COMMUTATION PROBLEMS INVOLVING RINGS OF INFINITE MATRICES

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1. Introduction

Let R be a ring and let J be the set of all integers. In the set $M(R)$ of all mappings $A: J \times J \to R$, let addition and multiplication be defined by

$$A+B = C, \text{ where } c_{ij} = a_{ij}+b_{ij}, \quad \text{.........................(1)}$$

$$AB = D, \text{ where } d_{ij} = \sum_{k \in J} a_{ik}b_{kj}. \quad \text{.......................(2)}$$

Here a_{ij} denotes the image of (i, j) under A and b_{ij}, c_{ij}, d_{ij} are similarly defined for the mappings B, C, D. In (2) we require A, B to be such that the sum $\sum_{k \in J} a_{ik}b_{kj}$ is defined and is in R. Thus, in general, $M(R)$ is not closed with respect to multiplication.

When addition and multiplication are defined in this way, it is natural to call the elements of $M(R)$ *infinite matrices* over R. The (i, j)th element, the ith row and the jth column of such a matrix are defined in the usual way. We say that an infinite matrix A is *row-finite* if for each $i \in J$ there exists a finite subset $N(i)$ of J, consisting of $n(i)$ elements of J, such that

$$a_{ik} = 0 \text{ whenever } k \notin N(i).$$

Thus all but a finite number of the elements of each row are zero. If $n(i)$ can be chosen to be independent of i (that is, if the set $\{n(i): i \in J\}$ is bounded above) then we say that A is *uniformly row-finite*. If the set $N(i)$ can be chosen to be independent of i, then we say that A is *row-bounded*. Clearly a row-bounded matrix is uniformly row-finite, but a uniformly row-finite matrix need not be row-bounded.

The set of all row-finite matrices over R is a ring $M_\rho(R)$ with respect to addition and multiplication defined by (1) and (2). The set of all uniformly row-finite matrices is a subring $M'_\rho(R)$ of $M_\rho(R)$ and the set of row-bounded matrices over R is a subring $M^*_\rho(R)$ of $M'_\rho(R)$. In fact, $M^*_\rho(R)$ is a two-sided ideal of both $M_\rho(R)$ and $M'_\rho(R)$.

Column-finite, uniformly column-finite and column-bounded matrices can be defined by analogy with the above.

Suppose that Φ is a mapping which associates with each ring R some uniquely determined subring $\Phi(R)$ of R; suppose also that F is a mapping which associates with each ring R some other ring $F(R)$ such that if S is a subring of R, then $F(S)$ is a subring of $F(R)$. By the commutation problem for Φ and F we shall mean the following: in what circumstances do $\Phi(F(R))$ and $F(\Phi(R))$ coincide?

In the present paper, we consider this problem when F is one of the mappings M_ρ, M'_ρ and M^*_ρ (where, for example, M_ρ means the mapping $R \rightarrow M_\rho(R)$). A case of some interest occurs when Φ is the mapping Γ given by $R \rightarrow \Gamma(R)$, where $\Gamma(R)$ is the Jacobson-Perlis radical (**1**) of R. For Γ, M^*_ρ and for Γ, M_ρ solutions to the commutation problem have been given in two previous papers (**2**), (**3**), in which the following results were proved.

(i) *The Jacobson-Perlis radical of* $M^*_\rho(R)$ *satisfies*

$$\Gamma(M^*_\rho(R)) = M^*_\rho(\Gamma(R))$$

for all rings R.

(ii) *The Jacobson-Perlis radical of* $M_\rho(R)$ *satisfies*

$$\Gamma(M_\rho(R)) = M_\rho(\Gamma(R))$$

if and only if $\Gamma(R)$ *is right-vanishing in the sense of Levitzki.*

In Section **2** we give a solution to the commutation problem for Γ and M'_ρ by proving the following theorem.

Theorem 1. *The Jacobson-Perlis radical of* $M'_\rho(R)$ *satisfies*

$$\Gamma(M'_\rho(R)) = M'_\rho(\Gamma(R))$$

if and only if Γ (R) *is nilpotent.*

Thus we have a stronger condition for Γ, M'_ρ than for either Γ, M_ρ or Γ, M^*_ρ, despite the fact that M'_ρ is "between" M^*_ρ and M_ρ in the sense that

$$M^*_\rho(R) \subset M'_\rho(R) \subset M_\rho(R).$$

The remainder of the paper is devoted to the case in which Φ is the mapping $R \rightarrow R^\alpha$, where R^α is the αth power of R ($\alpha \in J^+$, the set of positive integers) and F is one of the mappings M_ρ, M'_ρ, M^*_ρ. Again we obtain complete solutions to the commutation problems.

2. The radical of the ring of uniformly row-finite matrices

To prove Theorem 1, we require the following result.

Lemma. *Let R be a ring in which, given any sequence*

$$\sigma = \{x_i \colon i \in J^+,\ x_i \in R\},$$

there is an integer p (depending upon σ) such that every product

$$x_i x_{i+1} x_{i+2} \cdots x_{i+p-1} \quad (i \in J^+)$$

of p consecutive members of the sequence is zero. Then R is nilpotent.

Proof. If R is a non-nilpotent ring, then, given any positive integer q, there exist elements $y_{q1}, y_{q2}, \ldots, y_{qq}$ such that

$$y_{q1} y_{q2} \cdots y_{qq} \neq 0.$$

Consider the sequence $y_{11}, y_{21}, y_{22}, y_{31}, y_{32}, y_{33}, \ldots$. Given any positive integer q, there is a set of q consecutive elements of this sequence whose product is not zero. This proves the lemma.

Proof of Theorem 1. Suppose first that $\Gamma(R)$ is nilpotent. Then $M'_\rho(\Gamma(R))$ is a nilpotent ideal of $M'_\rho(R)$ (the index of nilpotency being the same as that of $\Gamma(R)$). Hence

$$M'_\rho(\Gamma(R)) \subset \Gamma(M'_\rho(R)).$$

By a standard argument (see (**2**), Theorem 1), we can show that

$$\Gamma(M'_\rho(R)) \subset M'_\rho(\Gamma(R))$$

for all rings R. Hence

$$\Gamma(M'_\rho(R)) = M'_\rho(\Gamma(R)).$$

Suppose conversely that this condition is satisfied. Let

$$\{x_i : i \in J^+\}$$

be a sequence of elements in $\Gamma(R)$ and let A be the matrix for which

$$a_{ii+1} = x_i (i \in J^+), \quad a_{ij} = 0 (i \notin J^+ \text{ or } j \neq i+1).$$

Then $A \in M'_\rho(\Gamma(R))$ and so, by hypothesis, $A \in \Gamma(M'_\rho(R))$. Therefore A is quasi-regular. Let B be the quasi-inverse of A. We have $BA = A+B$ and so the elements of B satisfy

$$0 = b_{ij+1} \quad (j \notin J^+), \quad \text{......(3)}$$

$$b_{ij}x_j = b_{ij+1} \quad (j \neq i,\ i, j \in J^+), \quad \text{......(4)}$$

$$b_{ii}x_i = x_i + b_{ii+1} \quad (i \in J^+). \quad \text{......(5)}$$

Suppose that $i \in J^+$. By (3) and (4), we have

$$b_{ij} = 0 \quad (j \leqq i).$$

Therefore, by (5),

$$b_{ii+1} = -x_i$$

and repeated application of (4) then gives

$$b_{ij} = -x_i x_{i+1} \dots x_{j-1}, \quad (j \geqq i+1). \quad \text{......(6)}$$

But B is uniformly row-finite and so there exists a positive integer n, independent of i, such that at least one of the elements $b_{ij} (j = i+1, i+2, \dots, i+n+1)$ is zero. It follows from (6) that, if $b_{ij} = 0$ and $j \geqq i+1$, then $b_{ij+1} = 0$. Therefore

$$b_{ii+n+1} = 0$$

and hence

$$x_i x_{i+1} \dots x_{i+n} = 0.$$

The integer n does not depend on i, but does depend on the matrix A or, equivalently, depends on the sequence $\{x_i\}$. Thus, given any sequence $\sigma = \{x_i : i \in J^+\}$ in $\Gamma(R)$, there exists an integer $p = p(\sigma) = n+1$, such that the product of any p consecutive members of σ is zero. Hence, by the lemma, $\Gamma(R)$ is nilpotent.

3. Commutation problems involving the powers of a ring

Let $R^\alpha (\alpha \geqq 2)$ be the αth power of the ring R: that is, the set of all elements of the form

$$\Sigma x_1 x_2 \dots x_\alpha \quad (x_1, x_2, \dots, x_\alpha \in R)$$

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in which the summation is over a finite number of terms. Suppose that Φ is the mapping $R \to R^\alpha$. Then it is natural to expect Φ to commute with the mappings M_ρ, M'_ρ and M^*_ρ. We shall show, however, that this is true without restriction on R only in the case of M_ρ. We deal first with this case, proving a result which is due to Dr C. St J. A. Nash-Williams, to whom I am indebted for permission to include it here.

Theorem 2. *For any ring R, we have*

$$\{M_\rho(R)\}^\alpha = M_\rho(R^\alpha).$$

Proof. Any element of $\{M_\rho(R)\}^\alpha$ can be expressed as a finite sum of products of the form $A_1A_2\ldots A_\alpha$, where the A's are row-finite matrices over R. Since each element of $A_1A_2\ldots A_\alpha$ is in R^α, we have

$$\{M_\rho(R)\}^\alpha \subset M_\rho(R^\alpha).$$

Suppose now that $A \in M_\rho(R^\alpha)$. We shall construct a set of row-finite matrices $A_1, A_2, \ldots, A_\alpha$ over R such that $A = A_1A_2\ldots A_\alpha$. These matrices are such that A_1 has at most one non-zero element in each column, $A_2, \ldots, A_{\alpha-1}$ are diagonal matrices (these do not occur in the case $\alpha = 2$) and A_α has at most one non-zero element in each row.

Consider the elements in row 1 of A. Each of these is a finite sum of the form $\Sigma x_1x_2\ldots x_\alpha$. The total number of products $x_1x_2\ldots x_\alpha$ involved is finite, since A is row-finite. Suppose that there are λ such products. Then we can set up a one-one correspondence between them and the set consisting of the integers $(1, 2, \ldots, \lambda)$. Suppose that $x_1x_2\ldots x_\alpha$ is a product arising from the element a_{1j} of A and that μ is the integer corresponding to $x_1x_2\ldots x_\alpha$. Then we define the element in position $(1, \mu)$ of A_1 to be x_1, those in position (μ, μ) of $A_2, \ldots, A_{\alpha-1}$ to be $x_2, \ldots, x_{\alpha-1}$ respectively, and that in position (μ, j) of A_α to be x_α. We do this for each μ in the set $1, 2, \ldots, \lambda$.

Next we consider the elements of row 2 in a similar manner. If in this case there are λ' non-zero products $x_1x_2\ldots x_\alpha$ involved, then we set up a one-one correspondence between them and the set of integers $\lambda+1, \lambda+2, \ldots, \lambda+\lambda'$. We then carry out a similar process to that described above, starting with the elements in row 2 of A_1.

Continuing in this way, we deal with each row of A in turn. At each stage we ensure that the non-zero products $x_1x_2\ldots x_\alpha$ involved in the particular row of A under consideration correspond to a finite set of integers which are not involved in the correspondences for the other rows. All the remaining elements of the matrices $A_1, \ldots, A_\alpha$ are defined to be zero.

Then we have $A = A_1\ldots A_\alpha$ and so

$$M_\rho(R^\alpha) \subset \{M_\rho(R)\}^\alpha.$$

Thus Theorem 2 is proved.

Analogous results do not hold for M'_ρ, M^*_ρ; the above construction for the matrix A_1 does not necessarily give a matrix which is in $M'_\rho(R)$ or $M^*_\rho(R)$ even when A is in $M'_\rho(R^\alpha)$ or $M^*_\rho(R^\alpha)$.

Let R^α_q denote the set of all members of R^α which are sums of at most q

terms of the form $x_1x_2\dots x_\alpha$, where $x_1, x_2, \dots, x_\alpha \in R$. Thus

$$R_q^\alpha = \left\{ \sum_{i=1}^{q} x_1^{(i)}x_2^{(i)}\dots x_\alpha^{(i)}\colon x_1^{(i)}, \dots, x_\alpha^{(i)} \in R \right\}.$$

Clearly $R^\alpha = \bigcup_{q \in J^+} R_q^\alpha$ and $R_q^\alpha \subset R_{q+1}^\alpha (q \in J^+)$.

Suppose that there exists an integer q such that

$$R_q^\alpha = R_{q+1}^\alpha. \quad \dots\dots(7)$$

Then it is easily verified that $R_q^\alpha = R^\alpha$. In this case, we shall write $\lambda(\alpha)$ for the least positive value of q for which (7) is satisfied. If there is no integer q such that (7) is satisfied, we shall write $\lambda(\alpha) = \infty$.

Theorem 3. *For any ring R, we have*

$$\{M_\rho'(R)\}^\alpha = \bigcup_{q \in J^+} M_\rho'(R_q^\alpha).$$

Proof. Suppose that $A_1, A_2, \dots, A_\alpha \in M_\rho'(R)$. Let n_β be the maximum number of non-zero elements in a row of $A_\beta (\beta = 1, 2, \dots, \alpha)$. Then the (i, j)th element of $A_1A_2\dots A_\alpha$ is a sum of at most $n = n_1n_2\dots n_\alpha$ products of the form $x_1x_2\dots x_\alpha$, where $x_1, x_2, \dots, x_\alpha \in R$. Hence $A_1A_2\dots A_\alpha \in M_\rho'(R_n^\alpha)$. It follows that any finite sum of the form $\Sigma X_1X_2\dots X_\alpha$, where $X_1, X_2, \dots, X_\alpha \in M_\rho'(R)$, belongs to $\bigcup_{q \in J^+} M_\rho'(R_q^\alpha)$. Therefore

$$\{M_\rho'(R)\}^\alpha \subset \bigcup_{q \in J^+} M_\rho'(R_q^\alpha).$$

Now suppose that $A \in M_\rho'(R_q^\alpha)$ for some $q \in J^+$. Then there exist $B_1, B_2, \dots, B_q \in M_\rho'(R_1^\alpha)$ such that $A = B_1 + B_2 + \dots + B_q$.

Given any matrix C such that $C \in M_\rho'(R_1^\alpha)$, we can write

$$C = C_1 + C_2 + \dots + C_n,$$

where each of the matrices $C_1, C_2, \dots, C_n$ has at most one non-zero element in each row, of the form $x_1x_2\dots x_\alpha$.

Let D be a matrix of this type. Thus, given $i \in J$, there exists $j \in J$ for which

$$d_{ij} = x_{i1}x_{i2}\dots x_{i\alpha}, \quad d_{ik} = 0 (k \neq j).$$

Define $D_\beta (\beta = 1, 2, \dots, \alpha - 1)$ to be the diagonal matrix whose (i, i)th element is $x_{i\beta}$ and whose remaining elements are zero; define D_α to be the matrix whose (i, j)th element is $x_{i\alpha}$ and whose remaining elements are zero. Then

$$D = D_1D_2\dots D_\alpha \in \{M_\rho'(R)\}^\alpha.$$

Hence $C \in \{M_\rho'(R)\}^\alpha$ and so each matrix $B_1, B_2, \dots, B_q$ belongs to $\{M_\rho'(R)\}^\alpha$. Therefore $A \in \{M_\rho'(R)\}^\alpha$ and thus

$$\bigcup_{q \in J^+} M_\rho'(R_q^\alpha) \subset \{M_\rho'(R)\}^\alpha.$$

This proves Theorem 3.

We can now give a necessary and sufficient condition for the mappings $R \to R^\alpha$ and M'_ρ to commute.

Theorem 4. *The* α*th power of the ring* $M'_\rho(R)$ *satisfies*

$$\{M'_\rho(R)\}^\alpha = M'_\rho(R^\alpha)$$

if and only if $\lambda(\alpha) < \infty$.

Proof. If $\lambda(\alpha) = \infty$ there exists a matrix $A \in M'_\rho(R^\alpha)$ such that $A \notin M'_\rho(R^\alpha_q)$ for any q. For example, choose A to be a diagonal matrix whose (i, i)th element (for $i \geqq 1$) is in R^α_i but not in R^α_j for $1 \leqq j \leqq i-1$. Hence

$$M'_\rho(R^\alpha) \neq \bigcup_{q \in J^+} M'_\rho(R^\alpha_q)$$

and therefore, by Theorem 3,

$$M'_\rho(R^\alpha) \neq \{M'_\rho(R)\}^\alpha.$$

If $\lambda(\alpha) < \infty$, then

$$\bigcup_{q \in J^+} M'_\rho(R^\alpha_q) = M'_\rho(R^\alpha)$$

since $R^\alpha_q = R^\alpha$ for $q = \lambda(\alpha)$. Hence, by Theorem 3, we have

$$\{M'_\rho(R)\}^\alpha = M'_\rho(R^\alpha).$$

By using similar arguments we can prove analogous results for the commutativity of the mappings $R \to R^\alpha$ and M^*_ρ. In particular, we have the following analogue of Theorem 4.

Theorem 5. *The* α*th power of the ring* $M^*_\rho(R)$ *satisfies*

$$\{M^*_\rho(R)\}^\alpha = M^*_\rho(R^\alpha)$$

if and only if $\lambda(\alpha) < \infty$.

In many cases, it is clear that the integer $\lambda(\alpha)$ defined above is finite for all α. For example, if R has an identity with respect to multiplication, then $\lambda(\alpha) = 1$ for all α. Moreover, the argument used in the proof of Theorem 2 shows that $\lambda(\alpha) = 1$ for the ring $M_\rho(R)$, where R is any ring.

On the other hand, if R is a free ring (without unity) on an infinite number of symbols, then it is not difficult to verify that $\lambda(\alpha) = \infty$ for all values of $\alpha > 1$. If β is an integer > 1, then R/R^β is a nilpotent ring for which $\lambda(\alpha) = \infty$ when $1 < \alpha < \beta$, and $\lambda(\alpha) = 1$ when $\alpha \geqq \beta$.

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