and $L(x) \not\subset R(x)$. Clearly $x \in L(x)$. Let $z \in L(x) \cap (\sigma R(x) - x)$ and let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in $(\sigma R(x) - x) \cap (R(z) - z)$ converging to x such that $x \notin L(x_\alpha)$ for each $\alpha \in \Gamma$. Hence there exists a cofinal set $\Gamma_z \subset \Gamma$ such that either $L(x_\alpha) \subset R(x)$ for each $\alpha \in \Gamma_z$ or $L(x_\alpha) \in \sigma R(x)$ for each $\alpha \in \Gamma_z$. If $L(x_\alpha) \subset R(x)$, then

$$L(x) \subset L(\{x_\alpha\}^*) \subset L(\{x_\alpha\})^* \subset R(x)$$

for each $\alpha \in \Gamma_z$, a contradiction.

A similar argument suffices if $L(x_\alpha) \subset \sigma R(x)$.

COROLLARY. *If* $x \in X - K_L$, *then either* $L(x) \subset R(x)$ *and* $\sup L(x)$ *exists or* $L(x) \subset \sigma R(x)$ *and* $\inf L(x)$ *exists.*

LEMMA 5. *If* M *is a connected subset of* $X - K_L$ *then either* $L(x) \subset R(x)$ *for each* $x \in M$ *or* $L(x) \subset \sigma R(x)$ *for each* $x \in M$.

Proof. Let $A = \{x \in M \mid L(x) \subset R(x)\}$, $B = \{x \in M \mid L(x) \subset \sigma R(x)\}$. Clearly $M = A \cup B$. Suppose $A \neq \emptyset$ and $B \neq \emptyset$. If $A^* \cap B \neq \emptyset$, then let $x \in A^* \cap B$ and let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in A converging to x. Since $x \in B$, there exists $y \in L(x)$ such that $x < y$. The continuity of L implies $L(x) \subset \liminf L(x_\alpha)$. Let U be an open connected set about x and let V be an open connected set about y such that $V \cap U = \emptyset$. For $x_\alpha \in A$, $\sup L(x_\alpha) \leqslant x_\alpha$. Hence for some $\beta \in \Gamma$, $x_\alpha \in U$ for $\alpha > \beta$. It follows that $L(x_\alpha) \cap V = \emptyset$ for $\alpha > \beta$ and therefore $y \notin \liminf L(x_\alpha)$, a contradiction. A similar argument will suffice for $A \cap B^* \neq \emptyset$. Therefore either $A = \emptyset$ or $B = \emptyset$ and the lemma is proved.

LEMMA 6. *If* M *is a connected subset of* $X - K_L$ *and* $L(x) \subset R(x)$ *for each* $x \in M$, *then the function* f *defined by* $f(x) = \sup L(x)$ *for each* $x \in M$ *is continuous.*

Proof. Suppose f is not continuous at $x \in M$. Let U be an open connected set about $f(x)$ and let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net converging to x such that $f(x_\alpha) \notin U$ for each $\alpha \in \Gamma$. The continuity of L implies that there exists a residual set $\Gamma_u \subset \Gamma$ such that $L(x_\alpha) \cap U \neq \emptyset$ for $\alpha \in \Gamma_u$. Let $z = \sup U$, and it follows that $z \in L(x_\alpha)$ for each $\alpha \in \Gamma_u$ and hence $z \in L(x)$, a contradiction.

LEMMA 7. *If* M *is an open connected subset of* $X - K_L$, $L(x) \subset R(x)$ *for each* $x \in M$, *and* $G = \{x \in M \mid \inf L(x)$ *exists*$\}$, *then* G *is open and the function* g *defined by* $g(x) = \inf L(x)$ *for each* $x \in G$ *is continuous.*

Proof. Let $N = M - G$ and suppose N is not closed relative to M. Let $x \in N^* - N$ so that $x \in G$ and let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in N converging to x. Let $y = \inf L(x)$ and let U be a connected set about y which is bounded below. The continuity of L implies that there exists a residual set $\Gamma_u \subset \Gamma$

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such that $L(x_\alpha) \cap U \neq \emptyset$ for $\alpha \in \Gamma_u$. Let $t \in X$ such that t is less than the greatest lower bound of U. It follows that $t \in L(x_\alpha)$ for $\alpha \in \Gamma_u$ and therefore $t \in L(x)$, a contradiction.

An argument similar to that given in the preceding lemma will suffice to prove that g is continuous.

In the following three lemmas it will be assumed that $p \in K_L \cap (X - K_L)^*$, $\{x_\alpha \mid \alpha \in \Gamma\}$ is a net in a connected subset M of $X - K_L$ converging to p and $L(x_\alpha) \subset R(x_\alpha)$ for each $\alpha \in \Gamma$.

LEMMA 8. *If* $\emptyset \neq L(p) = K_L \neq \{p\}$, *then* $\{\sup L(x_\alpha)\}$ *converges to* p.

Proof. Let U be an open connected set about $y \in L(p)^\circ$ such that $U \subset L(p) - p$. The continuity of L implies that there exists a residual set $\Gamma_y \subset \Gamma$ such that $U \cap L(x_\alpha) \neq \emptyset$ for $\alpha \in \Gamma_y$. Since $L(z) = L(p)$ for each $z \in L(p) = K_L$, it follows that $L(p) \subset L(x_\alpha)$ for each $\alpha \in \Gamma_y$. Hence $p \leqslant \sup L(x_\alpha)$ for each $\alpha \in \Gamma_y$ and since $L(x_\alpha) \subset R(x_\alpha)$ it follows that $p \leqslant \sup L(x_\alpha) \leqslant x_\alpha$ for each $\alpha \in \Gamma_y$. Therefore $\{\sup L(x_\alpha)\}$ converges to p since $\{x_\alpha\}$ converges to p.

LEMMA 9. *If* $L_p = \{p\}$ *and* $L(p) = \{p\}$, *then* $\{sup\, L(x_\alpha)\}$ *converges to* p.

Proof. Suppose $\{\sup L(x_\alpha)\}$ does not converge to p. Let U be an open connected set about p and let $\Gamma_p \subset \Gamma$ be a cofinal set such that $\sup L(x_\alpha) \notin U$ for each $\alpha \in \Gamma_p$. Since $\sup L(x_\alpha) \leqslant x_\alpha$, it follows that there exists a residual set $\Gamma_p' \subset \Gamma_p$ such that $\sup L(x_\alpha) < u$ for each $u \in U$ and $\alpha \in \Gamma_p'$. Hence

$$L(\{x_\alpha\}^*) \not\subset L(\{x_\alpha\})^*$$

for $\alpha \in \Gamma_p'$, a contradiction.

LEMMA 10. *If* $L_p = \{p\}$ *and* $L(p) = \emptyset$, *then either* $\{\sup L(x_\alpha)\}$ *converges to* $-\infty$ *or* $\{\sup L(x_\alpha)\}$ *converges to* $z \in K_L - p$.

Proof. Suppose $\{\sup L(x_\alpha)\}$ does not converge to $-\infty$. Then there exists $t \in X$ and a cofinal set $\Gamma_t \subset \Gamma$ such that $t \leqslant \sup L(x_\alpha)$ for each $\alpha \in \Gamma_t$. Clearly $t < p$. Let $A = \{x \in R(p) - p \mid$ there exists a cofinal set $\Gamma_x \subset \Gamma$ such that $\sup L(x_\alpha) \leqslant x$ for each $\alpha \in \Gamma_x\}$ and let $z = \sup A$. Suppose $\{\sup L(x_\alpha)\}$ does not converge to z. Then there exists an open connected set U about z and a cofinal set $\Gamma_u \subset \Gamma$ such that $\sup L(x_\alpha) \notin U$ for $\alpha \in \Gamma_u$. Clearly z is not R-minimal. Since $z = \sup A$, there exists a residual set $\Gamma_u' \subset \Gamma_u$ such that $f(x_\alpha) < u$ for each $u \in U$ and each $\alpha \in \Gamma_u'$. Also there exists a cofinal set $\Gamma_y \subset \Gamma$ such that $f(x_\alpha) \in U$ for $\alpha \in \Gamma_y$. For each $\alpha \in \Gamma_y$ there exists at least one $\beta \in \Gamma_u'$ such that $x_\beta \in R(x_\alpha) - x_\alpha$. Choose one such x_β for each $\alpha \in \Gamma_y$ and consider the set $\{\sigma R(x_\beta) \cap R(x_\alpha)\}$, where x_β is the chosen one for x_α. It follows that $\sup L(\sigma R(x_\beta) \cap R(x_\alpha))$ is connected and $y \in \sup L(\sigma R(x_\beta) \cap R(x_\alpha))$, where $y = \inf U$. Hence there exists at least one $y_\alpha \in \sigma R(x_\beta) \cap R(x_\alpha)$ for each $\alpha \in \Gamma_y$ such that $\sup L(y_\alpha) = y$. Choose one y_α

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for each $\alpha \in \Gamma_y$ and it follows that the net $\{y_\alpha \mid \alpha \in \Gamma_y\}$ converges to p. Hence L is closed above implies $y \in L(p)$, a contradiction. Therefore $\{\sup L(x_\alpha)\}$ converges to z.

Clearly if z is R-minimal, $z \in K_L$. Suppose $z \notin K_L$. There exists an open connected set U about y which is bounded below and a residual set $\Gamma_u \subset \Gamma$ such that $U \cap K_L = \emptyset$ and $\sup L(x_\alpha) \in U$ for each $\alpha \in \Gamma_u$. Clearly $\inf L(x_\alpha) \notin U$. Let $y = \inf U$ and it follows that $\{x_\alpha\} \subset \sigma L(y)$ and hence $y \in L(p)$, a contradiction. Therefore $z \in K_L$.

LEMMA 11. *If $L_x = \{x\}$ for each $x \in K_L$, then $L(K_L)$ is contained in and open relative to K_L.*

Proof. Clearly $L(K_L) \subset K_L$ and the continuity of L implies that the complement of $L(K_L)$ is closed.

LEMMA 12. *If M is an open connected subset of $X - K_L$ and*

$$\emptyset \neq L(x) = K_L \neq \{x\}$$

for each $x \in K_L$, then $K_L \subset L(x)$ for each $x \in M$.

Proof. The argument presented in Lemma 8 showed the existence of an element of $X - K_L$ that contained K_L. Using the connectedness of M and the continuity of L, it can be shown that $K_L \subset L(x)$ for each $x \in M$.

It should be noted that duals of lemmas are omitted. References will be made to the duals and the meaning will be clear from the context.

4. General results. This section contains two general theorems which completely characterize monotone, continuous, closed above and below, transitive binary relations on connected ordered spaces. Lemma 1 is used to divide these relations into two large classes. Theorem 1 considers the class where $\emptyset \neq L(x) = K_L \neq \{x\}$ for each $x \in K_L$ and Theorem 2 characterizes the class of relations where $L_x = \{x\}$ for each $x \in K_L$. The special case where $K_L = \emptyset$ is understood to be in the latter class.

THEOREM 1. *Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L be monotone, continuous, closed above and below, and transitive with $\{x\} \neq K_L = L(x) \neq \emptyset$ for each $x \in K_L$, $K_L \neq \emptyset$, is that there exist open connected sets N and M, open sets G and F, $G \subset M$, $F \subset N$, a closed connected non-empty set H, and continuous functions $f : M \cup F \cup H \to M^*$ and $g : N \cup G \cup H \to N^*$ such that*

(a) *$N \cup H \cup M = X$, $N \subset R(h)$ for every $h \in H$, $H \subset R(m)$ for every $m \in M$, $N^* \cap M^* = \emptyset$, H, M, and N are pairwise disjoint, and $p = \sup N$ and $q = \inf M$ are interior points of $F \cup H$ and $H \cup G$ respectively if $N \neq \emptyset$ and $M \neq \emptyset$;*

(b) *$f(x) \leqslant x$ for each $x \in M$ and $f(x) = q$ for each $x \in H$;*

(b′) *$x \leqslant g(x)$ for each $x \in N$ and $g(x) = p$ for each $x \in H$;*

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(c) *if* $x \in G \cup F$, $y \in M \cup N$, *and* $g(x) \leqslant y \leqslant f(x)$, *then* $y \in G \cup F$ *and* $g(x) \leqslant g(y) \leqslant f(y) \leqslant f(x)$;

(d) *if* $x \in M - G$ *and* $y \in N$, *then* $y \in F$ *and* $f(y) \leqslant f(x)$;

(d′) *if* $x \in N - F$ *and* $y \in M$, *then* $y \in G$ *and* $g(x) \leqslant g(y)$;

(e) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* G *converging to* $x \in M - G$, *then* $\{g(x_\alpha)\}$ *converges to* $-\infty$;

(e′) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* F *converging to* $x \in N - F$, *then* $\{f(x_\alpha)\}$ *converges to* $+\infty$;

(f)
$$\begin{aligned} L = \{x, y) \in X \times (G \cup F) \mid g(y) \leqslant x \leqslant f(y)\} \\ \cup \{(x, y) \in X \times (M - G) \mid x \leqslant f(y)\} \\ \cup \{(x, y) \in X \times (N - F) \mid g(y) \leqslant x\} \\ \cup \{(x, y) \in H \times H\}. \end{aligned}$$

Proof of necessity. Define $H = K_L$ and it follows that H is closed, connected, and non-empty. Let

$$N = \{x \in X - K_L \mid K_L \subset \sigma R(x)\}, \qquad M = \{x \in X - K_L \mid K_L \subset R(x)\},$$
$$G = \{x \in M \mid \inf L(x) \text{ exists}\}, \qquad F = \{x \in N \mid \sup L(x) \text{ exists}\}.$$

It follows that M and N are open and connected, and G and F are open. Define f by $f(x) = \sup L(x)$ and g by $g(x) = \inf L(x)$. By the definition and structure of K_L, (b) and (b′) are true. The transitivity of L implies (c), (d), and (d′).

Let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in G converging to $x \in M - G$. The continuity of L implies $L(x) \subset \liminf L(x_\alpha)$. For any $t \in X$ it follows that there exists $y \in L(x)$ and an open connected set U about y such that $t \notin U$ and $y \in R(t)$. Now $y \in \liminf L(x_\alpha)$ so that there exists a residual set $\Gamma_u \subset \Gamma$ such that $U \cap L(x_\alpha) \neq \emptyset$ for $\alpha \in \Gamma_u$. Hence $g(x_\alpha) < t$ for $\alpha \in \Gamma_u$ and it follows that $\{g(x_\alpha)\}$ converges to $-\infty$. This establishes (e) and by a similar argument (e′) is true.

Let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in M converging to $q = \inf M$ and suppose that there exists a cofinal set $\Gamma_q \subset \Gamma$ such that $\{x_\alpha\} \subset M - G$ for $\alpha \in \Gamma_q$. Let $t \in N$. It follows that $\{x_\alpha\} \subset \sigma L(t)$ for $\alpha \in \Gamma_q$; hence $q \in \sigma L(t)$ or $t \in L(q)$, a contradiction since $t < g(q)$ and $q \in K_L$. This implies $q \in (H \cup G)^\circ$ and $P \in (F \cup H)^\circ$ by a similar argument, and hence (a) is established. The definitions imply (f).

Lemmas 6 and 7 and their corresponding duals imply the continuity of f and g over M and N. Lemma 8 and its dual establishes continuity at $q = H \cap M^*$ and $p = H \cap N^*$. Clearly f and g are continuous on H°. This concludes the proof of necessity.

Proof of sufficiency. Clearly from condition (a), $F = \emptyset$ only if $N = \emptyset$, and $G = \emptyset$ only if $M = \emptyset$. Therefore, we assume M and N to be non-empty sets leaving special cases to the remarks. From (a) and (f) it follows that L is monotone and closed below, and where $K_L = H$ and $\{x\} \neq K_L = L(x) \neq \emptyset$ for each $x \in K_L$.

Let (x, y) and (y, z) be elements of L and consider all possible cases to show that L is transitive.

Case 1. If either $z \in K_L$ or $y \in K_L$, then $x \in K_L$ and (a), (b), (b′), and (f) imply $K_L \subset L(x)$ for each $x \in X$.

Case 2. If $z \in G \cup F$ and $y \in M \cup N$, then (f) implies $g(z) \leqslant y \leqslant f(z)$ and hence (c) and (f) implies $y \in G \cup F$ and $g(z) \leqslant g(y) \leqslant x \leqslant f(y) \leqslant f(z)$; therefore $(x, z) \in L$.

Case 3. If $z \in M - G$ and $y \in M$, then (b) and (f) imply $x \leqslant f(y) \leqslant y \leqslant f(z)$ and $(x, z) \in L$.

Case 4. If $z \in M - G$ and $y \in N$, then (d) and (f) imply $y \in F$ and $g(y) \leqslant x \leqslant f(y) \leqslant f(z)$ and $(x, z) \in L$.

Case 5. If $z \in N - F$ and $y \in N$, then (b′) and (f) imply $g(z) \leqslant y \leqslant g(y) \leqslant x$ and $(x, z) \in L$.

Case 6. If $z \in N - F$ and $y \in M$, then (d′) and (f) imply $y \in G$ and $g(z) \leqslant g(y) \leqslant x \leqslant f(y)$ and $(x, z) \in L$. Therefore L is transitive.

Now to show that L is continuous. Clearly $K_L \subset L(x)$ for each $x \in X$, hence for $y \in K_L$ and $\{y_\alpha \mid \alpha \in \Gamma\}$ a net converging to y, $L(y) = K_L \subset \lim\inf L(y_\alpha)$ and L is continuous over K_L.

Case 1. Suppose L is not continuous at $x \in G$. Then there exists a net $\{x_\alpha \mid \alpha \in \Gamma\}$ in G converging to x such that $L(x) \not\subset \lim\inf L(x_\alpha)$. Hence there exist $a \in L(x)$, an open set U about a, and a cofinal set $\Gamma_u \subset \Gamma$ such that $U \cap L(x_\alpha) = \emptyset$ for $\alpha \in \Gamma_u$. Clearly $a \neq f(x)$ and $a \neq g(x)$ since f and g are continuous functions. Thus, there exists an open connected set V about $f(x)$ and an open connected set V' about $g(x)$ such that $a \notin V \cup V'$. It follows that there exists $\beta \in \Gamma$ such that $g(x_\alpha) \in V'$ and $f(x_\alpha) \in V$ for $\alpha > \beta$. Since L is monotone, $a \in L(x_\alpha)$ for $\alpha > \beta$—a contradiction since $L(x_\alpha) \cap U = \emptyset$ for $\alpha \in \Gamma_u$ and $a \in U$. Therefore L is continuous on G.

Case 2. Suppose L is not continuous at $x \in M - G$. Since M is open, there exists a net $\{x_\alpha \mid \alpha \in \Gamma\}$ in M converging to x such that $L(x) \not\subset \lim\inf L(x_\alpha)$. Hence there exists $a \in L(x)$, an open connected set U about a, and a cofinal set $\Gamma_u \subset \Gamma$ such that $U \cap L(x_\alpha) = \emptyset$ for $\alpha \in \Gamma_u$. Clearly $\{x_\alpha \mid \alpha \in \Gamma_u\}$ is cofinal in either $M - G$ or G. Since f is continuous, $a \neq f(z)$. Let V be an open connected set about $f(x)$ such that $a \notin V$. If $\{x_\alpha \mid \alpha \in \Gamma_u\}$ is cofinal in $M - G$, then there exists $\beta \in \Gamma_u$ such that $f(x_\beta) \in V$, $x_\beta \in M - G$. Hence $a \in L(x_\beta)$, a contradiction since $L(x_\beta) \cap U' = \emptyset$. If $\{x_\alpha \mid \alpha \in \Gamma_u\}$ is cofinal in G, then there exists $\beta \in \Gamma_u$ such that $f(x_\beta) \in V$, $x_\beta \in G$, and $a \in L(x_\beta)$ by (e), a contradiction. Therefore L is continuous on $M - G$.

By arguments similar to those given in cases 1 and 2 it follows that L is continuous over N, hence L is continuous.

Now to show that L is closed above. For every $x \in K_L$, $\sigma L(x) = X$, so that $\sigma L(x)$ is closed.

Suppose $\sigma L(x)$ is not closed for some $x \in M$. Let $y \in \sigma L(x)^* - \sigma L(x)$. Now $x \notin L(y)$ implies $y \in F \cup M \cup H$ and $f(y) < x$ since $L(y)$ is connected and contains K_L. Let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in $\sigma L(x)$ converging to y. From (a) it

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follows that there exists $\beta \in \Gamma$ such that $x_\alpha \in F \cup M \cup H$ for $\alpha > \beta$. Let U be an open connected set about $f(y)$ such that $x \notin U$. The continuity of f implies that there exists $\beta' \in \Gamma$ such that $f(x_\alpha) \in U$ for $\alpha > \beta'$. Hence $f(x_\alpha) < x$ for $\alpha > \beta'$, a contradiction. By a similar argument it follows that $\sigma L(x)$ is closed for $x \in N$; hence L is closed above and this concludes the proof of Theorem 1.

Remarks. In considering the class of relations where $\emptyset \neq L(x) = K_L \neq \{x\}$ for each $x \in L_L$, $K_L \neq \emptyset$, it is clear that there exist three cases since L is monotone.

Case 1. Suppose $a < b$ and $K_L = \sigma R(a) \cap R(b)$ assuming a non-R-minimal and b non-R-maximal. Therefore $M \neq \emptyset$ and $N \neq \emptyset$ and this case was considered in Theorem 1.

Case 2. Suppose $K_L = R(a)$ for some non R-minimal $a \in X$. In this case it is clear that $N = \emptyset$, $F = \emptyset$ and either $G = \emptyset$ or $G = M$ depending on the existence or non-existence of an R-minimal element. Consequently, N, F, and G can be deleted from the theorem and the conditions involving these sets are vacuous and can be omitted. In this case Theorem 1 becomes:

Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L be monotone, continuous, closed above and below, and transitive with $K_L = R(a)$ for some non R-minimal $a \in X$ and

$$\emptyset \neq L(x) = K_L \neq \{x\}$$

for each $x \in K_L$ is that there exist an open connected set M, a closed connected non-empty set, H, and a continuous function $f : M \cup H \to M^*$ such that

(a) $H \cup M = X$, $H \subset R(m)$ *for every* $m \in M$ *and* $H \cap M = \emptyset$;

(b) $f(x) \leqslant x$ *for every* $x \in M$ *and* $f(x) = q = \inf M$ *for every* $x \in H$;

(c) $L = \{(x, y) \in X \times X \mid x \leqslant f(y)\}$.

Case 3. Suppose $K_L = \sigma R(b)$ for some non R-maximal $b \in X$, the obvious dual of case 2.

For this class of relations if L is given the additional property of being reflexive, Theorem 1 will apply with only the following two slight modifications. Condition (b) becomes: *$f(x) = x$ for every $x \in M$ and $f(x) = q$ for every $x \in H$ and* (b′) *becomes*: *$g(x) = x$ for every $x \in N$ and $g(x) = p$ for every $x \in H$.*

THEOREM 2. *Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L is monotone, continuous, closed above and below, and transitive with $L_x = \{x\}$ for each $x \in K_L$ is that there exist open sets N, M, G, and F, a set H contained in and open relative to $X - (M \cup N)$, and continuous functions $f : G \to X$, $g : M \to X$, $h : N \to X$, and $k : F \to X$ such that*

(a) *$F \subset N$, $G \subset M$, M and N are mutually separated, and H, N, and M are pairwise disjoint*;

(b) (i) $x \leqslant g(x)$ *for each* $x \in M$; (ii) $x \leqslant g(x) \leqslant f(x)$ *for each* $x \in G$; (iii) $h(x) \leqslant x$ *for each* $x \in N$; (iv) $k(x) \leqslant h(x) \leqslant x$ *for each* $x \in F$; *and* (v) *there exists no* $x \in X$ *such that* $x = g(x) = f(x)$ *or* $x = h(x) = k(x)$;

(c) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* G *converging to* $x \in M - G$, *then* $\{f(x_\alpha)\}$ *converges to* $+\infty$;

(c′) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* F *converging to* $x \in N - F$, *then* $\{k(x_\alpha)\}$ *converges to* $-\infty$;

(d) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* G *converging to* $x \in X$ *and there exists* $y \in X$ *such that* $g(x_\alpha) \leqslant y \leqslant f(x_\alpha)$ *for each* $\alpha \in \Gamma$, *then* $(y, x) \in L$ (see (k));

(d′) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* $M - G$ *converging to* $x \in X$ *and there exists* $y \in X$ *such that* $g(x_\alpha) \leqslant y$ *for each* $\alpha \in \Gamma$, *then* $(y, x) \in L$ (see (k));

(e) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* F *converging to* $x \in X$ *and there exists* $y \in X$ *such that* $k(x_\alpha) \leqslant y \leqslant h(x_\alpha)$ *for each* $\alpha \in \Gamma$, *then* $(y, x) \in L$ (see (k));

(e′) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* $N - F$ *converging to* $x \in X$ *and there exists* $y \in X$ *such that* $y \leqslant h(x_\alpha)$ *for each* $\alpha \in \Gamma$, *then* $(y, x) \in L$ (see (k));

(f) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* M *converging to* $x \in H$, *then* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is eventually in* G *and* $\{g(x_\alpha)\} \to x \leftarrow \{f(x_\alpha)\}$;

(f′) *if* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is a net in* N *converging to* $x \in H$, *then* $\{x_\alpha \mid \alpha \in \Gamma\}$ *is eventually in* F *and* $\{k(x_\alpha)\} \to x \leftarrow \{h(x_\alpha)\}$;

(g) *if* $x \in G$, $y \in M$, *and* $g(x) \leqslant y \leqslant f(x)$, *then* $y \in G$ *and*

$$g(x) \leqslant g(y) \leqslant f(y) \leqslant f(x);$$

(g′) *if* $x \in G$, $y \in N$, *and* $g(x) \leqslant y \leqslant f(x)$, *then* $y \in F$ *and*

$$g(x) \leqslant k(y) \leqslant h(y) \leqslant f(x);$$

(h) *if* $x \in F$, $y \in N$, *and* $k(x) \leqslant y \leqslant h(x)$, *then* $y \in F$ *and*

$$k(x) \leqslant k(y) \leqslant h(y) \leqslant h(x);$$

(h′) *if* $x \in F$, $y \in M$, *and* $k(x) \leqslant y \leqslant h(x)$, *then* $y \in G$ *and*

$$k(x) \leqslant g(y) \leqslant f(y) \leqslant h(x);$$

(i) *if* $x \in M - G$, $y \in M$, *and* $g(x) \leqslant y$, *then* $g(x) \leqslant g(y)$;

(i′) *if* $x \in M - G$, $y \in N$, *and* $g(x) \leqslant y$, *then* $y \in F$ *and* $g(x) \leqslant k(y) \leqslant h(y)$

(j) *if* $x \in N - F$, $y \in N$, *and* $y \leqslant h(x)$, *then* $h(y) \leqslant h(x)$;

(j′) *if* $x \in N - F$, $y \in M$, *and* $y \leqslant h(x)$, *then* $y \in G$ *and*

$$g(y) \leqslant f(y) \leqslant h(x);$$

(k) $$\begin{aligned} L = {} & \{(x, y) \in X \times G \mid g(y) \leqslant x \leqslant f(y)\} \\ & \cup \{(x, y) \in X \times (M - G) \mid g(y) \leqslant x\} \\ & \cup \{(x, y) \in X \times F \mid k(y) \leqslant x \leqslant h(y)\} \\ & \cup \{(x, y) \in X \times (N - F) \mid x \leqslant h(x)\} \\ & \cup \{(x, y) \in H \times H \mid y = x\}. \end{aligned}$$

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The proof is omitted since the methods of Theorem 1 apply with slight modification.

Remarks. In considering relations in this class, special cases are too numerous to discuss because of the many possibilities of the properties of K_L, such as K_L need not be connected as will be shown by an example. A few cases of special interest will be considered and examples will be used to illustrate others.

Case 1. Suppose $K_L = \emptyset$ or $K_L = \{a\}$ where a is either R-minimal or R-maximal. Lemma 5 restricts this case since $X - K_L$ is connected and Theorem 2 becomes:

Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L is monotone, continuous, closed above and below, and transitive with $K_L = \emptyset$ or $K_L = \{a\}$ where a is either R-minimal or R-maximal is that there exists a continuous function $f : X \to X$ such that

(a) $f(x) \leqslant x$ *for all* $x \in X$;

(b) $L = \{(x, y) \in X \times X \mid x \leqslant f(y)\}$.

or

(a′) $f(x) \geqslant x$ *for all* $x \in X$;

(b′) $L = \{(x, y) \in X \times X \mid x \geqslant f(y)\}$.

Case 2. Suppose $K_L = R(a)$ for some non R-minimal $a \in X$. Then by Lemma 5 only two cases could arise which are illustrated in Examples 4 and 5.

Case 3. Suppose $a < b$ and $K_L = \sigma R(a) \cap R(b)$ assuming a non R-minimal and b non R-maximal. The properties of L and Lemma 5, 9, and 10 imply six possibilities in this case. Examples 6 and 7, illustrate two possibilities and it is clear that their duals would illustrate two others. Example 10 considers another while 8 and 9 show characteristics of the remaining possibility.

Case 4. Suppose K_L is not connected. Example 11 illustrates this together with the special case where $M = \emptyset$, $G = \emptyset$, and $F = \emptyset$.

Examples 12 and 13 illustrate further the complexity of a possible relation that is characterized by Theorem 2.

Consider now the relations of this class if L is given the additional property of being reflexive. It can easily be shown that K_L is connected in this case, since $x \in L(x)$ for each $x \in X$ and $\{x\} = L(x)$ for each $x \in K_L$. In this case Theorem 2 becomes:

Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L is reflexive, monotone, continuous, closed above and below, and transitive with $L(x) = \{x\}$ for each $x \in K_L$, $K_L \neq \emptyset$, and $K_L \neq \{a\}$, where a is either R-minimal or R-maximal is that there exist open connected sets M and N, open sets G and F, $G \subset M$, $F \subset N$, a closed set H, and continuous functions $f : M \cup F \cup H \to M \cup H$ and $g : N \cup G \cup H \to N \cup H$ such that

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(a) *$N \cup M \cup H = X$, $N \subset R(h)$ for each $h \in H$, $H \subset R(m)$ for each $m \in M$, H, M, and N are pairwise disjoint, and $p = \sup N$ and $q = \inf M$ are interior points of $F \cup H$ and $G \cup H$ respectively;*

(b) *$f(x) = x$ for each $x \in M \cup H$;*

(b′) *$g(x) = x$ for each $x \in N \cup H$;*

(c) *if $x \in G \cup F$, $y \in M \cup N$, and $g(x) \leqslant y \leqslant f(x)$, then $y \in G \cup F$ and $g(x) \leqslant g(y) \leqslant f(y) \leqslant f(x)$;*

(d) *if $x \in M - G$ and $y \in N$, then $y \in F$ and $f(y) \leqslant f(x)$;*

(d′) *if $x \in N - F$ and $y \in M$, then $y \in G$ and $g(x) \leqslant g(y)$;*

(e) *if $\{x_\alpha \mid \alpha \in \Gamma\}$ is a net in G converging to $x \in M - G$, then $\{g(x_\alpha)\}$ converges to $-\infty$;*

(e′) *if $\{x_\alpha \mid \alpha \in \Gamma\}$ is a net in F converging to $x \in N - F$, then $\{f(x_\alpha)\}$ converges to $+\infty$;*

(f)
$$\begin{aligned} L = &\{(x, y) \in X \times (G \cup F) \mid g(y) \leqslant x \leqslant f(y)\} \\ &\cup \{(x, y) \in X \times (M - G) \mid x \leqslant f(y)\} \\ &\cup \{(x, y) \in X \times (N - F) \mid g(y) \leqslant x\} \\ &\cup \{(x, x) \in H \times H\}. \end{aligned}$$

If $K_L = \emptyset$ or $K_L = \{a\}$ where a is either R-minimal or R-maximal the theorem will be very similar to the theorem stated in case 1. It will become:

Let (X, R) be a connected ordered space and let L be a relation on X. A necessary and sufficient condition that L be reflexive, monotone, continuous, closed above and below, and transitive with $K_L = \emptyset$ or $K_L = \{a\}$ where a is either R-minimal or R-maximal is that $L = \{(x, y) = X \times X \mid x \leqslant y\}$ or $L = \{(x, y) \in X \mid y \leqslant x\}$. Note that $L = R$ or $L = \sigma R$ and this theorem is equivalent to the theorem in **(2)**.

It should be noted that the relation L is closed for the class characterized in Theorem 1. Also for the class of relations characterized in Theorem 2, L is necessarily closed if K_L is at most a single point.

5. Examples. The first three examples correspond to Theorem 1 and the remaining ones illustrate Theorem 2.

In each of the following examples X is a connected subset of the real numbers R, and the usual meaning of less than and greater than applies.

Example 1. Let $X = \{x \in R \mid 0 < x < 3\}$ and let

$$L = \{(x, y) \in X \times X \mid x \leqslant 1\} \cup \{(x, y) \in X \times X \mid x \leqslant \tfrac{1}{2}(y + 1)\}.$$

Example 2. Let $X = \{x \in R \mid 0 < x < 3\}$ and let

$$\begin{aligned} L = &\{(x, y) \in X \times X \mid 2 \leqslant x \leqslant 3\} \\ &\cup \{(x, y) \in X \times X \mid 3 < x \leqslant \tfrac{1}{2}(y + 3)\} \\ &\cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y + 2) \leqslant x < 2\}. \end{aligned}$$

Example 3. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 2 \leqslant x \leqslant 3\} \cup \{(x, y) \in X \times X \mid 3 < x \leqslant \tfrac{1}{2}(y + 3)\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y + 2) \leqslant x < 2\} \cup \{(x, y) \in X \times X \mid 2y - 4 \geqslant x \geqslant -2y + 8\}.$$

Example 4. Let $X = \{x \in R \mid 0 < x < 3\}$ and let

$$L = \{(x, y) \in X \times X \mid y = x < \tfrac{1}{2}\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 4y + 9) \leqslant x < 3\}.$$

Example 5. Let $X = \{x \in R \mid 0 < x < 3\}$ and let

$$L = \{(x, y) \in X \times X \mid y = x \leqslant 1\} \cup \{(x, y) \in X \times X \mid y < 3 \text{ and } \tfrac{1}{2}(-y + 3) \leqslant x \leqslant \tfrac{1}{2}(y + 1)\}.$$

Example 6. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 9/4 < x = y < 11/4\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 2y + 10) \leqslant x < 5\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 8y + 25) \leqslant x < 5\}.$$

Example 7. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 9/4 \leqslant x = y \leqslant 11/4\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 2y) \geqslant x > 0\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 8y + 25) \leqslant x < 5\}.$$

Example 8. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 2 \leqslant x = y \leqslant 3\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y + 2) \leqslant x \leqslant -\tfrac{1}{2}(y - 6)\} \cup \{(x, y) \in X \times X \mid -\tfrac{1}{2}(y - 9) \leqslant x \leqslant \tfrac{1}{2}(y + 3)\}.$$

Example 9. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 2 \leqslant x = y \leqslant 3\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y + 2) \leqslant x \leqslant -\tfrac{1}{2}(y - 6)\} \cup \{(x, y) \in X \times X \mid -3y + 12 \leqslant x \leqslant \tfrac{1}{2}(y + 3)\}.$$

Example 10. Let $X = \{x \in R \mid 0 < x < 5\}$ and let

$$L = \{(x, y) \in X \times X \mid 2 \leqslant x = y < 5/2\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y + 2) \leqslant x \leqslant -\tfrac{1}{2}(y - 6)\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 8y + 25) \leqslant x < 5\}.$$

Example 11. Let $X = \{x \in R \mid 0 < x < 4\}$ and let

$$L = \{(x, y) \in X \times X \mid y = x < \tfrac{1}{2}\} \cup \{(x, y) \in X \times X \mid y = x > 7/2\} \cup \{(x, y) \in X \times X \mid \tfrac{1}{2}(y^2 - 4y + 11) \leqslant x < 4\}.$$

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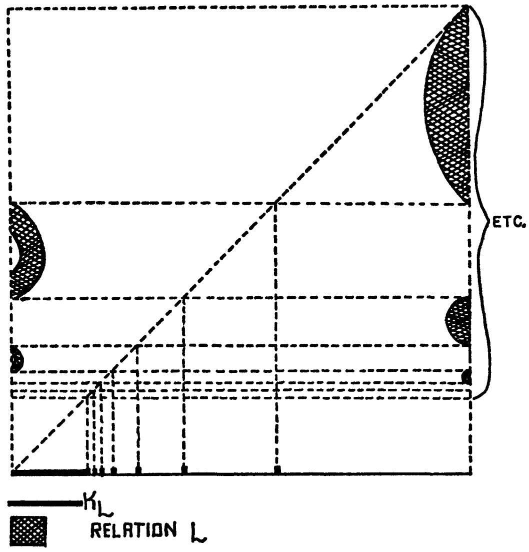
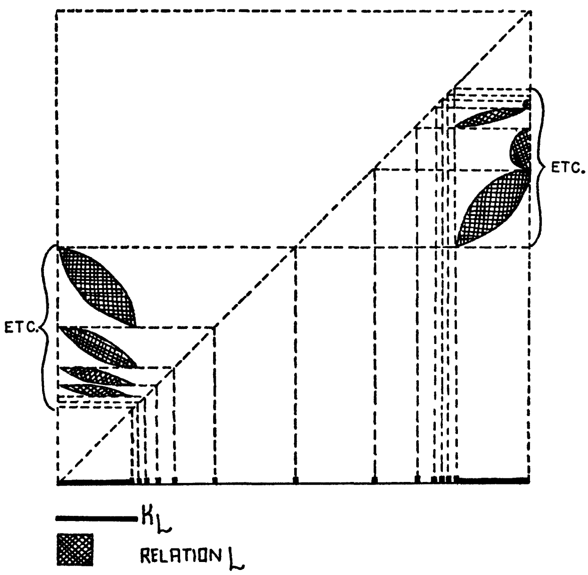
Examples 12 *and* 13. The last two examples are only suggestions of possible relations where K_L neither is connected nor has but a finite number of components. Because of the structure of K_L, diagrams will be used.



EXAMPLE 12



EXAMPLE 13

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References

1. John L. Kelley, *General topology* (Princeton, 1955).
2. I. S. Krule, *Concerning binary relations on connected ordered spaces*, Can. J. Math., *11* (1959), 107–111.
3. ——— *Structs on the 1-sphere*, Duke Math. J., *24* (1957), 405–414.
4. W. L. Strother, *Continuity for multi-valued functions*, Tulane University Dissertation (1952).
5. A. D. Wallace, *Struct ideals*, Proc. Amer. Math. Soc., *6* (1955), 634–638.

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