

BOUNDARY UNIQUE CONTINUATION THEOREMS UNDER ZERO NEUMANN BOUNDARY CONDITIONS

XIANGXING TAO AND SONGYAN ZHANG

Let u be a solution to a second order elliptic equation with singular potentials belonging to the Kato–Fefferman–Phong’s class in Lipschitz domains. We prove the boundary unique continuation theorems and the doubling properties for u^2 near the boundary under the zero Neumann boundary condition.

1. INTRODUCTION

The following boundary unique continuation theorem was proved in [1]: if u is a harmonic function on a connected $C^{1,1}$ domain Ω in \mathbb{R}^n whose normal derivative vanishes everywhere on an open subset Γ of $\partial\Omega$ and whose gradient vanishes on a subset of Γ with positive surface measure, then u must be identically constant on Ω . In fact, the unique continuation problem for second order partial differential equations has been receiving increasing attention from both workers in partial differential equations and mathematical physics. In particular, this attention has been focusing on second order equations in which the coefficients of the lower-order terms are allowed to be singular, which is suggested by situations of physical interest; see for instance the extensive survey papers [3, 11].

A useful approach to the unique continuation for the elliptic equations is based on a combination of geometric and variational methods that exploits the following local doubling properties of solutions u of the elliptic equations. The original idea goes back to Garofala and Lin [6] who dealt with the inner unique continuation for the equation $\operatorname{div}(A\nabla u) = 0$, and Adolffson and Escauriaza [1, 2] who dealt with the boundary unique continuation for harmonic functions. Suppose

$$(1.1) \quad \int_{B_{2r}(x_0)} |u(x)|^2 dx \leq C \int_{B_r(x_0)} |u(x)|^2 dx$$

Received 9th February, 2005

The work of the first author is supported by National Nature Science Foundation of China (No.10471069), and Zhejiang Provincial Natural Science Foundation of China (No.102066). The second author is supported by Scientific Research Fund of Zhejiang Provincial Education Department (No.20040962) and Doctoral Foundation of Ningbo City (No.2004A610003).

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/05 \$A2.00+0.00.

for any ball $B_{2r}(x_0) \subset \mathbb{R}^n$ with $x_0 \in \overline{\Omega}$ and $0 < r < r_0$, r_0 is a positive number. If $B_{2r_0}(x_0) \subset \Omega$ or $x_0 \in \Omega$, (1.1) is the so called inner doubling property; and if $x_0 \in \partial\Omega$, (1.1) is the so called boundary doubling property.

In this paper we shall extend all the above results, under the zero Neumann boundary condition, to second order elliptic operators of form,

$$(1.2) \quad Lu(x) = -\operatorname{div}(A(x)\nabla u(x)) + \vec{b}(x) \cdot \nabla u(x) + V(x)u(x), \quad x \in \Omega,$$

in a connected Lipschitz domain Ω , where $A(x) = (a_{jk}(x))_{j,k=1}^n$ is a real symmetric matrix function satisfying the ellipticity condition and Lipschitz continuity, $\vec{b}(x)$ is a singular vector-valued function, and $V(x)$ is a real-valued potential satisfying some Kato type conditions.

It may be worthwhile to remark that for a nonnegative solution u to the equation $Lu = 0$, the doubling property (1.1) is a simple consequence of Harnack’s inequality, see [4]. However, if u has arbitrary sign the situation is drastically different, as one has to control the zeros of u . So the main thrust in (1.1) consists in the fact that no sign assumption is made on u .

There have been many results about the inner unique continuation and the boundary unique continuation under the zero Dirichlet boundary condition and the assumption $V \in K_n(\Omega)$, the Kato’s class, refer to [5, 7, 9, 10, 12]. In this paper, we shall study the singular potential V which belongs to a large class Q_t , the Kato-Fefferman-Phong’s class. We shall derive the boundary unique continuation under the zero Neumann boundary condition.

To state our results precisely, we first need to introduce Kato’s class, $K_n(\Omega)$, and Fefferman-Phong’s class, $F_t(\Omega)$.

DEFINITION 1.1: We say a measurable function $V \in L^1_{loc}(\Omega)$ belongs to Kato’s class $K_n(\Omega)$ if

$$\lim_{r \rightarrow 0} \rho^K(r; V) = 0, \quad \rho^K(r; V) = \sup_{x \in \mathbb{R}^n} \int_{B_r(x) \cap \Omega} \frac{|V(y)|}{|x - y|^{n-2}} dy,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y - x| \leq r\}$ is the ball in \mathbb{R}^n . For $1 \leq t \leq n/2$, $V \in L^1_{loc}(\Omega)$ is said to be in Fefferman-Phong’s class $F_t(\Omega)$ if

$$\|V\|_{F_t} = \sup_{x \in \mathbb{R}^n, r > 0} r^2 \left(\frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |V(y)|^t dy \right)^{1/t} < +\infty.$$

We note that $K_n \subset F_1$ and $F_{n/2} = L^{n/2} \subset F_t \subset F_s$ for $1 \leq s \leq t \leq n/2$; but $L^{n/2}(\Omega)$ and $K_n(\Omega)$ are incomparable for $n \geq 3$.

For $1 < t \leq n/2$, we define the function space $Q_t(\Omega)$ by

$$Q_t(\Omega) = \left\{ V = V_1 + V_2 : V_1 \in K_n(\Omega), V_2 \in F_t(\Omega) \right\}$$

and set

$$\eta(r; x_0; V) = \inf_{V=V_1+V_2 \in Q_t(\Omega)} \left\{ \varrho^K(r; \chi_{B_r(x_0) \cap \Omega} V_1) + \|\chi_{B_r(x_0) \cap \Omega} V_2\|_{F_t} \right\}.$$

The assumptions on A , \vec{b} and V in this paper are the following.

ASSUMPTION (A). For any $x_0 \in \bar{\Omega}$, there exists a $\lambda > 1$ such that for every $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2.$$

There exists a constant $C > 0$ and a nondecreasing function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{r \rightarrow 0} f(r) = 0$ and for every $x \in B_1(x_0) \cap \Omega$,

$$(1.3) \quad |\nabla A(x)| \leq C \frac{f(|x - x_0|)}{|x - x_0|}, \quad |A(x) - A(x_0)| \leq C f(|x - x_0|);$$

$$(1.4) \quad |\vec{b}(x)| \leq C \frac{f(|x - x_0|)}{|x - x_0|}.$$

ASSUMPTION (B). For any $x_0 \in \bar{\Omega}$ and any sufficiently small positive number ε , there exists a t with $1 < t \leq n/2$ such that

$$(i) \quad V \in Q_t(\Omega), \quad (2(x - x_0)/|x - x_0|V + |x - x_0|\nabla V) \in Q_t(\Omega), \quad \text{and}$$

$$\lim_{r \rightarrow 0} \left[\eta(r; x_0; V^-) + \eta\left(r; x_0; \left(2 \frac{x - x_0}{|x - x_0|} V + |x - x_0|\nabla V\right)\right) \right] \leq \varepsilon,$$

where V^- denotes the negative part of the function V .

$$(ii) \quad (|x - x_0|V^-)^2 \in Q_t(\Omega), \quad \text{when } \vec{b} \neq 0.$$

ASSUMPTION (C). For every $x_0 \in \bar{\Omega}$,

$$\int_0^1 \frac{f(r)}{r} dr < \infty, \quad \int_0^1 \frac{\eta(r; x_0; (2(x - x_0)/|x - x_0|V + |x - x_0|\nabla V))}{r} dr < \infty.$$

We would like to remark here that the above assumptions are weaker than those required in [1, 5, 7, 10, 12]. One can see that the potential

$$V = |x - x_0|^{-2} - f(|x - x_0|)|x - x_0|^{-2}$$

belongs to $Q_t(\Omega)$ and satisfies Assumption (B) above. But it does not belong to Kato's class and does not satisfy the assumptions in [1, 5, 7, 10, 12].

In this paper, we always denote a Carleson region for the boundary point Q , $Q \in \partial\Omega$ by $\Delta_r(Q) = B_r(Q) \cap \partial\Omega$ a surface ball, and by $T_r(Q) = B_r(Q) \cap \Omega$. Taking a boundary point $Q_0 \in \partial\Omega$, we may assume $T_3(Q_0) \subset \Omega$, and write

$$\varrho^K(r; g) = \varrho^K(r; 0; g\chi_{T_3(Q_0)}), \quad \eta_0(r; g) = \eta(r; 0; g\chi_{T_3(Q_0)})$$

for brevity.

The main results in this paper are the following doubling properties near the boundary of the Lipschitz domain.

THEOREM 1.2. *Let Ω be a Lipschitz domain, L be the operator in (1.2) satisfying Assumptions (A), (B) and (C), and let $u \in H^2_{loc}(\Omega)$ be a solution to $Lu = 0$ in Ω whose conormal derivative vanishes almost everywhere on $\Delta_3(Q_0)$ for some $Q_0 \in \partial\Omega$. Suppose that there exists a positive number r_0 and a point $x_0 \in B_1(Q_0) \cap \bar{\Omega}$ such that $A(x_0) = I$, the unit matrix, and*

$$(1.5) \quad A(Q)(Q - x_0) \cdot \vec{\nu}(Q) = 0, \quad \text{for almost everywhere } Q \in \partial\Omega \cap B_{2r_0}(x_0),$$

where $\vec{\nu}(Q)$ is the outward unit normal vector at $Q \in \partial\Omega$. Then

$$(1.6) \quad \int_{B_{2r}(x_0) \cap \Omega} |u(x)|^2 dx \leq 2^{C(r_0)} \int_{B_r(x_0) \cap \Omega} |u(x)|^2 dx$$

for all $0 < r < r_0$, where $C(r_0)$ is a constant independent of x_0 and r .

THEOREM 1.3. *Suppose the same conditions as in Theorem 1.2 hold except Assumption (C). Then there exist absolute constants C_1 and C_2 independent of $0 < r < r_0$, and $x_0 \in B_1(Q_0) \cap \bar{\Omega}$ such that*

$$(1.7) \quad \int_{B_{2r}(x_0) \cap \Omega} |u(x)|^2 dx \leq \exp\left(\frac{C_1}{r^{C_2 \varepsilon}}\right) \int_{B_r(x_0) \cap \Omega} |u(x)|^2 dx.$$

with some small positive number $\varepsilon = \varepsilon(r_0)$ satisfying $\varepsilon(r) \rightarrow 0$ if $r \rightarrow 0$.

We remark that if $B_{2r_0}(x_0) \subset\subset \Omega$, then the condition (1.5) is trivial, and the inequality (1.6) and (1.7) are the doubling properties in the interior of domain Ω , from which we can deduce the following inner unique continuation, results of Corollary 1.4 and 1.5. Also see [6, 7] where the potential

$$V(x) = |x - x_0|^{-2} f(|x - x_0|)$$

with Dini function f or

$$V(x) = V(|x - x_0|) \in K_n(\Omega)$$

have been considered.

COROLLARY 1.4. *Suppose Assumptions (A), (B) and (C) hold, and Ω is a connected domain in \mathbb{R}^n . Then the operator L has the following strong inner unique continuation property: if $u \in H^2_{loc}(\Omega)$ is a solution to $Lu = 0$ and satisfies for a point $x_0 \in \Omega$ and every $m > 0$,*

$$\int_{B_r(x_0)} |u(x)| dx = O(r^m), \quad r \rightarrow 0,$$

then $u \equiv 0$ in Ω .

COROLLARY 1.5. *Suppose Assumptions (A) and (B) hold, and Ω is a connected domain in \mathbb{R}^n . Then the operator L has the following inner unique continuation property: if $u \in H^2_{loc}(\Omega)$ is a solution to $Lu = 0$ and satisfies for a point $x_0 \in \Omega$ and two positive numbers K and ε ,*

$$\int_{B_r(x_0)} |u(x)| dx = O(\exp(-K/r^\varepsilon)), \quad r \rightarrow 0,$$

then $u \equiv 0$ in Ω .

In particular, L has the weak inner unique continuation property: if u vanishes on an open subset Ω_0 of Ω , then $u \equiv 0$ in Ω .

Another result of this paper is the following $B_2(d\sigma)$ weight property of the solution to $Lu = 0$ at a Lipschitz boundary.

THEOREM 1.6. *Let Ω be a Lipschitz domain in \mathbb{R}^n , L be the operator in (1.2) with coefficients satisfying Assumption (A), (B) and (C). If $u \in H^2_{loc}(\Omega)$ is a solution in Ω to $Lu = 0$ whose conormal derivative vanishes almost everywhere on $\Delta_3(Q_0)$, $Q_0 \in \partial\Omega$, and u vanishes on a subset S of $\Delta_3(Q_0)$ where S has positive surface measure. Assume that there exists a constant M , possibly depending on u , such that for all $Q \in \Delta_1(Q_0)$ and $0 < r < 1$ we have*

$$(1.8) \quad \int_{T_{2r}(Q)} |u(x)|^2 dx \leq M \int_{T_r(Q)} |u(x)|^2 dx.$$

Then there exists a constant C and $r_0 > 0$ depending on M , the Lipschitz character of Ω and n , such that for all $Q \in \Delta_1(Q_0)$ and $0 < r < r_0$

$$(1.9) \quad \left(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} |\nabla u|^2 d\sigma \right)^{1/2} \leq \frac{C}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} |\nabla u| d\sigma.$$

That is, $|\nabla u|$ is a $B_2(d\sigma)$ weight when restricted to $\Delta_1(Q_0)$.

From the theorems above, we shall deduce the following boundary unique continuation theorem for the solution u to $Lu = 0$ under zero Neumann boundary condition on $C^{1,1}$ domains.

THEOREM 1.7. *Let Ω be a connected $C^{1,1}$ domain in \mathbb{R}^n , L be the operator in (1.2) with coefficients satisfying Assumption (A), (B) and (C). If $u \in H^2_{loc}(\Omega)$ is a solution in Ω to $Lu = 0$ whose conormal derivative vanishes almost everywhere on an open subset Γ of the boundary $\partial\Omega$, and u vanishes on a subset of Γ which has positive surface measure, then u must be identically zero in Ω .*

REMARK 1.8. By using an approximation argument we can prove unique continuation theorems similar to those above even for H^1_{loc} -solutions.

In this paper, the letter C always denotes a positive constant which may depend on λ , n , the Q_i norm, and the Lipschitz character of Ω , but may change at different occurrences. By the notation $h = O(f)$, we mean that $|h| \leq C|f|$ for a constant C .

2. DOUBLING PROPERTIES WITH ZERO NEUMANN BOUNDARY CONDITION

The purpose of this section is to establish the doubling properties for elliptic operators with singular potential, Theorems 1.2 and 1.3. We first recall some lemmas concerning Kato’s class and Fefferman-Phong’s class which will be useful in this paper.

LEMMA 2.1. *Let Ω be a Lipschitz domain with $x_0 \in \bar{\Omega} \subset \mathbb{R}^n$, $n > 2$, and $u \in H^1_{loc}(\Omega)$. Then*

$$(2.1) \quad \int_{B_r(x_0) \cap \Omega} \frac{|u(x)|^2}{|x - x_0|^2} dx \leq C_n \left(\int_{B_r(x_0) \cap \Omega} |\nabla u|^2 dx + \frac{1}{r} \int_{\partial B_r(x_0) \cap \Omega} |u|^2 d\sigma \right)$$

for all r, x_0 and u .

PROOF: This lemma is a variation of the Heisenberg’s uncertainty principle. We can deduce it as in [7]:

$$\begin{aligned} & \int_{B_r(x_0) \cap \Omega} \frac{|u(x)|^2}{|x - x_0|^2} dx \\ &= \int_0^r \left(\int_{\partial B_\rho \cap \Omega} u^2(x_0 + \rho\omega) d\omega \right) \rho^{n-3} d\rho \\ &\leq \frac{1}{(n-2)r} \int_{\partial B_r(x_0) \cap \Omega} u^2(x) d\sigma - \frac{2}{n-2} \int_{B_r(x_0) \cap \Omega} \frac{\nabla u(x) \cdot (x - x_0)}{|x - x_0|^2} u(x) dx. \end{aligned}$$

This, by Hölder inequality, implies the inequality (2.1) and the lemma. □

LEMMA 2.2. ([5]) *Let Ω be a Lipschitz domain, $g \in K_n(\Omega)$, $u \in H^1_{loc}(\Omega)$, and let $B = B_r(x_0)$ for $x_0 \in \bar{\Omega}$ and $r > 0$. Then there exists a dimensional constant C_n independent of r, x_0 and u such that*

$$(2.2) \quad \int_{B \cap \Omega} |g| |u|^2 dx \leq C_n \varrho^K(r; g\chi_{B \cap \Omega}) \left(\int_{B \cap \Omega} |\nabla u|^2 dx + \frac{1}{r^2} \int_{B \cap \Omega} |u|^2 dx \right).$$

LEMMA 2.3. ([9]) *Let Ω be a Lipschitz domain, $u \in H^1_{loc}(\Omega)$, $g \in F_t(\Omega)$ with $1 \leq t \leq n/2$, and let $B = B_r(x_0)$ for $x_0 \in \bar{\Omega}$ and $r > 0$. Then there exists a dimensional constant $C = C(n, t)$ independent of r, x_0 and u such that*

$$(2.3) \quad \int_{B \cap \Omega} |g| |u|^2 dx \leq C \|g\chi_{B \cap \Omega}\|_{F_t} \left(\int_{B \cap \Omega} |\nabla u|^2 dx + \frac{1}{r^2} \int_{B \cap \Omega} |u|^2 dx \right).$$

The two lemmas could be proved with the same arguments as in [5, 9]. In particular, some modifications are needed for the proof of Lemma 2.2, so we shall give the line of the proof in the appendix of the paper for completeness.

We now start the proof of Theorem 1.2 and 1.3. Without loss of generality, we may assume $x_0 = 0$ is the origin and write $B_r = B_r(0)$. Thus the condition (1.5) in Theorem 1.2 or 1.3 can be rewritten as

$$(2.4) \quad A(Q)Q \cdot \vec{\nu}(Q) = 0, \quad \text{for almost everywhere } Q \in \partial\Omega \cap B_{2r_0}.$$

We introduce the function μ and vector field β defined as

$$\mu(x) = A(x)x \cdot x/|x|^2, \quad \beta(x) = A(x)x/\mu(x),$$

and from Assumption (A) we have for $|x| = r$,

$$(2.5) \quad \lambda^{-1} \leq \mu(x) \leq \lambda, \quad |\nabla\mu(x)| \leq O(f(r)/r), \quad \mu(x) = 1 + O(f(r)),$$

$$(2.6) \quad |\beta(x)| = O(r), \quad \operatorname{div}(Ax) = n + O(f(r)), \quad (\partial/\partial x_j)\beta_k = \delta_{jk} + O(f(r)),$$

where the constants depend only on λ and n . For u as in Theorem 1.2 or 1.3, and $0 < r < 2$, we consider the following functions:

$$(2.7) \quad \begin{aligned} I_1(r) &= \int_{B_r \cap \Omega} A \nabla u \cdot \nabla u \, dx, & I_2(r) &= \int_{B_r \cap \Omega} (\vec{b} \cdot \nabla u) u \, dx, \\ I_3(r) &= \int_{B_r \cap \Omega} V |u|^2 \, dx, & I(r) &= I_1(r) + I_2(r) + I_3(r), \\ H(r) &= \int_{\partial B_r \cap \Omega} \mu |u|^2 \, d\sigma, & N(r) &= \frac{r I(r)}{H(r)}. \end{aligned}$$

Since

$$H(r) = \int_{\partial B_r \cap \Omega} \mu |u|^2 \, d\sigma = \int_{B_r \cap \Omega} \operatorname{div} \left(\frac{Ax}{|x|} |u|^2 \right) \, dx,$$

differentiating $H(r)$, we can get from (2.5) and (2.6) that

$$\begin{aligned} H'(r) &= \int_{\partial B_r \cap \Omega} \operatorname{div} \left(\frac{Ax}{|x|} |u|^2 \right) \, d\sigma \\ &= \int_{\partial B_r \cap \Omega} \operatorname{div} \left(\frac{Ax}{|x|} \right) |u|^2 \, d\sigma + \int_{\partial B_r \cap \Omega} 2u \nabla u \cdot \frac{Ax}{|x|} \, d\sigma \\ &= \left(\frac{n-1}{r} + O\left(\frac{f(r)}{r}\right) \right) H(r) + 2 \int_{\partial B_r \cap \Omega} u \frac{\partial u}{\partial \vec{\nu}_A} \, d\sigma. \end{aligned}$$

We note that u is a solution to equation $-\operatorname{div}(A \nabla u) + \vec{b} \cdot \nabla u + Vu = 0$ on domain Ω , and $\frac{\partial u}{\partial \vec{\nu}_A} = 0$ on $\partial\Omega \cap B_1$. Then by the divergence theorem,

$$(2.8) \quad I(r) = \int_{B_r \cap \Omega} \operatorname{div}(u A \nabla u) \, dx = \int_{\partial B_r \cap \Omega} u \frac{\partial u}{\partial \vec{\nu}_A} \, d\sigma.$$

Thus we have

$$(2.9) \quad H'(r) = 2I(r) + \left[\frac{n-1}{r} + O\left(\frac{f(r)}{r}\right) \right] H(r).$$

LEMMA 2.4. *For every $0 < r < 1$, there exists an absolute constant C depending only on λ , n and the Q_t norms of V such that*

$$(2.10) \quad |I_2(r)| + |I_3(r)| \leq C(f(r) + \eta_0(r; V)) \left(\frac{H(r)}{r} + I_1(r) \right).$$

Further, there exists a small number $r_0 > 0$ such that

$$(2.11) \quad I_1(r) \leq \frac{H(r)}{r} + 2I(r), \quad \text{for all } 0 < r < 2r_0.$$

PROOF: Using Assumption (A) and (B), one can see that the inequality (2.10) is a simple result of Lemma 2.1, 2.2 and 2.3. Moreover,

$$I(r) \geq I_1(r) - C(f(r) + \eta_0(r; V^-)) \left(\frac{H(r)}{r} + I_1(r) \right).$$

Thus if we take a positive number r_0 small enough, we obtain the inequality (2.11) for any $0 < r < 2r_0$. □

LEMMA 2.5. For every $r \in (0, 2r_0)$, $H(r) > 0$ unless $u \equiv 0$ in $B_r \cap \Omega$.

PROOF: Assume that $H(r) = 0$ for a certain r sufficiently small. Noting (2.8) we have $I(r) = 0$. This and (2.11) imply $I_1(r) = 0$, and so we obtain $|\nabla u(x)| = 0$ for almost everywhere $x \in B_r \cap \Omega$. Thus, $H(r) = 0$ implies $u \equiv 0$ in $B_r \cap \Omega$. □

Our next task is to find the size of frequency function $N(r)$. From Lemma 2.5, one can see that the function $N(r)$ is almost everywhere, differentiable. We consider the differentiation of the function $I(r)$ and $N(r)$. Our argument is based on the following identity.

LEMMA 2.6. For every $0 < r < 1$,

$$(2.12) \quad \begin{aligned} I_1'(r) &= \int_{\partial B_r \cap \Omega} A \nabla u \cdot \nabla u d\sigma \\ &= 2 \int_{\partial B_r \cap \Omega} \frac{1}{\mu} |A \nabla u \cdot \vec{\nu}|^2 d\sigma + \left[\frac{n-2}{r} + O\left(\frac{f(r)}{r}\right) \right] \int_{B_r \cap \Omega} A \nabla u \cdot \nabla u dx \\ &\quad - \frac{2}{r} \int_{B_r \cap \Omega} \beta \cdot \nabla u \vec{b} \cdot \nabla u dx - \frac{2}{r} \int_{B_r \cap \Omega} \beta \cdot \nabla u V u dx, \end{aligned}$$

where $\vec{\nu}$ is the outward unit normal vector on ∂B_r or $\partial \Omega$.

PROOF: From a direct computation, we have the following Rellich-Necás identity

$$(2.13) \quad \begin{aligned} \operatorname{div}(\beta A \nabla u \cdot \nabla u) - 2 \operatorname{div}(\beta \cdot \nabla u A \nabla u) &= \operatorname{div}(\beta) A \nabla u \cdot \nabla u \\ &\quad + \beta_l \frac{\partial a_{jk}}{\partial x_l} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} - 2a_{jk} \frac{\partial \beta_l}{\partial x_k} \frac{\partial u}{\partial x_l} \frac{\partial u}{\partial x_j} - 2\beta_l \frac{\partial u}{\partial x_l} \frac{\partial}{\partial x_k} \left(a_{jk} \frac{\partial u}{\partial x_j} \right) \end{aligned}$$

We recall that $\beta \cdot \vec{\nu} = r$ on ∂B_r , and $\beta \cdot \nabla u A \nabla u \cdot \vec{\nu} = (r/\mu) |A \nabla u \cdot \vec{\nu}|^2$ on ∂B_r . Also we note $\beta \cdot \vec{\nu} = 0$ and $A \nabla u \cdot \vec{\nu} = 0$ almost everywhere on $B_r \cap \partial \Omega$. Therefore, integrating the Rellich-Necás identity (2.13) over $B_r \cap \Omega$, we obtain Lemma 2.6. □

LEMMA 2.7. Let $\theta_0(r) = f(r) + \eta_0(r; (2V + \beta \nabla V)^-)$ and $Z(r) = N(r) + 1$, then there exists an absolute constant C and a positive number r_0 such that

$$(2.14) \quad Z'(r) \geq -C \frac{\theta_0(r)}{r} Z(r), \quad \text{for all } 0 < r < 2r_0.$$

PROOF: We introduce the following quantities

$$F(r) = 2 \int_{\partial B_r \cap \Omega} \frac{1}{\mu} |A \nabla u \cdot \vec{\nu}|^2 d\sigma,$$

and

$$J(r; \vec{b}) = -\frac{2}{r} \int_{B_r \cap \Omega} \beta \cdot \nabla u \vec{b} \cdot \nabla u \, dx, \quad J(r; V) = -\frac{2}{r} \int_{B_r \cap \Omega} \beta \cdot \nabla u V u \, dx,$$

and therefore (2.12) can be rewritten

$$(2.15) \quad I'_1(r) = F(r) + \left[\frac{n-2}{r} + O\left(\frac{f(r)}{r}\right) \right] I_1(r) + J(r; \vec{b}) + J(r; V)$$

Using divergence theorem, we can get that

$$(2.16) \quad \begin{aligned} J(r; V) &= \frac{1}{r} \int_{B_r \cap \Omega} \operatorname{div}(\beta V) u^2 \, dx - \frac{1}{r} \int_{\partial(B_r \cap \Omega)} \beta \cdot \vec{\nu} V u^2 \, d\sigma \\ &= \frac{n-2 + O(f(r))}{r} I_3(r) + \frac{1}{r} \int_{B_r \cap \Omega} (2V + \beta \nabla V) u^2 \, dx - I'_3(r). \end{aligned}$$

It's not difficult to see that

$$|(2V + \beta \nabla V)| \leq \lambda^2 \left| ((2x/|x|)V + |x| \nabla V) \right| \in Q_t(\Omega).$$

From (2.15), (2.16) and Lemma 2.1, 2.2, 2.3 and 2.4, we obtain that

$$(2.17) \quad \begin{aligned} I'(r) &\geq I'_2(r) + F(r) + \frac{n-2}{r} I(r) + J(r; \vec{b}) \\ &\quad - C \frac{f(r) + \eta_0(r; (2V + \beta \nabla V)^-)}{r} \left(\frac{H(r)}{r} + I_1(r) \right). \end{aligned}$$

Therefore, in the case $\vec{b} = 0$, the above inequality implies

$$I'(r) \geq F(r) + \frac{n-2}{r} I(r) - C \frac{\theta_0(r)}{r} \left(\frac{H(r)}{r} + I(r) \right).$$

By this inequality and (2.9), and the quotient rule we obtain

$$(2.18) \quad \begin{aligned} Z'(r) &= \frac{I(r)H(r) + rI'(r)H(r) - rI(r)H'(r)}{H(r)^2} \\ &\geq \frac{rF(r)H(r) - 2rI(r)^2}{H(r)^2} - C \frac{\theta_0(r)}{r} Z(r) \\ &\geq -C \frac{\theta_0(r)}{r} Z(r), \end{aligned}$$

with an absolute constant $C > 0$ independent of $r \in (0, 2r_0)$, where we have used the fact $F(r)H(r) - 2I(r)^2 \geq 0$ by Hölder's inequality. Thus we have deduced the inequality (2.14) in the case $\vec{b} = 0$.

For the case $\vec{b} \neq 0$, some more careful estimates are needed. First by the assumptions for \vec{b} we have

$$(2.19) \quad \begin{aligned} |J(r; \vec{b})| &\leq \frac{C}{r} \int_{B_r, \Omega} |x| |\vec{b}| |\nabla u|^2 dx \leq C \frac{f(r)}{r} \int_{B_r, \Omega} |\nabla u|^2 dx \\ &\leq C \frac{f(r)}{r} I_1(r) \end{aligned}$$

where the positive constant C independent of r .

An analogous estimate as in (2.16) and Lemma 2.2, 2.3 and 2.4 give

$$(2.20) \quad \begin{aligned} -I'_3(r) &\leq \int_{\partial B_r, \Omega} V^- u^2 dx \\ &= \frac{n-2 + O(f(r))}{r} \int_{B_r, \Omega} V^- u^2 dx \\ &\quad + \frac{1}{r} \int_{B_r, \Omega} (2V^- + \beta \nabla V^-) u^2 dx + \frac{2}{r} \int_{B_r, \Omega} (\beta \nabla u) V^- u dx \\ &\leq \frac{C}{r} \left(I(r) + \frac{H(r)}{r} \right), \end{aligned}$$

where we have used the Hölder inequality and the assumption $(|x - x_0| V^-)^2 \in Q_t(\Omega)$ for the integral $\int_{B_r, \Omega} (\beta \nabla u) V^- u dx$.

Now from the inequalities (2.15), (2.16), (2.19) and (2.20), we have

$$(2.21) \quad \begin{aligned} I'_1(r) &\leq -I'_3(r) + F(r) + \frac{n-2}{r} (I_1(r) + I_3(r)) \\ &\quad + C \frac{f(r) + \eta_0(r; 2V + \beta \nabla V)}{r} \left(\frac{H(r)}{r} + I_1(r) \right) \\ &\leq F(r) + \frac{C}{r} \left(I(r) + \frac{H(r)}{r} \right), \end{aligned}$$

and moreover,

$$(2.22) \quad \begin{aligned} |I'_2(r)| &\leq \int_{\partial B_r, \Omega} |\vec{b} \nabla u| |u| d\sigma \leq C \sqrt{\int_{\partial B_r, \Omega} |\nabla u|^2 d\sigma} \sqrt{\int_{\partial B_r, \Omega} |\vec{b}|^2 u^2 d\sigma} \\ &\leq \frac{C f(r)}{r} \sqrt{I'_1(r) H(r)} \\ &\leq \frac{C f(r)}{r} \left(\sqrt{F(r) H(r)} + I(r) + \frac{H(r)}{r} \right). \end{aligned}$$

Combining (2.17), (2.19), (2.22) and Lemma 2.4, we have

$$(2.23) \quad I'(r) \geq F(r) + \frac{n-2}{r} I(r) - C \frac{\theta_0(r)}{r} \left(I(r) + \frac{H(r)}{r} \right) - \frac{C f(r)}{r} \sqrt{F(r) H(r)}$$

where $\theta_0(r) = f(r) + \eta_0(r; (2V + \beta \nabla V)^-)$.

By (2.23), (2.9) and the quotient rule we obtain

$$(2.24) \quad \begin{aligned} Z'(r) &= \frac{I(r)H(r) + rI'(r)H(r) - rI(r)H'(r)}{H(r)^2} \\ &\geq \frac{rF(r)H(r) - 2rI(r)^2}{H(r)^2} - \frac{Cf(r)}{H(r)}\sqrt{F(r)H(r)} - C\frac{\theta_0(r)}{r}Z(r) \end{aligned}$$

with an absolute constant $C > 0$ independent of $r \in (0, r_0)$.

Since $F(r)H(r) - 2I(r)^2 \geq 0$, so if $F(r)H(r) \leq 4I(r)^2$, then

$$\frac{Cf(r)}{H(r)}\sqrt{F(r)H(r)} \leq \frac{Cf(r)}{r}Z(r),$$

and thus the desired differential inequality (2.14) holds. Otherwise, we assume $F(r)H(r) \geq 4I(r)^2$. We note that

$$\frac{Cf(r)}{H(r)}\sqrt{F(r)H(r)} \leq \frac{Cf(r)}{H(r)}\left(rF(r) + \frac{H(r)}{r}\right),$$

and so the inequality (2.24) implies

$$\begin{aligned} Z'(r) &\geq \frac{rF(r)H(r) - 2rI(r)^2}{H(r)^2} - \frac{Cf(r)rF(r)}{H(r)} - C\frac{\theta_0(r)}{r}Z(r) \\ &\geq \frac{(1 - Cf(r))rF(r)H(r) - 2rI(r)^2}{H(r)^2} - C\frac{\theta_0(r)}{r}Z(r) \\ &\geq -C\frac{\theta_0(r)}{r}Z(r) \end{aligned}$$

for sufficiently small r satisfying $Cf(r) < 1/2$, which yields the inequality (2.14). □

LEMMA 2.8. *Let L be an operator as in (1.2) satisfying Assumption (A) and (B), and let u be a solution to $Lu = 0$ in Ω whose conormal derivative vanishes on $\Delta_3(Q_0)$. With the notation above, if the condition (1.5) in Theorem 1.2 or 1.3 holds for $x_0 = 0$ and small positive number r_0 , and if $A(0) = I$, then there exists an absolute constant $C > 0$ such that*

$$Z(r) \exp\left\{-C \int_r^{2r_0} \frac{\theta_0(t)}{t} dt\right\}$$

is nondecreasing in $r \in (0, 2r_0)$. Moreover,

- (1) If $\int_0^1 (\theta_0(r)/r) dr < +\infty$, then $N(r) \leq C(r_0)$ for all $r \in (0, 2r_0)$,
- (2) In general, for every $r \in (0, 2r_0)$, $N(r) \leq (C_1(r_0))/(r^{C_2(r_0)\varepsilon(r_0)})$ where $C(r_0)$, $C_1(r_0)$ and $C_2(r_0)$ are bounded constants independent of r , and $\varepsilon(r_0) = \max_{0 < r < 2r_0} \theta_0(r)$.

PROOF: Recalling the inequality (2.14) above, we have

$$(2.25) \quad \frac{d}{dr} \log Z(r) \geq -C\frac{\theta_0(r)}{r}, \quad \text{for all } 0 < r < 2r_0,$$

which shows that

$$Z(r) \exp \left\{ -C \int_r^{2r_0} (\theta_0(t)/t) dt \right\}$$

is nondecreasing. Further, we integrate (2.25) between r and r_0 to get

$$\frac{Z(r)}{Z(2r_0)} \leq \exp \left\{ C \int_r^{2r_0} \frac{\theta_0(t)}{t} dt \right\},$$

which yields the assertion. □

This lemma and (2.9) imply Theorem 1.2 and 1.3 by a standard argument. For the details see [2, 5, 6].

3. \mathcal{B}_2 WEIGHT PROPERTY ON THE BOUNDARY

Before proving the \mathcal{B}_2 weight property on the boundary, Theorem 1.6, we need to prove several lemmas. Using Lemma 2.1, we can first deduce the following Cozziopoli inequality ([8]).

LEMMA 3.1. *Let Ω be a Lipschitz domain with $Q_0 \in \partial\Omega$ and L be an operator as in (1.2) satisfying Assumption (A) and (B). Suppose $u \in H^1_{loc}(\Omega)$ is a solution to $Lu = 0$ whose conormal derivative vanishes almost everywhere on $\Delta_3(Q_0)$. Then there exist constants C and $0 < r_0 < 1$ such that for all $0 < r < r_0$ and $x_0 \in B_1(Q_0) \cap \bar{\Omega}$,*

$$(3.1) \quad \int_{B_r(x_0) \cap \Omega} |\nabla u|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0) \cap \Omega} |u|^2 dx.$$

PROOF: Take $0 < r < 1$, and let $\phi \in C^\infty_0(\mathbb{R}^n)$ be a real function, $\phi \equiv 1$ on $B_r(x_0)$, $\text{supp} \phi \subset B_{2r}(x_0)$, $|\nabla \phi| \leq C/r$. Since $\frac{\partial u}{\partial \bar{\nu}_A} \equiv 0$ on $B_{2r}(x_0) \cap \partial\Omega$, $u\phi^2 \in H^1(B_{2r}(x_0) \cap \Omega)$. Thus

$$\int_{B_{2r}(x_0) \cap \Omega} [A \nabla u \cdot \nabla (u\phi^2) + (\bar{b} \nabla u) u \phi^2 + V u u \phi^2] dx = 0.$$

By the assumptions and Hölder’s inequality, one can see from Lemma 2.1, 2.2 and 2.3 that

$$(3.2) \quad \begin{aligned} & \int_{B_{2r}(x_0) \cap \Omega} |\nabla (u\phi)|^2 dx \\ & \leq C_{\lambda,n} \int_{B_{2r}(x_0) \cap \Omega} |u|^2 |\nabla \phi|^2 dx + C \int_{B_{2r}(x_0) \cap \Omega} |\bar{b} u \phi|^2 dx + C \int_{B_{2r}(x_0) \cap \Omega} V^- |u\phi|^2 dx \\ & \leq \frac{C}{r^2} \int_{B_{2r}(x_0) \cap \Omega} |u|^2 dx + C \{f(r) + \eta_0(2r; V^-)\} \int_{B_{2r}(x_0) \cap \Omega} |\nabla (u\phi)|^2 dx \end{aligned}$$

Taking $0 < r_0 < 1$ such that $C \{f(r) + \eta_0(2r; V^-)\} \leq 1/2$ for all $r \in (0, r_0)$, then from (3.2) we can get (3.1). The lemma is proved. □

LEMMA 3.2. *Let Ω be a Lipschitz domain in \mathbb{R}^n with $Q_0 \in \partial\Omega$, and let u be a nonconstant solution in $T_3(Q_0)$ to $Lu = 0$ whose conormal derivative vanishes almost everywhere on $\Delta_3(Q_0)$, where L is the operator as in (1.2) with its coefficients satisfying Assumption (A), (B) and (C). If the doubling property (1.8) holds, then there exist constants C and $0 < r_0 < 1$ such that for any $Q \in \Delta_1(Q_0)$ and all $0 < r < 2r_0$,*

$$(3.3) \quad \left\{ \int_{\Delta_r(Q)} |\nabla u|^2 d\sigma \right\}^{1/2} \leq Cr^{-(n+3)/2} \int_{T_r(Q)} |u(x)| dx.$$

PROOF: Without loss of generality we may assume $Q = 0$ and $A(0) = I$, and that Ω is the set of points $x = (x', x_n)$ in the unit cylindrical body of \mathbb{R}^n such that $x_n > \varphi(x')$, where φ is a Lipschitz function in \mathbb{R}^{n-1} verifying $\varphi(0) = 0$ and

$$|\nabla\varphi(x') - \nabla\varphi(0)| \leq \varrho(|x'|)$$

for all $x' \in \mathbb{R}^{n-1}$, where ϱ is a Dini function. From the mean value theorem we get

$$(3.4) \quad x' \nabla\varphi(x') - \varphi(x') \geq -2|x'| \varrho(|x'|) \geq -\frac{1}{2}r, \quad \text{for all } |x'| \leq 2r,$$

with some small positive number r_0 and $0 < r < r_0$.

We take a nonnegative function $\phi \in C_0^\infty(B_{2r}(0))$ such that $\phi \equiv 1$ in $B_r(0)$ and $|\nabla\phi| \leq C/r$ for some positive constant C . Let $x_0 = (0, r)$ and $\gamma(x) = (x - x_0/r)\phi^3(x)$, a vector field supported in $B_{2r}(0)$, then one can see $|\nabla\gamma(x)| \leq C_1/r$ in $T_{2r} = B_{2r}(0) \cap \Omega$. So we can see from (3.4) that $\gamma \cdot \vec{\nu} \geq C_2$ on $\Delta_r = B_r(0) \cap \partial\Omega$ for some positive constants C_1 and C_2 depending on the Lipschitz character of Ω , and that $\gamma \cdot \vec{\nu} \geq 0$ on Δ_{2r} . Recalling the Rellich-Necas identity (2.13) and integrating over T_{2r} gives

$$(3.5) \quad \begin{aligned} & \int_{\Delta_{2r}} \gamma \cdot \vec{\nu} A \nabla u \cdot \nabla u d\sigma \\ &= \frac{O(1)}{r} \int_{T_{2r}} |\nabla u|^2 dx - 2 \int_{T_{2r}} \gamma \cdot \nabla u [\vec{b} \cdot \nabla u + Vu] dx \\ &= \frac{O(1)}{r} \int_{T_{2r}} |\nabla u|^2 dx - 2 \int_{T_{2r}} \gamma \cdot \nabla u Vu dx \\ &= \frac{O(1)}{r} \int_{T_{2r}} |\nabla u|^2 dx - \int_{\Delta_{2r}} \gamma \cdot \vec{\nu} Vu^2 d\sigma + \int_{T_{2r}} \operatorname{div}(\gamma V) |u|^2 dx. \end{aligned}$$

where the last equality follows from the divergence theorem. Moreover, since $V \in Q_t$ and

$$|2V + (x - x_0)\nabla V| \leq |2(x - x_0)/|x - x_0|V + |x - x_0|\nabla V| \in Q_t,$$

we can get from Lemma 2.1, Lemma 2.3, and the Caccioppoli inequality (3.1) in Lemma 3.1 that

$$\begin{aligned}
 (3.6) \quad & \int_{T_{2r}} \operatorname{div}(\gamma V) |u|^2 dx \\
 &= \int_{T_{2r}} \left(\frac{n}{r} \phi^3 V + 3 \frac{x-x_0}{r} \phi^2 \nabla \phi V + \frac{x-x_0}{r} \phi^3 \nabla V \right) |u|^2 dx \\
 &= \frac{O(1)}{r} \int_{T_{2r}} |V| |\phi u|^2 dx - \frac{1}{r} \int_{T_{2r}} (2V + (x-x_0) \nabla V) \phi |\phi u|^2 dx \\
 &\leq \frac{C}{r} \int_{T_{2r}} |V| |\phi u|^2 dx + \frac{1}{r} \int_{B_{3r}(x_0) \cap \Omega} \left| 2 \frac{x-x_0}{|x-x_0|} V + |x-x_0| \nabla V \right| |\phi u|^2 dx \\
 &\leq \frac{C}{r} \left(\int_{B_{3r}(x_0) \cap \Omega} |\nabla u|^2 dx + \frac{1}{r^2} \int_{B_{3r}(x_0) \cap \Omega} |u|^2 dx \right) \\
 &\leq \frac{C}{r^3} \int_{T_{3r}} |u|^2 dx.
 \end{aligned}$$

Using the uncertainty principle, Lemma 2.2 and 2.3 on Δ_{2r} , we have

$$(3.7) \quad - \int_{\Delta_{2r}} \gamma \cdot \bar{\nu} V u^2 d\sigma \leq C\eta(r; V^-) \int_{\Delta_{2r}} |\nabla u|^2 d\sigma.$$

Now combining inequalities (1.8), (3.5), (3.6) and (3.7), we get that

$$(3.8) \quad C_2 \lambda^{-1} \int_{\Delta_r} |\nabla u|^2 d\sigma \leq \frac{C}{r^3} \int_{T_{r/2}} |u|^2 dx,$$

where the constants C_2 and C are independent of $r < 2r_0$, r_0 is a sufficiently small positive number. Now using the maximum principle, we can obtain the lemma from (3.8). \square

LEMMA 3.3. *Let Ω be a Lipschitz domain in \mathbb{R}^n , L be the operator in (1.2) whose coefficients satisfy Assumption (A), (B) and (C). If $Q \in \partial\Omega$, $0 < r < 1$, and u is a solution to $Lu = 0$ on $T_{2r}(Q)$ vanishing continuously on a subset S of $\Delta_r(Q)$ and S has positive surface measure, and the conormal derivative of the solution u vanishes almost everywhere on $T_{2r}(Q)$. Then for each $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that the following holds:*

$$\int_{T_r(Q)} |u| dx \leq C(\varepsilon) r^2 \int_{\Delta_{2r}(Q)} |\nabla u| d\sigma + \varepsilon \int_{T_{2r}(Q)} |u| dx.$$

PROOF: After a translation and dilation, we may assume $Q = 0$ and $r = 1$. Following from the similar idea of [1, Lemma 3.1] we let \mathcal{F} denote the set of Lipschitz mappings φ on \mathbb{R}^{n-1} verifying $\varphi(0) = 0$ and $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})} \leq m$ for $m > 0$, and \mathcal{L} denote the set of all the operators L as in (1.2) whose coefficient matrix A satisfies $A(0) = I$ and $\|\nabla A\|_{L^\infty(\mathbb{R}^{n-1})} \leq m$. For each $\varphi \in \mathcal{F}$ we denote $\Omega(\varphi) = \{(x', x_n); x_n > \varphi(x')\}$.

If Lemma 3.3 were false, we could find $\varepsilon > 0$, a sequence $\{\varphi_k\}$ in \mathcal{F} , $\{L_k\}$ in \mathcal{L} with

$$L_k w = \operatorname{div}(A_k \nabla w) + \vec{b}_k \cdot \nabla w + V_k w,$$

and a sequence $\{u_k\}$ of functions verifying that $L_k u_k = 0$ on $B_2 \cap \Omega(\varphi_k)$, $\frac{\partial u_k}{\partial \vec{\nu}_A} = 0$ on $B_2 \cap \partial\Omega(\varphi_k)$, $u_k = 0$ on $S_k \subset B_1 \cap \partial\Omega(\varphi_k)$,

$$(3.9) \quad \int_{B_1 \cap \Omega(\varphi_k)} |u_k| \, dx = 1,$$

and

$$(3.10) \quad \varepsilon \int_{B_2 \cap \Omega(\varphi_k)} |u_k| \, dx + k \int_{B_2 \cap \partial\Omega(\varphi_k)} |\nabla u_k| \, d\sigma \leq 1.$$

For each k we let w_k and $f_k = (f_k^1, \dots, f_k^n)$ denote the zero extensions to the whole ball B_2 outside of domain $\Omega(\varphi_k)$ of u_k and ∇u_k , respectively. Since all the above sequences are compact in the proper topologies, we can find subsequences that we can assume are the whole sequences such that $\varphi_k \rightarrow \varphi \in \mathcal{F}$ and $A_k \rightarrow A$ uniformly over compact sets, $\vec{b}_k \rightarrow \vec{b}$ and $V_k \rightarrow V$ weakly in $L^2(B_{3/2})$, and $w_k \rightarrow w$, $f_k \rightarrow f = (f^1, \dots, f^n)$ weakly in $L^2(B_{3/2})$ and uniformly over compact sets contained in $B_{3/2} \setminus \partial\Omega(\varphi)$. From the divergence theorem, the Poincaré inequality on Δ_2 and (3.10), we find that there is a constant C such that for all $\psi \in C_0^\infty(B_{3/2})$ and $1 \leq j \leq n$

$$(3.11) \quad \begin{aligned} & \left| \int \left(w_k \frac{\partial \psi}{\partial x_j} + f_k^j \psi \right) dx \right| + \left| \int (A_k f_k \nabla \psi + \vec{b}_k \cdot f_k \psi + V_k w_k \psi) dx \right| \\ &= \left| \int_{B_{3/2} \cap \partial\Omega(\varphi_k)} u_k \psi \nu_j \, d\sigma \right| \\ &\leq C \|\psi\|_{L^\infty} \int_{B_{3/2} \cap \partial\Omega(\varphi_k)} \left| u_k - \frac{1}{|S_k|} \int_{S_k} u_k \, d\sigma \right| d\sigma \\ &\leq \frac{C}{k} \|\psi\|_{L^\infty}. \end{aligned}$$

Taking limits in (3.9) and (3.11), we find that the limit w satisfies the following: $w \in W^{1,2}(B_{3/2})$, w is the solution to $Lw = 0$ on $B_{3/2}$,

$$(3.12) \quad \int_{B_1 \cap \Omega(\varphi)} |w| \, dx = 1,$$

and w vanishes on $B_1 \setminus \Omega(\varphi)$. But an operator $L \in \mathcal{L}$ has the interior unique continuation property; thus $w = 0$ on $B_{3/2}$. This contradicts (3.12) and proves the lemma. \square

PROOF OF THEOREM 1.6: Using doubling property (1.8) and choosing the ε in Lemma 3.3 sufficiently small, we can deduce that

$$(3.13) \quad \int_{T_{2r}(Q)} |u(x)| \, dx \leq Cr^2 \int_{\Delta_r(Q)} |\nabla u| \, d\sigma.$$

for all $0 < r < r_0$, where $r_0 > 0$ and the constant C depends only on r_0 . (3.13) and Lemma 3.2 yield the $\mathcal{B}_2(d\sigma)$ property (1.9) for $|\nabla u|$. \square

4. BOUNDARY UNIQUE CONTINUATION FOR $C^{1,1}$ DOMAIN

Our ultimate aim is to establish the unique continuation at the boundary, Theorem 1.7. We first have the following lemma by the same argument as in [1].

LEMMA 4.1. *Let Ω be a $C^{1,1}$ domain, L be the operator in (1.2) satisfying Assumptions (A), (B) and (C), and let $u \in H^2_{loc}(\Omega)$ be a solution to $Lu = 0$ in Ω whose conormal derivative vanishes almost everywhere on $\Delta_3(Q_0)$ for some $Q_0 \in \partial\Omega$. Then there exists a positive number r_0 depending on the $C^{1,1}$ character of Ω and n , and a constant M depending on n , the $C^{1,1}$ character of Ω and u , such that the doubling property (1.8) holds for all $Q \in \Delta_1(Q_0)$ and $0 < r < r_0$.*

PROOF: The proof of theorem 0.8 in [1] can be used here with some obvious modifications. After a translation we may let $Q = 0 \in \Delta_3(Q_0)$, we can construct a proper $C^{1,1}$ diffeomorphism $\Psi : B_{r_1}(0) \rightarrow B_{r_0}(0)$, where r_1 and r_0 are two small positive numbers. Defining $\tilde{u}(x) = u \circ \Psi(x)$ and $\tilde{\Omega} = \Psi^{-1}(\Omega)$, we have that \tilde{u} is a solution to

$$\tilde{L}\tilde{u} = -\operatorname{div}(\tilde{A}\nabla\tilde{u}) + \tilde{b} \cdot \nabla\tilde{u} + \tilde{V}\tilde{u} = 0,$$

on $B_{r_1}(Q_0) \cap \tilde{\Omega}$, where $\tilde{\Omega} = \{(x', x_n); x_n > 0\}$ and

$$\begin{aligned} \tilde{A}(x) &= \det J\Psi(x)J\Psi^{-t}(x)A \circ \Psi(x)J\Psi^{-1}(x) \\ \tilde{b}(x) &= \det J\Psi(x)J\Psi^{-1}(x)\tilde{b} \circ \Psi(x) \\ \tilde{V}(x) &= \det J\Psi(x)V \circ \Psi(x). \end{aligned}$$

Moreover, the operator \tilde{L} , \tilde{u} and $\tilde{\Omega}$ satisfy all the assumptions in Theorem 1.2, thus the doubling property (1.6) with $x_0 = 0$, and then (1.8), holds for \tilde{u} and as a consequence for u , which implies the lemma. □

PROOF OF THEOREM 1.7: Without loss of generality we may assume that $\Gamma = \Delta_3(Q_0)$ and

$$S = \{Q \in \Gamma : u(Q) = 0\} \subset \Delta_1(Q_0)$$

for some $Q_0 \in \partial\Omega$. We suppose $u \in H^2_{loc}(\Omega)$ is a solution to $Lu = 0$ as in Theorem 1.7 and that $Q \in \Delta_1(Q_0)$ denotes a density point of the set $E = \{Q \in \Delta_1(Q_0) : \nabla u(Q) = 0\}$.

By Lemma 4.1 and Theorem 1.6, we obtain from Hölder's inequality that

$$\begin{aligned} \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} |\nabla u| d\sigma &\leq \left(\frac{\sigma(\Delta_r(Q) \setminus E)}{\sigma(\Delta_r(Q))} \right)^{1/2} \left(\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} |\nabla u|^2 d\sigma \right)^{1/2} \\ &\leq \left(\frac{\sigma(\Delta_r(Q) \setminus E)}{\sigma(\Delta_r(Q))} \right)^{1/2} \frac{C}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} |\nabla u| d\sigma, \end{aligned}$$

for all $0 < r < r_0$. This implies that there is a positive constant C such that

$$(4.1) \quad \frac{\sigma(\Delta_r(Q) \setminus E)}{\sigma(\Delta_r(Q))} \geq C^{-2}$$

for any small $\tau > 0$. But one can find that if $\sigma(E) > 0$ the inequality (4.1) is impossible by taking $\tau \rightarrow 0$. On the other hand, since $\frac{\partial u}{\partial \vec{v}_A} = 0$, $\sigma(S) > 0$ implies $\sigma(E) > 0$. Therefore, we have $\sigma(S) = 0$ if $u \neq 0$ in Ω , and then complete the proof of Theorem 1.7. \square

5. APPENDIX

LEMMA 5.1. *Let Ω be a Lipschitz domain, $B_r = B_r(x_0)$, and assume*

$$V \in K_n(\Omega) \cap L^\infty(\Omega \cap B_r),$$

and $V > 0$. Then the following Neumann problem

$$(5.1) \quad \begin{cases} \Delta \psi = V, & \text{in } \Omega \cap B_r, \\ \frac{\partial \psi}{\partial \vec{v}} = 0, & \text{on } \partial \Omega \cap B_r, \\ \frac{\partial \psi}{\partial \vec{v}} = \frac{1}{|\Omega \cap \partial B_r|} \int_{\Omega \cap B_r} V(y) dy, & \text{on } \Omega \cap \partial B_r, \end{cases}$$

is solvable, where \vec{v} denotes the unit outer normal vector. Moreover, the solution ψ to the above Neumann problem satisfies the following estimate

$$(5.2) \quad \|\psi\|_{L^\infty(\Omega \cap B_r)} \leq C_n \varrho^K(r; V \chi_{B_r \cap \Omega})$$

with a dimensional constant C_n .

PROOF: For $g \in C(\partial(B_r \cap \Omega))$ and $P, Q \in \partial(B_r \cap \Omega)$, we define

$$K^*g(P) = p.v. \frac{1}{\omega_n} \int_{\partial(B_r \cap \Omega)} \frac{(P - Q) \cdot \vec{v}(P)}{|P - Q|^n} g(Q) dQ,$$

where ω_n denotes the measure of the unit sphere ∂B_1 in \mathbb{R}^n . By known results (see [13]), the operator $S = (-I + K^*)/2$ maps $L^2_0(\partial(B_r \cap \Omega))$ into itself. Let

$$g_x(Q) = \begin{cases} \frac{(x - Q) \cdot \vec{v}(Q)}{\omega_n |x - Q|^n}, & Q \in \partial \Omega \cap B_r, \\ \frac{(x - Q) \cdot \vec{v}(Q)}{\omega_n |x - Q|^n} + \frac{1}{|\Omega \cap \partial B_r|}, & Q \in \Omega \cap \partial B_r, \end{cases}$$

then $g_x(Q) \in L^2_0(\partial(B_r \cap \Omega))$. We can now define the Neumann function,

$$(5.3) \quad N(x, y) = \frac{1}{\omega_n(2 - n)} \frac{1}{|x - y|^{n-2}} - \frac{1}{\omega_n(2 - n)} \int_{\partial(B_r \cap \Omega)} \frac{S(g_x)(Q)}{|y - Q|^{n-2}} dQ,$$

and one can see from [13] that the solution of the Neumann problem (5.1) is given by

$$(5.4) \quad \psi(x) = \int_{\Omega \cap B_r} N(x, y) V(y) dy.$$

Substituting (5.3) into (5.4), one can see that because of the assumption $V \in K_n$ the terms corresponding to the addend in (5.3) have L^∞ norm $\Omega \cap B_r$ that satisfies the bound (5.2) by the same arguments as that in [5]. \square

LEMMA 5.2. ([5]) *Let Ω be a Lipschitz domain, $V \in K_n(\Omega)$, and $u \in H^1_{loc}(\Omega)$. Then there exists a dimensional constant C_n , independent of $B = B_r(x_0)$ and u , such that*

$$(5.5) \quad \int_{B \cap \Omega} |V| |u|^2 dx \leq C_n \varrho^K(r; V \chi_{B \cap \Omega}) \left(\int_{B \cap \Omega} |\nabla u|^2 dx + \frac{1}{r} \int_{\partial B \cap \Omega} |u|^2 d\sigma \right).$$

PROOF: Denote $\varrho^K(r; V \chi_{B \cap \Omega})$ by $\varrho(r)$, and set

$$h(r) = \frac{1}{|\Omega \cap \partial B_r|} \int_{\Omega \cap B_r} |V(y)| dy,$$

we can see that $h(r) \leq C(\varrho(r)/r)$. From the divergence theorem and Lemma 5.1, we have

$$(5.6) \quad \begin{aligned} \int_{B \cap \Omega} |V| |u|^2 dx &= \int_{B \cap \Omega} \Delta \psi u^2 dx = h(r) \int_{\partial B \cap \Omega} u^2 d\sigma - 2 \int_{B \cap \Omega} u \nabla \psi \cdot \nabla u dx \\ &\leq h(r) \int_{\partial B \cap \Omega} u^2 d\sigma + 2 \left(\int_{B \cap \Omega} u^2 |\nabla \psi|^2 dx \right)^{1/2} \left(\int_{B \cap \Omega} |\nabla u|^2 dx \right)^{1/2}. \end{aligned}$$

Using the divergence theorem and Cauchy inequality, we can deduce that

$$(5.7) \quad \begin{aligned} \int_{B \cap \Omega} u^2 |\nabla \psi|^2 dx &= \frac{1}{2} \int_{B \cap \Omega} u^2 \Delta(\psi^2) dx - \int_{B \cap \Omega} |V| u^2 \psi dx \\ &\leq \frac{1}{2} \int_{\partial B \cap \Omega} u^2 \frac{\partial(\psi^2)}{\partial \nu} d\sigma \\ &\quad - \frac{1}{2} \int_{B \cap \Omega} \nabla(u^2) \cdot \nabla(\psi^2) dx + \|\psi\|_{L^\infty} \int_{B \cap \Omega} |V| u^2 dx \\ &\leq \frac{C \varrho^2(r)}{r} \int_{\partial B \cap \Omega} u^2 d\sigma + \frac{1}{2} \int_{B \cap \Omega} u^2 |\nabla \psi|^2 dx \\ &\quad + 2 \int_{B \cap \Omega} |\nabla \psi|^2 |\nabla u|^2 dx + C \varrho(r) \int_{B \cap \Omega} |V| u^2 dx. \end{aligned}$$

Inequality (5.7) yields

$$(5.8) \quad \begin{aligned} \int_{B \cap \Omega} u^2 |\nabla \psi|^2 dx &\leq C \varrho^2(r) \left[\frac{1}{r} \int_{\partial B \cap \Omega} u^2 d\sigma + \int_{B \cap \Omega} |\nabla u|^2 dx \right] \\ &\quad + C \varrho(r) \int_{B \cap \Omega} |V| u^2 dx \end{aligned}$$

Now we replace (5.8) in (5.6), we can deduce (5.5) and obtain the lemma. \square

Finally, we can prove Lemma 2.2 by Lemma 5.2 and the following inequality

$$(5.9) \quad \int_{\partial B_r(x_0) \cap \Omega} |u|^2 d\sigma \leq \frac{C_n}{r} \int_{B_r(x_0) \cap \Omega} |u|^2 dx + C_n r \int_{B_r(x_0) \cap \Omega} |\nabla u|^2 dx,$$

which can be deduced as (2.1) of Lemma 2.1.

REFERENCES

- [1] V. Adolfsson and L. Escauriaza, ' $C^{1,\alpha}$ domains and unique continuation at the boundary', *Comm. Pure Appl. Math.* **50** (1997), 935–969.
- [2] V. Adolfsson, L. Escauriaza and C. Kenig, 'Convex domains and unique continuation at the boundary', *Rev. Mat. Iberoamericana* **11** (1995), 513–525.
- [3] M. Aizenman and B. Simon, 'Brownian motion and Harnack's inequality for Schrödinger operators', *Comm. pure Appl. Math.* **35** (1982), 209–273.
- [4] F. Chiarenza, E. B. Fabes and N. Garofalo, 'Harnack's inequality for Schrödinger operators and the continuity of solutions', *Proc. Amer. Soc.* **98** (1986), 415–425.
- [5] E. Fabes, N. Garofalo and F-H. Lin, 'A partial answer to a conjecture of B. Simon concerning unique continuation', *J. Funct. Anal.* **88** (1990), 194–210.
- [6] N. Garofalo and F-H. Lin, 'Monotonicity properties of variational integrals, A_p weights and unique continuation', *Indiana Univ. Math. J.* **35** (1986), 245–268.
- [7] N. Garofalo and F-H. Lin, 'Unique continuation for elliptic operators: a geometric-variational approach', *Comm. Pure Appl. Math.* **40** (1987), 347–366.
- [8] C.E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics **83** (American Mathematical Society, Providence R.I., 1994).
- [9] K. Kurata, 'A unique continuation theorem for uniformly elliptic equations with strongly singular potentials', *Comm. Partial Differential Equations* **18** (1993), 1161–1189.
- [10] K. Kurata, 'A unique continuation theorem for the Schrödinger equation with singular magnetic field', *Proc. Amer. Soc. Math.* **125** (1997), 853–860.
- [11] B. Simon, 'Schrödinger semigroups', *Bull. Amer. Math. Soc.* **7** (1982), 447–521.
- [12] X.X. Tao, 'Doubling properties and unique continuation at the boundary for elliptic operators with singular magnetic fields', *Studia Math.* **151** (2002), 31–48.
- [13] G. Verchota, 'Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains', *J. Funct. Anal.* **59** (1984), 572–611.

Department of Mathematics
Faculty of Science
Ningbo University
Ningbo, 315211
Peoples Republic of China
e-mail: taoxiangxing@nbu.edu.cn