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Uniformly Continuous Functionals and M-Weakly Amenable Groups

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Abstract. Let G be a locally compact group. Let $A_M(G)$ ($A_0(G)$)denote the closure of $A(G)$, the Fourier algebra of G in the space of bounded (completely bounded) multipliers of $A(G)$. We call a locally compact group M-weakly amenable if $A_M(G)$ has a bounded approximate identity. We will show that when G is M-weakly amenable, the algebras $A_M(G)$ and $A_0(G)$ have properties that are characteristic of the Fourier algebra of an amenable group. Along the way we show that the sets of topologically invariant means associated with these algebras have the same cardinality as those of the Fourier algebra.

1 Introduction

For an abelian locally compact group G the Fourier transform yields an isometric isomorphism of the commutative function algebra $A(G)$, the so-called Fourier algebra of G, and the convolution algebra $L^1(\widehat{G})$, where $\widehat{G}$ is the dual group of G. In [5] Pierre Eymard introduced the Fourier algebra $A(G)$ for nonabelian locally compact groups. Shortly after the publication of [5], H. Leptin showed that the amenable locally compact groups can be characterized in terms of the Banach algebra $A(G)$ [15]. In particular, he showed that the group G is amenable if and only if $A(G)$ has a bounded approximate identity. The boundedness of the approximate identity is important. Indeed, the free group on two generators F_2 is certainly not amenable, but nonetheless $A(F_2)$ does have an approximate identity. While this approximate identity is necessarily unbounded in the usual norm, it is bounded if we replace the usual norm on $A(G)$ by the norm obtained by viewing elements of $A(G)$ as multiplication operators on itself. In fact, a similar statement holds if we consider the completely bounded norm of such multiplication operators.

If we replace the Fourier algebra of F_2 by its closure $A_M(F_2)$ in the multiplier algebra of $A(F_2)$, then by making use of the existence of this approximate identity it can be shown that the new algebra $A_M(F_2)$ has automatically continuous derivations into any Banach $A_M(F_2)$-bimodule. For the Fourier algebra this property characterizes amenability of the group G. This was the first piece of evidence that for the class of locally compact groups for which $A(G)$ admits an approximate identity that is bounded in the (completely bounded) multiplier norm, the algebra $A_M(G)$ ($A_0(G)$) behaves much like the Fourier algebra of an amenable group. As further evidence to support this thesis we note that Ruan [17] showed that the Fourier algebra $A(G)$ is

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operator amenable if and only if the group G is amenable. More recently, the first author together with V. Runde and N. Spronk showed that the quantized Banach algebra $A_0(F_2)$ obtained by closing $A(G)$ in its completely bounded multiplier algebra is operator amenable [8].

In this paper, we will give further evidence that whenever the Fourier algebra $A(G)$ admits an approximate identity that is bounded in the (completely bounded) multiplier norm, then $(A_0(G))$ $A_M(G)$ has properties that for the Fourier algebra are characteristic of the class of amenable groups. Along the way, we will give special attention to the nature of the uniformly continuous functionals on $A(G)$ and on $A_M(G)$ and $A_0(G)$. In particular, we will show that the set of topologically invariant means on the duals of all three of theses algebras have the same cardinality.

2 Preliminaries and Notation

Let G be a locally compact group. We let $A(G)$ and $B(G)$ denote the Fourier and Fourier–Stieltjes algebras of G, which are Banach algebras of continuous functions on G and were introduced in [5]. A *multiplier* of $A(G)$ is a (necessarily bounded and continuous) function $v\colon G \to \mathbb{C}$ such that $vA(G) \subseteq A(G)$. For each multiplier v of $A(G)$, the linear operator M_v on $A(G)$ defined by $M_v(u) = vu$ for each $u \in A(G)$ is bounded via the Closed Graph Theorem. The *multiplier algebra* of $A(G)$ is the closed subalgebra

$$M(A(G)) := \{M_v : v \text{ is a multiplier of } A(G)\}$$

of $B(A(G))$, where $B(A(G))$ denotes the algebra of all bounded linear operators from $A(G)$ to $A(G)$. Throughout this paper we will generally use v in place of the operator M_v, and we will write $\|v\|_M$ to represent the norm of M_v in $B(A(G))$.

Let G be a locally compact group and let $VN(G)$ denote its group von Neumann algebra. The duality $A(G) = VN(G)_*$ equips $A(G)$ with a natural operator space structure. With this operator space structures we can define the *cb-multiplier algebra* of $A(G)$ to be

$$M_{cb(A(G))} := CB(A(G)) \cap M(A(G)),$$

where $CB(A(G))$ denotes the algebra of all completely bounded linear maps from $A(G)$ into itself. We let $\|v\|_{cb}$ denote the cb-norm of the operator M_v. It is well known that $M_{cb}(A(G))$ is a closed subalgebra of $CB(A(G))$ and is thus a (quantized) Banach algebra with respect to the norm $\|\cdot\|_{cb}$.

It is known that in general,

$$A(G) \subseteq B(G) \subseteq M_{cb}(A(G)) \subseteq M(A(G))$$

and that for $v \in A(G)$

$$\|v\|_{A(G)} = \|v\|_{B(G)} \geq \|v\|_{cb} \geq \|v\|_M.$$

In the case where G is an amenable group, we have

$$B(G) = M_{cb}(A(G)) = M(A(G))$$

and that

$$\|v\|_{B(G)} = \|v\|_{cb} = \|v\|_M$$

for any $v \in B(G)$.

In [7], the first author introduced the algebra

$$A_0(G) \stackrel{\text{def}}{=} A(G)^{-\|\cdot\|_{cb(G)}} \subseteq M_{cb}(A(G))$$

which was denoted in that paper by $A_{M_0}(G)$. The free group on two generators $\mathbb{F}_2$ is such that $A(\mathbb{F}_2)$ has an approximate identity that, while unbounded in $\|\cdot\|_{A(G)}$, is bounded in $\|\cdot\|_{M_0(G)}$ [3]. Groups for which $A(G)$ has an approximate identity that is bounded in $\|\cdot\|_{A_0(G)}$ have been extensively studied and are now widely referred to as being *weakly amenable* (see [2, 3, 10]).

In this paper, we will be interested in multipliers that may or may not be completely bounded but can nonetheless be approximated by elements of $A(G)$. This leads us naturally to the following additional definition.

Definition 2.1 Given a locally compact group G let

$$A_M(G) \stackrel{\text{def}}{=} A(G)^{-\|\cdot\|_M} \subseteq M(A(G)).$$

We say that the locally compact group G is *multiplier-weakly amenable*, or *M-weakly amenable*, if there is an approximate identity $\{u_\alpha\}_{\alpha\in I}$ in $A(G)$ that is bounded in the norm $\|\cdot\|_M$.

As we have seen, if G is amenable, then $A(G) = A_0(G) = A_M(G)$ with equality holding for the various norms. Moreover, Losert has shown that G is amenable if and only if $A(G) = A_M(G)$. In fact, Losert showed that G is amenable whenever the $\|\cdot\|_{B(G)}$ and the $\|\cdot\|_M$ norms are equivalent on $A(G)$ ([16]). We will now use this to show that for nonamenable groups, $A_M(G)$ must also contain elements that are not even in the Fourier–Stieltjes algebra of G.

Proposition 2.2 *$A_M(G) \subseteq B(G)$ if and only if G is amenable.*

Proof If G is amenable, this follows immediately from Losert's result above.

Conversely, let $A_M(G) \subseteq B(G)$. Then $A_M(G)$ is closed in $B(G)$ in the $\|\cdot\|_{B(G)}$-norm topology. In fact, let $u_n \in A_M(G)$ and $b \in B(G)$ be such that $\|u_n - b\|_{B(G)} \to 0$. Then it is clear that $\|u_n - b\|_M \to 0$ and hence that $b \in A_M(G)$.

Since $\|\cdot\|_{B(G)} \geq \|\cdot\|_M$ on $B(G)$, the Open Mapping Theorem shows that $\|\cdot\|_{B(G)}$ and $\|\cdot\|_M$ are equivalent on $A_M(G)$ and hence on $A(G)$. It follows that G is amenable. ■

We will make frequent use of the following rather straightforward proposition. The first part of the proposition is [6, Proposition 1]. The proof of the second assertion follows exactly as in [6, Proposition 1].

Proposition 2.3

(i) *G is weakly amenable if and only if $A_0(G)$ has a bounded approximate identity.*
(ii) *G is M-weakly amenable if and only if $A_M(G)$ has a bounded approximate identity.*

Finally, let $\mathcal{A}$ be $A(G)$, $A_0(G)$ or $A_M(G)$. Then $X = \mathcal{A}^*$ is Banach-$\mathcal{A}$ module via the action

$$\langle v \cdot T, w\rangle = \langle T, vw\rangle$$

for each $v, w \in \mathcal{A}$ and $T \in X$.

We will follow standard convention and call the space

$$\overline{\operatorname{span}\{v \cdot T \mid v \in \mathcal{A}, T \in X\}}$$

the *uniformly continuous functionals on* $\mathcal{A}$. We will write

$$UCB(\widehat{G}) = \overline{\operatorname{span}\{v \cdot T \mid v \in A(G), T \in VN(G)\}},$$

$$UCB_0(\widehat{G}) = \overline{\operatorname{span}\{v \cdot T \mid v \in A_0(G), T \in A_0(G)^*\}},$$

$$UCB_M(\widehat{G}) = \overline{\operatorname{span}\{v \cdot T \mid v \in A_M(G), T \in A_M(G)^*\}}.$$

3 Topologically Invariant Means and Uniformly Continuous Functionals

We will begin this section by showing that the duals of $A(G)$, $A_0(G)$, and $A_M(G)$ admit the same number of topologically invariant means.

Definition 3.1 Let $X = VN(G)$, $A_M(G)^*$ or $A_0(G)^*$. Then $m \in X^*$ is called a Topologically Invariant Mean (TIM) on X if

(i) $\|m\| = \langle m, I\rangle = 1$, where $I = L_e$ is the identity operator in X;
(ii) $\langle m, v \cdot T\rangle = \langle m, T\rangle$ for $T \in X$, $v \in X_*$ with $v(e) = 1$.

Note that property (ii) in Definition 3.1 is equivalent to

$$\langle m, v \cdot T\rangle = v(e)\langle m, T\rangle$$

for $T \in X, v \in X_*$.

Definition 3.2 Let $\mathcal{A}$ be a Banach algebra. An element $T \in \mathcal{A}$ is *weakly almost periodic* if

$$\{u \cdot T \mid \|u\|_{\mathcal{A}} \leq 1\}$$

is relatively weakly compact. We denote the closed subspace of all weakly almost periodic functionals by $WAP(\mathcal{A})$.

In what follows, we will restrict our attention to $A_M(G)$. Analogous statements can be obtained for $A_0(G)$ by making the obvious modifications.

Consider the inclusion map and its adjoints:

$$i\colon A(G) \to A_M(G), \quad i^*\colon A_M(G)^* \to VN(G), \quad i^{**}\colon VN(G)^* \to A_M(G)^{**}.$$

Since i has dense range, i^* is injective, and as such is invertible with inverse ${i^*}^{-1}$ on *Range*(i^*). Moreover, i^* is simply the restriction map. It will also be useful to view all of the above maps as embeddings. With this in mind we observe that when G is nonamenable, $A_M(G)^*$ can be viewed as a proper subset of $VN(G)$ and $VN(G)^*$ as a proper subset of $A_M(G)^{**}$.

The following proposition will prove useful.

Proposition 3.3 *Let $A_M(G) \cdot VN(G) = \{u \cdot T : u \in A_M(G), T \in VN(G)\}$.*

(i) $A_M(G) \cdot VN(G) \subseteq UCB(\widehat{G})$.
(ii) $i^*(UCB_M(\widehat{G})) \subseteq UCB(\widehat{G})$.
(iii) *If G is M-weakly amenable, then $A_M(G) \cdot VN(G) = UCB(\widehat{G})$.*
(iv) *$u \cdot T \in i^*(A_M(G)^*)$ for each $u \in A(G), T \in VN(G)$.*
(v) *$i^*(v \cdot T) = v \cdot i^*(T)$ for each $v \in A_M(G), T \in A_M(G)^*$.*

Proof (i) To establish this we need only show that for any sequence $\{v_n\} \subset A(G)$ and $v \in A_M(G)$ with $\|v_n - v\|_M \to 0$ and any $T \in VN(G)$, we have

$$\|v_n \cdot T - v \cdot T\|_{VN(G)} \to 0.$$

However, this follows immediately, since for any $u \in A(G)$

$$\begin{aligned}|\langle v_n \cdot T - v \cdot T, u\rangle| &= |\langle (v_n - v) \cdot T, u\rangle| = |\langle T, (v_n - v)u\rangle| \\ &\leq \|T\|_{VN(G)} \|v_n - v\|_M \|u\|_{A(G)}.\end{aligned}$$

(ii) This follows immediately from (i), since $i^*(v \cdot T) \in A_M(G) \cdot VN(G)$ for any $v \in A_M(G)$ and $T \in A_M(G)^*$.
(iii) Since $A_M(G)$ has a bounded approximate identity, it follows from Cohen's Factorization Theorem and from (i) that $A_M(G) \cdot VN(G)$ is a closed subspace of $UCB(\widehat{G})$ (see also [9, p. 373]). However, since $A(G) \cdot VN(G) \subseteq A_M(G) \cdot VN(G)$, it is also clear that $A_M(G) \cdot VN(G)$ is dense in $UCB(\widehat{G})$. So (iii) is true.
(iv) Let $u \in A(G), T \in VN(G)$. Then we can define a linear functional on $A_M(G)$ by $\varphi_{u,T}(v) = \langle T, uv\rangle$ for each $v \in A_M(G)$. It is also clear that $\varphi_{u,T}$ has norm at most $\|u\|_{A(G)} \|T\|_{VN(G)}$. Moreover, this linear functional agrees with $u \cdot T$ on $A(G)$ and as such $u \cdot T = i^*(\varphi_{u,T})$.
(v) Let $v \in A_M(G), T \in A_M(G)^*$. Then for each $u \in A(G)$, we have

$$\begin{aligned}\left\langle i^*(v \cdot T), u \right\rangle &= \langle v \cdot T, i(u)\rangle = \left\langle T, vi(u)\right\rangle = \left\langle T, i(vu)\right\rangle \\ &= \langle i^*(T), vu\rangle = \langle v \cdot i^*(T), u\rangle.\end{aligned}$$ ■

Theorem 3.4 *For any locally compact group, $i^{**}(TIM(A(G)) \subseteq TIM(A_M(G))$. Moreover, $i^{**}\colon TIM(A(G)) \to TIM(A_M(G))$ is a bijection.*

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Proof We will first show that $i^{**}(TIM(A(G))) \subseteq TIM(A_M(G))$.

Let $m \in TIM(A(G))$. Let $v \in A_M(G)$ and $T \in A_M(G)^*$. Then there exists $\{u_n\} \subset A(G)$ such that $\|u_n - v\|_M \to 0$. Since $\|u_n - v\|_\infty \leq \|u_n - v\|_M$, it follows that $u_n(e) \to v(e)$. We also see as in the proof of the previous proposition (i) that $u_n \cdot T \to v \cdot T$ in the $\|\cdot\|_M$-norm for each $T \in A_M(G)^*$. It follows that

$$\begin{aligned}\big\langle i^{**}(m), v\cdot T\big\rangle &= \lim_{n\to\infty}\big\langle i^{**}(m), u_n\cdot T\big\rangle = \lim_{n\to\infty}\big\langle m, i^*(u_n\cdot T)\big\rangle\\ &= \lim_{n\to\infty}\big\langle m, u_n\cdot i^*(T)\big\rangle = \lim_{n\to\infty} u_n(e)\langle m, i^*(T)\rangle\\ &= \lim_{n\to\infty} v(e)\big\langle m, i^*(T)\big\rangle = v(e)\langle i^{**}(m), T\rangle.\end{aligned}$$

This shows that $i^{**}(TIM(A(G))) \subseteq TIM(A_M(G))$.

We next show that $i^{**}\colon TIM(A(G)) \to TIM(A_M(G))$ is injective. To see this we first note that if $m_1, m_2 \in TIM(A(G))$ with $m_1 \neq m_2$, then there exists an $T \in VN(G)$ for which $\langle m_1, T\rangle \neq \langle m_2, T\rangle$.

Next choose $u_0 \in A(G)$ with $u_0(e) = 1$. Then

$$\langle m_1, u_0\cdot T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, u_0\cdot T\rangle.$$

Since $u_0 \cdot T \in A_M(G)^*$, we have

$$\begin{aligned}\big\langle i^{**}(m_1), u_0\cdot T\big\rangle &= \big\langle m_1, i^*(u_0\cdot T)\big\rangle = \big\langle m_1, u_0\cdot T\big\rangle\\ &\neq \langle m_2, u_0\cdot T\rangle\\ &= \big\langle m_2, i^*(u_0\cdot T)\big\rangle = \big\langle i^{**}(m_2), u_0\cdot T\big\rangle\end{aligned}$$

so that $i^{**}(m_1) \neq i^{**}(m_2)$.

Finally, we show that $i^{**}\colon TIM(A(G)) \to TIM(A_M(G))$ is surjective.

Let $M \in TIM(A_M(G))$. First note that if $u, v \in A(G)$, with $u(e) = 1 = v(e)$ and if $T \in VN(G)$, then $u \cdot T$ and $v \cdot T$ are in $A_M(G)^*$ and

$$\langle M, u\cdot T\rangle = \big\langle M, v\cdot(u\cdot T)\big\rangle = \big\langle M, u\cdot(v\cdot T)\big\rangle = \langle M, v\cdot T\rangle.$$

Pick a $u_0 \in A(G)$ with $\|u_0\|_{A(G)} = 1$ and $u_0(e) = 1$. We can define $m_M \in A(G)^{**}$ by

$$\langle m_M, T\rangle = \langle M, u_0\cdot T\rangle$$

for $T \in VN(G)$.

It is clear from the observation above that if $v \in A(G)$ is such that $v(e) = 1$, then $\langle m_M, v\cdot T\rangle = \langle m_M, T\rangle$. We also have that

$$\langle m_M, I\rangle = \langle M, u_0\cdot I\rangle = \langle M, u_0(e)I\rangle = \langle M, I\rangle = 1.$$

That is, $m_M \in TIM(A(G))$.

Finally, if $T \in A_M(G)^*$, then

$$\big\langle i^{**}(m_M), T\big\rangle = \big\langle m_M, i^*(T)\big\rangle = \big\langle M, u_0\cdot i^*(T)\big\rangle = \big\langle M, i^*(T)\big\rangle = \langle M, T\rangle.$$

Therefore, $i^{**}(m_M) = M$. ∎

Definition 3.5 Given a locally compact group G we let $b(G)$ denote the smallest cardinality of a neighbourhood basis at the identity e for G.

The next corollary follows immedately from the previous theorem and from Hu [11].

Corollary 3.6 *Let G be a nondiscrete locally comapct group. Then* $|TIM(A_M(G))| = 2^{2^{b(G)}}$. *In particular,* $A_M(G)$ *admits a unique topological invariant mean if and only if G is discrete.*

Recall that a Banach algebra $\mathcal{A}$ is said to be *Aren's regular* if the two Arens products on $\mathcal{A}^{**}$ agree. It is well known that a Banach algebra $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}) = A^*$.

It is clear that if $\mathcal{A} = A(G)$, $\mathcal{A} = A_0(G)$, or $\mathcal{A} = A_M(G)$ and if X is a norm closed $\mathcal{A}$-submodule of $\mathcal{A}^*$ that contains $I = L_e$, then it makes sense to talk about topologically invariant means on X. Moreover, each $m \in TIM(\mathcal{A})$ restricts to a topologically invariant mean on X. In particular, $X = WAP(\mathcal{A})$ is such a space, as is $UCB(\mathcal{A})$.

Proposition 3.7 *The restriction map* $R\colon TIM(A_M(G)) \to TIM(UCB_M(\widehat{G}))$ *is a bijection. In particular* $|TIM(A_M(G))| = |TIM(UCB_M(\widehat{G}))|$.

Proof If $M_1, M_2 \in TIM(A_M(G))$ with $M_1 \neq M_2$, then there exists a $T \in A_M(G)^*$ for which $\langle M_1, T\rangle \neq \langle M_2, T\rangle$. As in the proof of the previous theorem, we see that if we choose $u_0 \in A(G)$ with $u_0(e) = 1$, then

$$\langle M_1, u_0 \cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, u_0 \cdot T\rangle.$$

This shows that $R(M_1) \neq R(M_2)$, and hence R is injective.

Next, let $m \in TIM(UCB_M(\widehat{G}))$. Pick a $u_0 \in A(G)$ with $\|u_0\|_{A(G)} = 1 = u_0(e)$. Define $M \in A_M(G)^{**}$ by

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle, \qquad T \in A_M(G)^*.$$

Since $u_0(e) = 1$, it follows that

$$\langle M, L_e\rangle = \langle m, u_0 \cdot L_e\rangle = \langle m, L_e\rangle = 1.$$

From this and the fact that $\|u_0\|_{A(G)} = 1$, we get that $\|M\| = 1$.

Next, if $v \in A_M(G)$, $T \in A_M(G)^*$ with $v(e) = 1$, then

$$\big\langle M, v \cdot T\big\rangle = \big\langle m, u_0 \cdot (v \cdot T)\big\rangle = \big\langle m, v \cdot (u_0 \cdot T)\big\rangle = \langle m, u_0 \cdot T\rangle = \langle M, T\rangle.$$

This shows that $M \in TIM(A_M(G))$.

Finally, if $T \in UCB_M(\widehat{G})$, then

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle = \langle m, T\rangle,$$

since $T \in UCB_M(\widehat{G})$ and $m \in TIM(UCB_M(\widehat{G}))$. Therefore, $R(M) = m$ and R is surjective. ∎

The following result is well known for $A(G)$ (see [4, 9]). The proof we give is a modification of that given for $A(G)$.

Proposition 3.8 *For any locally compact group, there is a unique topologically invariant mean on $WAP(A_M(G))$.*

Proof Let M be a topologically invariant mean on $A_M(G)^*$. Then the restriction of M to $WAP(A_M(G))$ is clearly a topologically invariant mean on $WAP(A_M(G))$. By Goldstine's Theorem we can find a net $\{u_\alpha\}_{\alpha\in I} \subset A_M(G)$ with $\tau(u_\alpha) \to M$ in the weak* topology of $A_M(G)^{**}$, where τ is the canonical injection of $A_M(G)$ into $A_M(G)^{**}$. In particular, the restriction of $\tau(u_\alpha)$ converges in the weak* topology of $WAP(A_M(G))^*$ to M. Moreover, since $M(L_e) = 1$, $u_\alpha(e) \to 1$.

Now let m be any topologically invariant mean on $WAP(A_M(G))$, let $u \in A_M(G)$ with $\|u\|_{A_M(G)} = 1$, and let $T \in WAP(A_M(G))$. Then

$$\langle u_\alpha \cdot T, u\rangle = \langle u \cdot T, u_\alpha\rangle \to \langle M, u \cdot T\rangle = u(e)\langle M, T\rangle.$$

It follows that the net $\{u_\alpha \cdot T\}_{\alpha\in I}$ converges in the weak*-topology of $A_M(G)^*$ to $\langle M, T\rangle L_e$. But since T is weakly almost periodic, we in fact have that $\{u_\alpha \cdot T\}_{\alpha\in I}$ converges in the $\sigma(A_M(G)^*, A_M(G)^{**})$ topology to $\langle M, T\rangle L_e$. From this we can conclude that there is a sequence $\{v_n\}$ of convex combinations of the u_α's such that $v_n(e) \to 1$ and $v_n \cdot T \to \langle M, T\rangle L_e$ in norm. Hence

$$\langle m, T\rangle = \lim_{n\to\infty} v_n(e)\langle m, T\rangle = \lim_{n\to\infty} \langle m, v_n \cdot T\rangle = \langle m, \langle M, T\rangle L_e\rangle = \langle M, T\rangle. \qquad \blacksquare$$

Corollary 3.9 *If $A_M(G)$ is Arens regular, then G is discrete. Moreover, in this case every amenable subgroup of G is finite.*

Proof If $A_M(G)$ is Arens regular, then $WAP(A_M(G)) = A_M(G)^*$. In particular, $A_M(G)^*$ must have a unique invariant mean by Proposition 3.8. It then follows that G must be discrete by Corollary 3.6.

If $A_M(G)$ is Arens regular and H is an amenble subgroup, then the restriction map is a contractive homomorphism of $A_M(G)$ onto $A(H)$. It follows that $A(H)$ is also Arens regular and hence that H is finite (see Lau and Wong [14]). ■

Corollary 3.10 *Let G be a locally comapct group. If $UCB_M(\widehat{G}) \subseteq WAP(A_M(G))$, then G is discrete.*

Proof Assume that $UCB_M(\widehat{G}) \subseteq WAP(A_M(G))$. Then it follows that $UCB_M(\widehat{G})$ admits a unique topologically invariant mean. By Proposition 3.7 and Corollary 3.6, G is discrete. ■

4 Uniformly Continuous Functionals and Operators Commuting with A(G)

Let A be a commutative Banach algebra and let X and Y be commutative Banach A-modules. As usual, we let $B(X, Y)$ denote the space of bounded linear maps from X

to Y with $B(X) = B(X, X)$. We let $B_A(X, Y)$ denote the subspace of $B(X, Y)$ consisting of all Γ that commute with the action of A, that is, all $\Gamma \in B(X, Y)$ such that $\Gamma(u \cdot x) = u \cdot \Gamma(x)$ for all $u \in A$ and all $x \in X$.

In [13] Lau showed that if G is an amenable group, then there is an isometric algebra isomorphism between the space of bounded operators on $VN(G)$ that commute with the action of $A(G)$ and the space $UCB(\widehat{G})^*$, when the latter space is given a multiplication that is analogous to one of the Arens products. In this section we will prove the converse of this result. In addition, we will show that if G is M-weakly amenable, then $UCB(\widehat{G})^*$ can be identified with $B_{A(G)}(VN(G), A_M(G)^*)$. We will also generalize these results to the algebra $A_M(G)$.

We begin with the following extension theorem for M-amenable groups, which is of independent interest.

Theorem 4.1 *Assume that G is M-weakly amenable. Let $\Gamma \in B(UCB(\widehat{G}))$. Then there exists $\Gamma_0 \in B(VN(G))$ such that $\Gamma_{0|_{UCB(\widehat{G})}} = \Gamma$. Moreover, if Γ commutes with the action of $A(G)$ on $UCB(\widehat{G})$, then Γ_0 commutes with the action of $A(G)$ on $VN(G)$.*

Proof Let $\{u_\alpha\}_{\alpha \in I}$ be an approximate identity for $A(G)$ that is bounded in $\|\cdot\|_M$ with $\|u_\alpha\| \leq C$ for each $\alpha \in I$. Given $\alpha \in I$ define a bilinear form $\Lambda_\alpha \colon VN(G) \times A(G) \to \mathbb{C}$ by

$$\Lambda_\alpha(T, u) = \left\langle \Gamma(u_\alpha \cdot T), u \right\rangle.$$

Then $\|\Lambda_\alpha\| \leq C\|\Gamma\|$. Hence by the Banach–Alaoglu Theorem, there exists a subnet $\{\Lambda_{\alpha_k}\}$ of $\{\Lambda_\alpha\}$ and a bilinear form $\Lambda_0 \colon VN(G) \times A(G) \to \mathbb{C}$ such that $\|\Lambda_0\| \leq C\|\Gamma\|$ and $\{\Lambda_{\alpha_k}\}$ converges pointwise to Λ_0. Next define $\Gamma_0 \colon VN(G) \to VN(G)$ by

$$\left\langle \Gamma_0(T), u \right\rangle = \Lambda_0(T, u)$$

for every $T \in VN(G)$ and $u \in A(G)$. Then $\|\Gamma_0\| \leq C\|\Gamma\|$.

Let $T \in UCB(\widehat{G})$. Since $\{u_{\alpha_k}\}_{\alpha \in I}$ is a bounded approximate identity for $A_M(G)$ and by Proposition 3.3(iii), $\|u_{\alpha_k} \cdot T - T\|_{VN(G)} \to 0$. Hence

$$\lim_{\alpha_k} \left\langle \Gamma(u_{\alpha_k} \cdot T), u \right\rangle = \left\langle \Gamma(T), u \right\rangle$$

for every $u \in A(G)$. Therefore, $\Gamma_{0|_{UCB(\widehat{G})}} = \Gamma$.

Assume that Γ commutes with the action of $A(G)$ on $UCB(\widehat{G})$. Let $v \in A(G)$ and let $T \in VN(G)$. Then

$$\begin{aligned}\left\langle \Gamma_0(v \cdot T), u \right\rangle &= \lim_{\alpha_k} \left\langle \Gamma(u_{\alpha_k} \cdot (v \cdot T)), u \right\rangle = \lim_{\alpha_k} \left\langle \Gamma(v \cdot (u_{\alpha_k} \cdot T)), u \right\rangle \\ &= \lim_{\alpha_k} \left\langle v \cdot \Gamma(u_{\alpha_k} \cdot T), u \right\rangle = \left\langle v \cdot \Gamma_0(T), u \right\rangle.\end{aligned}$$

Therefore, Γ_0 commutes with the action of $A(G)$ on $VN(G)$. ■

In [13], Lau showed that any $m \in UCB(\widehat{G})^*$ induces an operator $\widehat{m}_L \colon VN(G) \to VN(G)$ such that $\widehat{m}_L(u \cdot T) = u \cdot T$ for every $u \in A(G)$ and $T \in VN(G)$. In this case,

$$\langle \widehat{m}_L(T), u \rangle = \langle m, u \cdot T \rangle$$

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for every $u \in A(G)$ and $T \in VN(G)$. Furthermore, if G is amenable, Lau shows that $\|m\|_{UCB(\widehat{G})} = \|\widehat{m}_L\|_{B_{A(G)}(VN(G))}$.

We can improve on the above result by showing that the range of $\widehat{m}_L$ is contained in $A_M(G)^*$ and that in fact $\widehat{m}_L \in B_{A(G)}(VN(G), A_M(G)^*)$.

Lemma 4.2 *Let $m \in UCB(\widehat{G})^*$. Define*

$$\left\langle \widehat{m}_L(T), v \right\rangle = \langle m, v \cdot T \rangle$$

for $v \in A_M(G)$ and $T \in VN(G)$. Then $\widehat{m}_L(T) \in A_M(G)^*$. Moreover,

$$\widehat{m}_L \in B_{A(G)}(VN(G), A_M(G)^*)$$

with $\|\widehat{m}_L\|_{B_{A(G)}(VN(G), A_M(G)^*)} \leq \|m\|_{UCB(\widehat{G})^*}$.

Proof First observe that the definition of $\widehat{m}_L(T)$ makes sense by Proposition 3.3(i). We also have that

$$\begin{aligned} |\langle \widehat{m}_L(T), v \rangle| = |\langle m, v \cdot T \rangle| &\leq \|m\|_{UCB(\widehat{G})^*} \|v \cdot T\|_{VN(G)} \\ &\leq \|m\|_{UCB(\widehat{G})^*} \|v\|_{A_M(G)} \|T\|_{VN(G)}. \end{aligned}$$

From this it follows immediately that $\widehat{m}_L(T) \in A_M(G)^*$ with

$$\|\widehat{m}_L(T)\|_{A_M(G)^*} \leq \|m\|_{UCB(\widehat{G})^*} \|T\|_{VN(G)}.$$

Since the map $T \to \widehat{m}_L(T)$ is clearly linear we get that $\widehat{m}_L \in B(VN(G), A_M(G)^*)$ with $\|\widehat{m}_L\|_{B(VN(G), A_M(G)^*)} \leq \|m\|_{UCB(\widehat{G})^*}$. To see that $\widehat{m}_L \in B_{A(G)}(VN(G), A_M(G)^*)$ observe that

$$\begin{aligned} \left\langle \widehat{m}_L(u \cdot T), v \right\rangle &= \left\langle m, v \cdot (u \cdot T) \right\rangle = \left\langle m, (vu) \cdot T \right\rangle \\ &= \left\langle \widehat{m}_L(T), uv \right\rangle = \left\langle u \cdot \widehat{m}_L(T), v \right\rangle \end{aligned}$$

for $u \in A(G)$, $v \in A_M(G)$ and $T \in VN(G)$. ∎

Theorem 4.3 *Let $\Lambda \colon UCB(\widehat{G})^* \to B_{A(G)}(VN(G), A_M(G)^*)$ be given by $\Lambda(m) = \widehat{m}_L$ for $m \in UCB(\widehat{G})^*$. Then Λ is linear, contractive and one-to-one. If G is M-weakly amenable, then Λ is an isomorphism onto $B_{A(G)}(VN(G), A_M(G)^*)$. Moreover, if $A_M(G)$ has a bounded approximate identity $\{v_\alpha\}_{\alpha \in I}$ such that $\|v_\alpha\|_{A_M(G)} \leq C$ for some C and every $\alpha \in I$, then $\|\Lambda^{-1}\| \leq C$.*

Proof It is easy to see that Λ is linear, and the previous lemma shows that it is contractive. To see that Λ is one-to-one, let $m_1, m_2 \in UCB(\widehat{G})^*$ with $m_1 \neq m_2$. Since $A(G) \cdot VN(G)$ has a dense span in $UCB(\widehat{G})$, there exists $u \in A(G)$ and $T \in VN(G)$ such that $\langle m_1, u \cdot T \rangle \neq \langle m_2, u \cdot T \rangle$. Then it follows immediately that $\Lambda(m_1) \neq \Lambda(m_2)$.

Assume that G is M-weakly amenable. We must show that Λ is surjective. Let $\Gamma \in B_{A(G)}(VN(G), A_M(G)^*)$. Then $\Gamma^* \in B(A_M(G)^{**}, VN(G)^*)$. We may view $\{v_\alpha\}_{\alpha \in I}$ as a bounded net in $A_M(G)^{**}$. It follows that $\{\Gamma^*(v_\alpha)\}_{\alpha\in I}$ is a bounded net in $VN(G)^*$. By passing to a subnet if necessary, we may assume $\{\Gamma^*(v_\alpha)\}_{\alpha\in I}$ converges in the *weak** topology to some $\phi \in VN(G)^*$. Moreover,

$$\|\phi\|_{VN(G)^*} \le \sup\{\|\Gamma^*\|_{B(A_M(G)^{**},VN(G)^*)}\|v_\alpha\|_{A_M(G)}\} \le C\|\Gamma^*\|_{B(A_M(G)^{**},VN(G)^*)}.$$

Let $m = \phi_{|UCB(\widehat{G})}$. Then

$$\|m\|_{UCB(\widehat{G})^*} \le C\|\Gamma\|_{B(VN(G),A_M(G)^*)}.$$

Let $v \in A_M(G)$ and $T \in VN(G)$. Then

$$\begin{aligned}\left\langle \widehat{m}_L(T), v\right\rangle &= \langle \phi, v\cdot T\rangle = \lim_{\alpha\in I}\left\langle \Gamma^*(v_\alpha), v\cdot T\right\rangle \\ &= \lim_{\alpha\in I}\left\langle v_\alpha, \Gamma(v\cdot T)\right\rangle = \lim_{\alpha\in I}\left\langle v_\alpha, v\cdot\Gamma(T)\right\rangle \\ &= \left\langle \Gamma(T), v_\alpha v\right\rangle = \left\langle \Gamma(T), v\right\rangle.\end{aligned}$$

It follows that $\widehat{m}_L(T) = \Gamma(T)$ for every $T \in VN(G)$ and hence that Λ is surjective. Finally, since $\|m\|_{UCB(\widehat{G})^*} \le C\|\Gamma\|_{B(VN(G),A_M(G)^*)}$, we have that $\|\Lambda^{-1}\| \le C$. ■

We saw in Proposition 3.3(iv) that if $u \in A(G)$ and $T \in VN(G)$, then $u\cdot T \in A_M(G)^*$. We can now improve upon this result.

Lemma 4.4 *Assume that $u \in A(G)$ and $T \in VN(G)$. Then $u\cdot T \in UCB_M(\widehat{G})$ and $\|u\cdot T\|_{A_M(G)^*} \le \|T\|_{VN(G)}\|u\|_{A(G)}$.*

Proof We know that $u\cdot T \in A_M(G)^*$ for any $u \in A(G)$, $T \in VN(G)$. Moreover, the final inequality is obvious.

We must show that $u\cdot T \in UCB_M(\widehat{G})$. To do this we first assume that $u_0 \in A(G)\cap C_{00}(G)$ and that $T \in VN(G)$ is arbitrary. Then we can find $v \in A(G) \subseteq A_M(G)$ such that $vu_0 = u_0$. It follows that $u_0\cdot T = (vu_0)\cdot T = v\cdot(u_0\cdot T) \in UCB_M(\widehat{G})$.

Next, we let $u \in A(G)$, $T \in VN(G)$ be arbitrary. Let $\epsilon > 0$. We know that there exists a $u_0 \in A(G)\cap C_{00}(G)$ such that $\|u-u_0\|_{A(G)} < \epsilon$. Moreover, from the above we see that $u_0\cdot T \in UCB_M(\widehat{G})$. Next, if $v \in A_M(G)$ with $\|v\|_M \le 1$, we have

$$\begin{aligned}|\langle u\cdot T, v\rangle - \langle u_0\cdot T, v\rangle| &= |\langle (u-u_0)\cdot T, v\rangle| \\ &\le \|T\|_{VN(G)}\|u-u_0\|_{A(G)}\|v\|_M \le \epsilon\|T\|_{VN(G)}.\end{aligned}$$

Hence, $\|u\cdot T - u_0T\|_{A_M(G)^*} \le \epsilon\|T\|_{VN(G)}$. This shows that $u\cdot T \in UCB_M(\widehat{G})$. ■

Let $M \in UCB_M(\widehat{G})^*$. Following [13] we can define a map $\widehat{M}_L\colon VN(G) \to VN(G)$ by

$$\left\langle \widehat{M}_L(T), u\right\rangle = \langle M, u\cdot T\rangle, \quad \left(T \in VN(G) \text{ and } u \in A(G)\right).$$

We will now show that every $\widehat{M}_L$ defines a bounded operator on $VN(G)$ that commutes with the action of $A(G)$ on $VN(G)$.

Lemma 4.5 *Let $M \in UCB_M(\widehat{G})^*$. If $u \in A(G)$ and $T \in VN(G)$, then $\widehat{M}_L(u \cdot T) = u \cdot \widehat{M}_L(T)$ and $\|\widehat{M}_L\|_{B(VN(G))} \leq \|M\|_{UCB_M(\widehat{G})^*}$.*

Proof Let $u, v \in A(G)$ and $T \in VN(G)$. Then

$$\begin{aligned}\left\langle \widehat{M}_L(u \cdot T), v\right\rangle &= \left\langle M, v \cdot (u \cdot T)\right\rangle = \langle M, vu \cdot T\rangle \\ &= \left\langle \widehat{M}_L(T), uv\right\rangle = \left\langle u \cdot \widehat{M}_L(T), v\right\rangle.\end{aligned}$$

By Lemma 4.4, it is also easy to see that

$$|\langle \widehat{M}_L(T), u\rangle| = |\langle M, u \cdot T\rangle| \leq \|M\|_{UCB_M(\widehat{G}))^*}\|u\|_{A(G)}\|T\|_{VN(G)}$$

so that $\|\widehat{M}_L\|_{VN(G)} \leq \|M\|_{UCB_M(\widehat{G})^*}$. ■

Proposition 4.6 *Let $\Phi\colon UCB_M(\widehat{G}))^* \to B_{A(G)}(VN(G))$ be given by $\Phi(M) = \widehat{M}_L$. Then Φ is linear, contractive and one-to-one.*

Proof It is easy to see that Φ is linear. The previous lemma shows that it is contractive.

To see that Φ is one-to-one, let $M_1,M_2 \in UCB_M(\widehat{G})^*$ with $M_1 \neq M_2$. Since $A(G) \cdot VN(G)$ has a dense span in $UCB_M(\widehat{G})$, there exists $u \in A(G)$ and $T \in VN(G)$ such that $\langle M_1, u \cdot T\rangle \neq \langle M_2, u \cdot T\rangle$. Then it follows immediately that $\Phi(M_1) \neq \Phi(M_2)$. ■

Lemma 4.7 *We have that $i^*(UCB_M(\widehat{G})) = UCB(\widehat{G})$ if and only if G is amenable.*

Proof It follows from Proposition 3.3 that $i^*(UCB_M(\widehat{G}) \subseteq UCB(\widehat{G})$. If G is amenable, then $A_M(G) = A(G)$, so clearly $i^*(UCB_M(\widehat{G})) = UCB(\widehat{G})$.

Conversely, assume that $i^*(UCB_M(\widehat{G})) = UCB(\widehat{G})$. Let τ be the restriction of i^* onto $UCB_M(\widehat{G})$. Then $\tau^*\colon UCB(\widehat{G})^* \to UCB_M(\widehat{G})^*$ is also an isomorphism. Let $u \in A_M(G)$. Then $u \in UCB_M(\widehat{G})^*$, and since τ^* is onto, $u = \tau^*(v)$ for some $v \in UCB(\widehat{G})^*$. Moreover, since $C^*_\rho(G) \subseteq UCB(\widehat{G})$, the function u determines a bounded linear functional on $C^*_\rho(G)$ with $\langle u, f\rangle = \int_G u(x)f(x)dx$ for each $f \in L^1(G)$ (see [5] for the reduced group C^* algebra $C^*_\rho(G)$). In particular, this shows that $u \in B_\rho(G) \subseteq B(G)$ (see [5] for $B_\rho(G)$). It follows that $A_M(G) \subseteq B(G)$, and hence that G is amenable by Proposition 2.2. ■

Theorem 4.8 *Let G be a locally compact group. Let $\Upsilon\colon UCB(\widehat{G})^* \to B_{A(G)}(VN(G))$ be given by $\Upsilon(m) = \widehat{m}_L$ for each $m \in UCB(\widehat{G})^*$. Then Υ is surjective if and only if G is amenable.*

Proof If G is amenable, then Υ is surjective by [12, Theorem 6.2].

We have seen that i^* maps $UCB_M(\widehat{G})$ into $UCB(\widehat{G})$ by Proposition 3.3. As in the preceding, we let let $\tau = i^*|UCB_M(\widehat{G})$. We also let $\Phi\colon UCB_M(\widehat{G})^* \to B_{A(G)}(VN(G))$ be given by $\Phi(M) = \widehat{M}_L$.

Then it is easy to see that the following diagram commutes:

$$\begin{tikzcd} & B_{A(G)}(VN(G)) \\ UCB(\widehat{G})^* \arrow[ur, "\Upsilon"] \arrow[r, "\tau^*"'] & UCB_M(\widehat{G})^*. \arrow[u, "\Phi"'] \end{tikzcd}$$

Assume that $\Upsilon\colon UCB(\widehat{G})^* \to B_{A(G)}(VN(G))$ is onto. Then since Φ is one-to-one, τ^* must also be onto. However, this would mean that i^* maps $UCB_M(\widehat{G})$ onto $UCB(\widehat{G})$, since τ^* is one-to-one by Lemma 4.4 (see Rudin [18, p. 103]). In particular, G is amenable by Lemma 4.7. ■

Let $M \in UCB_M(\widehat{G})^*$. Then it is easy to see that the operator $\widehat{M}_L$ defined by $\langle \widehat{M}_L(T), u\rangle = \langle M, u \cdot T\rangle$ for $T \in A_M(G)^*$ and $u \in A_M(G)$ is in $B_{A_M(G)}(A_M(G)^*)$. The following result is an analogue of Theorem 4.8. In the following the tensor product is the projective tensor product as Banach spaces.

Theorem 4.9 *Let the map $\Phi\colon UCB_M(\widehat{G})^* \to B_{A_M(G)}(A_M(G)^*)$ be defined by $\Phi(M) = \widehat{M}_L$. Then Φ is an isomorphism if and only if G is M-weakly amenable.*

Proof Let

$$\eta\colon A_M(G) \otimes A_M(G)^*/\mathcal{H} \to UCB_M(\widehat{G})$$

be defined by

$$\eta\Big(\sum_{i=1}^{\infty} u_i \otimes T_i + \mathcal{H}\Big) = \sum_{i=1}^{\infty} u_i T_i \quad \text{for} \quad \sum_{i=1}^{\infty} u_i \otimes T_i \in A_M(G) \otimes A_M(G)^*,$$

where

$$\mathcal{H} = \operatorname{span}\big\{ (uv) \otimes f - u \otimes (vf) : u, v \in A(G), f \in VN(G)\big\}.$$

Then it is clear that η is well defined and $\|\eta\| \le 1$. It follows from Banach space theory that

$$B_{A_M(G)}(A_M(G)^*) = \big(A_M(G) \otimes A_M(G)^*/\mathcal{H}\big)^*.$$

Hence we have

$$\eta^*\colon UCB_M(\widehat{G})^* \to B_{A_M(G)}\big(A_M(G)^*\big).$$

It is routine to check that $\eta^*(M) = \widehat{M}_L$ for each $M \in UCB_M(\widehat{G})^*$. So $\eta^* = \Phi$. If G is M-weakly amenable, then $A_M(G)$ has a bounded approximate identity. By Cohen's factorization theorem, $UCB_M(\widehat{G}) = A_M(G) \cdot A_M(G)^*$ (see also [9, p. 373]). Hence η is surjective and so $\Phi = \eta^*$ is 1 - 1 (see Rudin [18, p. 103]). To show that Φ is surjective, let $m \in B_{A_M(G)}(A_M(G)^*)$ and let $\{a_\alpha\}$ be a bounded approximate identity in $A_M(G)$ with $\|a_\alpha\|_{A_M(G)} \le C$. For any $uf \in UCB_M(\widehat{G})$ with $\|uf\|_{UCB_M(\widehat{G})} \le 1$, where $u \in A_M(G)$ and $f \in A_M(G)^*$, define $\widetilde{m} \in UCB_M(\widehat{G})^*$ by

$$\langle \widetilde{m}, uf\rangle = \lim_{\alpha}\big\langle m, a_\alpha \otimes (uf) + \mathcal{H}\big\rangle = \lim_{\alpha}\big\langle m, (a_\alpha u) \otimes f + \mathcal{H}\big\rangle = \langle m, u \otimes f + \mathcal{H}\rangle.$$

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Hence $|\langle \widetilde{m}, uf\rangle| \leq \|a_\alpha\|_{A_M(G)}\|uf\|_{A_M(G)^*} \leq C$. So $\|\widetilde{m}\|_{UCB_M(\widehat{G})^*} \leq C$. Thus, $m = \eta^*(\widetilde{m})$ and so Φ is surjective.

Conversely, let Φ be an isomorphism. Then Φ is surjective and so the identity operator $I = \Phi(\iota)$ for some $\iota \in UCB_M(\widehat{G})^*$. Since $UCB_M(\widehat{G})$ is a subspace of $A_M(G)^*$, we extend ι to a functional $\widetilde{\iota}$ on $A_M(G)^*$ with the same norm. Since $\widetilde{\iota} \in A_M(G)^{**}$, by Goldstine's theorem, there is a net $\{u_\alpha\}$ in $A_M(G)$ such that $\|u_\alpha\|_{A_M(G)} \leq \|\widetilde{\iota}\|$ for all α and $u_\alpha \to \widetilde{\iota}$ in the $\sigma(A_M(G)^{**}, A_M(G)^*)$-topology. Hence for every $a \in A_M(G)$ and $f \in A_M(G)^*$, since $\Phi = \eta^*$, we have

$$\langle a, f\rangle = \langle I, a\otimes f\rangle = \big\langle \eta^*(\iota), a\otimes f\big\rangle = \big\langle \iota, \eta(a\otimes f)\big\rangle = \langle \widetilde{\iota}, af\rangle.$$

Hence we have

$$\langle u_\alpha a - a, f\rangle = \langle u_\alpha, af\rangle - \langle \widetilde{\iota}, af\rangle \to 0.$$

Thus, $\{u_\alpha\}$ is a bounded weak approximate identity for $A_M(G)$. Hence $A_M(G)$ has a bounded approximate identity (see Bonsall and Duncan [1, p. 58]). Therefore $A(G)$ is M-weakly amenable. ■

The proof of the following result is a modification of the proof of Theorem 4.9.

Theorem 4.10 *If the map $\Phi\colon UCB_M(\widehat{G})^* \to B_{A(G)}(VN(G))$ defined by $\Phi(M) = \widehat{M}_L$ is surjective, then G is M-weakly amenable.*

Proof It follows from Lemma 4.4 that map

$$\eta\colon A(G)\otimes VN(G)/\mathcal{H} \to UCB_M(\widehat{G})$$

defined by

$$\eta\Big(\sum_{i=1}^{\infty} u_i\otimes T_i + \mathcal{H}\Big) = \sum_{i=1}^{\infty} u_i T_i, \quad \text{for} \quad \sum_{i=1}^{\infty} u_i\otimes T_i \in A(G)\otimes VN(G)$$

is well defined and $\|\eta\| \leq 1$, where

$$\mathcal{H} = \operatorname{span}\big\{ (uv)\otimes f - u\otimes(vf) : u, v \in A(G), f \in VN(G)\big\}.$$

Then it follows from Banach space theory that

$$B_{A(G)}(VN(G)) = \big(A(G)\otimes VN(G)/\mathcal{H}\big)^* \quad \text{and} \quad \eta^*\colon UCB_M(\widehat{G})^* \to B_{A(G)}(VN(G)).$$

It is easy to see that $\Phi = \eta^*$. Let I be the identity map in $B_{A(G)}(VN(G))$. If Φ is surjective, then $I = \Phi(\iota)$ for some $\iota \in UCB_M(\widehat{G})^*$. Since $UCB_M(\widehat{G})$ is a subspace of $A_M(G)^*$, we extend ι to a functional $\widetilde{\iota}$ on $A_M(G)^*$ with the same norm. Since $\widetilde{\iota} \in A_M(G)^{**}$, by the Goldstine's theorem, there is a net $\{u_\alpha\}$ in $A_M(G)$ such that $\|u_\alpha\|_{A_M(G)} \leq \|\widetilde{\iota}\|$ for all α and $u_\alpha \to \widetilde{\iota}$ in the $\sigma(A_M(G)^{**}, A_M(G)^*)$-topology. For every $a \in A_M(G)$ and $f \in A_M(G)^*$, let $a_n \in A(G)$ such that $\|a_n - a\|_M \to 0$. Since f is also in $VN(G)$, we have

$$\langle a_n, f\rangle = \langle I, a_n\otimes f\rangle = \big\langle \Phi(\iota), a_n\otimes f\big\rangle = \big\langle \eta^*(\iota), a_n\otimes f\big\rangle = \langle \widetilde{\iota}, a_n f\rangle.$$

Let $n \to \infty$ we get $\langle \tilde{\iota}, af \rangle = \langle a, f \rangle$. Hence we have

$$\langle u_\alpha a - a, f \rangle = \langle u_\alpha, af \rangle - \langle \tilde{\iota}, af \rangle \to 0.$$

Thus, $\{u_\alpha\}$ is a bounded weak approximate identity for $A_M(G)$. Hence $A_M(G)$ has a bounded approximate identity (see Bosall and Duncan [1, P. 58]). Therefore $A(G)$ is M-weakly amenable. ■

References

[1] F. F. Bonsall and J. Duncan, *Complete normed algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 80, Springer-Verlag, New York-Heidelberg, 1973.

[2] M. Cowling and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one.* Invent. Math. **96**(1989), no. 3, 507–549. http://dx.doi.org/10.1007/BF01393695

[3] J. De Cannière and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups.* Amer. J. Math. **107**(1985), no. 2, 455–500. http://dx.doi.org/10.2307/2374423

[4] C. Dunkl and D. Ramirez, *Weakly almost periodic functionals on the Fourier algebra.* Trans. Amer. Math. Soc. **185**(1973), 501–514. http://dx.doi.org/10.1090/S0002-9947-1973-0372531-2

[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact.* Bull. Soc. Math. France **92**(1964), 181–236.

[6] B. E. Forrest, *Completely bounded multipliers and ideals in $A(G)$ vanishing on closed subgroups.* In: Banach algebras and their applications, Contemp. Math., 363, American Mathematical Society, Providence, RI, 2004, pp. 89–94.

[7] ———, *Some Banach algebras without discontinuous derivations.* Proc. Amer. Math. Soc. **114**(1992), no. 4, 965–970. http://dx.doi.org/10.1090/S0002-9939-1992-1068120-4

[8] B. Forrest, V. Runde, and N. Spronk, *Operator amenability of the Fourier algebra in the cb-multiplier norm.* Canad. J. Math. **59**(2007), no. 5, 966–980. http://dx.doi.org/10.4153/CJM-2007-041-9

[9] E. E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group.* Trans. Amer. Math. Soc. **189**(1974), 371–382. http://dx.doi.org/10.1090/S0002-9947-1974-0336241-0

[10] U. Haagerup and J. Kraus, *Approximation properties for group C^*-algebras and group von Neumann algebras.* Trans. Amer. Math. Soc. **344**(1994), no. 2, 667–699.

[11] Z. Hu, *On the set of topologically invariant means on the von Neumann algebra* $\mathrm{VN}(G)$. Illinois J. Math. **39**(1995), no. 3, 463–490.

[12] A. T. Lau, *Uniformly continuous functionals on Banach algebras.* Colloq. Math. **51**(1987), 195–205.

[13] ———, *Uniformly continuous functionals on the Fourier algebra of any locally compact group.* Trans. Amer. Math. Soc. **251**(1979), 39–59. http://dx.doi.org/10.1090/S0002-9947-1979-0531968-4

[14] A. T. Lau and J. C. S. Wong, *Weakly almost periodic elements in $L_\infty(G)$ of a locally compact group.* Proc. Amer. Math. Soc. **107**(1989), no. 4, 1031–1036.

[15] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact.* C.R. Acad. Sci. Paris Sér A-B **266**(1968), A1180–A1182.

[16] V. Losert, *Properties of the Fourier algebra that are equivalent to amenability.* Proc. Amer. Math. Soc. **92**(1984), no. 3, 347–354. http://dx.doi.org/10.1090/S0002-9939-1984-0759651-8

[17] Z.-J. Ruan, *The operator amenability of $A(G)$.* Amer. J. Math. **117**(1995), no. 6, 1449–1474. http://dx.doi.org/10.2307/2375026

[18] W. Rudin, *Functional analysis.* Second Ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

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