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Bump Functions with Hölder Derivatives

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Abstract. We study the range of the gradients of a $C^{1,\alpha}$-smooth bump function defined on a Banach space. We find that this set must satisfy two geometrical conditions: It can not be too flat and it satisfies a strong compactness condition with respect to an appropriate distance. These notions are defined precisely below. With these results we illustrate the differences with the case of C^1-smooth bump functions. Finally, we give a sufficient condition on a subset of X^* so that it is the set of the gradients of a $C^{1,1}$-smooth bump function. In particular, if X is an infinite dimensional Banach space with a $C^{1,1}$-smooth bump function, then any convex open bounded subset of X^* containing 0 is the set of the gradients of a $C^{1,1}$-smooth bump function.

1 Introduction

A function from a Banach space X to $\mathbb{R}$ with a bounded nonempty support is called a bump. For $\alpha \in\ ]0,1]$ we say that a map with bounded support $g\colon X \to X^*$ is Hölder(α) if there exists $K > 0$ with $\|g(y) - g(x)\| \leq K\|y - x\|^\alpha$ for all $(x, y) \in X^2$. We then denote by $\omega_\alpha(g)$ the smallest constant K satisfying the above inequality. A Hölder(1) map g is just a Lipschitzian map and then $\omega_1(g)$ is Lip(g). We say that a function $f\colon X \to \mathbb{R}$ is $C^{1,\alpha}$-smooth if f is C^1-smooth and f' is Hölder(α). We define

$$\mathcal{S}_\alpha = \{f'(X)\ ;\ f\colon X \to \mathbb{R} \text{ is a } C^{1,\alpha}\text{-smooth bump}\},$$

$$\mathcal{S} = \{f'(X)\ ;\ f\colon X \to \mathbb{R} \text{ is a } C^{1}\text{-smooth bump}\}.$$

We notice that these sets can be empty. In fact, there exists $\alpha \in\ ]0,1]$ such that $\mathcal{S}_\alpha$ is not empty if and only if X is superreflexive. This follows from Lemma IV.5.3 and Theorem V.3.2 of [6]. The set $\mathcal{S}$ has been studied by many authors during the last years. A set F in $\mathcal{S}$ is a connected subset of X^*, compact if X is finite dimensional and analytic if X is infinite dimensional. Moreover, it can be proved, with Ekeland's variational principle ([6], Theorem I.2.4), that the norm closure of F is a neighbourhood of 0. It was shown by D. Azagra and R. Deville in [1] that $X^* \in \mathcal{S}$, provided X is an infinite dimensional Banach space which admits a C^1-smooth and Lipschitzian bump. Sufficient conditions on bounded closed subsets of X^* to be in $\mathcal{S}$ were obtained by J. M. Borwein, M. Fabian and P. D. Loewen in [4] when X is infinite dimensional, and by the same authors and I. Kortezov in [3] in the finite dimensional case. These results have been improved by D. Azagra, M. Fabian and M. Jimenez-Sevilla in [2]. The results of [2] will be detailed in the following of this paper. If X is infinite dimensional, there exist analytic subsets of X^*, neither closed nor open, which belong to $\mathcal{S}$. This was first done for convex sets by T. Gaspari in [8] and then with more general

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conditions by M. Fabian, O. Kalenda and J. Kolář in [7]. We recall the extension of Darboux's theorem by J. Malý in [10]: If $f\colon X \to \mathbb{R}$ is Fréchet differentiable, then $f'(X)$ is connected. This property does not remain true if the function is not real valued, as remarked in Problem 8.5.4 in [5] (see also [10] and [12]). We now introduce some notations.

$B(x,r)$ denotes the closed ball of center x and radius r, $S(x,r)$ is the sphere of center x and radius r. We sometimes write B_X instead of $B(0,1)$. The convex hull of a set M will be denoted by $\mathrm{co}(M)$. We recall that a function $f\colon X \to \mathbb{R}$ is said to be Fréchet differentiable at $x_0 \in X$ if there exists $f'(x_0)$ in X^* such that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)(h)}{\|h\|} = 0.$$

Then $f'(x_0)$ is called the derivative, or the gradient, of f at x_0. The set $f'(X) = \{f'(x) \,;\, x \in X\}$ is the range of the derivative of f. If f is a function from X to $\mathbb{R}$, the support of f is $\mathrm{Supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$. As said before, f is called a bump if its support is nonempty and bounded.

The symbol $\mathbb{N}$ means the set $\{1,2,\dots\}$. We denote by $\mathbb{N}^{<\mathbb{N}}$ the set of finite sequences of natural numbers and $\mathbb{N}^{\mathbb{N}}$ the set of infinite sequences of natural numbers. If $s = (s_1,\dots,s_k) \in \mathbb{N}^{<\mathbb{N}}$, k is called the length of s and we write $k = |s|$. If $k \geq 2$ we put $s_- = (s_1,\dots,s_{k-1})$. If $j \in \{1,\dots,k\}$, $s|j = (s_1,\dots,s_j)$. If $r = (r_1,\dots,r_m) \in \mathbb{N}^{<\mathbb{N}}$, then $s\hat{\ }r = (s_1,\dots,s_k,r_1,\dots,r_m)$. If $\sigma = (\sigma_j)_{j\geq 1} \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$, then $\sigma|j = (\sigma_1,\dots,\sigma_j)$.

Section 2 is devoted to the study of $\mathcal{S}$. Sufficient conditions on closed subsets of X^* so that they belong to $\mathcal{S}$ have been obtained by D. Azagra, M. Fabian and M. Jimenez-Sevilla in [2]. We observe in our two results of Section 2 that these conditions are almost optimal.

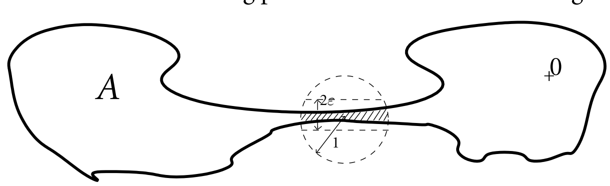
In Section 3 we study the sets $\mathcal{S}_\alpha$, $\alpha \in\,]0,1]$ and their differences with $\mathcal{S}$. We find two necessary conditions on a subset A of X^* containing 0 so that it belongs to $\mathcal{S}_\alpha$. The first one deals with the α-flatness of A.

Definition 1.1

(i) $\mathcal{F}_\alpha(A) = \sup\big\{ l^{1+\alpha}\varepsilon^{-\alpha} \,;\, (l,\varepsilon) \in \mathbb{R}^{+*2}$ such that there exist $z^* \in A$ and $e \in S_X$
with $B(z^*,l) \cap \{y^* \in A \,;\, |\langle y^* - z^*, e\rangle| = \varepsilon\} = \varnothing$
and $0 \notin B(z^*,l) \cap \{y^* \in A \,;\, |\langle y^* - z^*, e\rangle| \leq \varepsilon\}\big\}$.

(ii) $\mathcal{F}_\alpha(A)$ is called the α-*flatness* of A.

This definition is not translation-invariant, hence we should have called it the α-flatness of the set A according to the point 0. But for clarity we will simply write the α-flatness of the set A. The following picture illustrates the meaning of the flatness.



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We prove that if $A \in \mathcal{S}_\alpha$ then the α-flatness of A is finite (Theorem 3.1).

For the second condition we need the following definitions. If $\gamma\colon [0,1] \to X^*$ is continuous we define, for $\alpha \in \,]0,1]$, the α-length of γ by

$$l^{(\alpha)}(\gamma) = \sup\Big\{ \Big(\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|^{\frac{1}{\alpha}}\Big)^{\alpha} \,;\, n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1 \Big\}.$$

When $\alpha = 1$, $l^{(1)}(\gamma)$ is the usual length of the arc $\gamma([0,1])$ and will be written $l(\gamma)$. Now, for x and y in A we define

$$d_A^{(\alpha)}(x,y) = \inf\Big\{ l^{(\alpha)}(\gamma) \,;\, \gamma\colon [0,1] \to A \text{ is continuous and } (\gamma(0),\gamma(1)) = (x,y) \Big\}.$$

Clearly, for all $\alpha \in \,]0,1]$, $d_A^{(\alpha)}$ is a distance on A and we have, for all $0 < \beta \leq \alpha \leq 1$, $d_A^{(\beta)}(x,y) \leq d_A^{(\alpha)}(x,y)$. For $n \geq 1$ we define the index

$$M_n^{(\alpha)}(A) = \sup_{(y_1,\dots,y_n) \in A^n} \Big\{ \inf\{ d_A^{(\alpha)}(y_i,y_j) \,;\, 1 \leq i < j \leq n \} \Big\}$$

which measures the degree of precompactness of A for the distance $d_A^{(\alpha)}$. In particular the condition $M_n^{(\alpha)}(A) \to 0$ means that A equipped with the metric $d_A^{(\alpha)}$ is precompact. If $\alpha = 1$ we will write $d_A(x,y) = d_A^{(1)}(x,y)$ and $M_n(A) = M_n^{(1)}(A)$. In Theorem 3.2 we obtain that if $A \in \mathcal{S}_\alpha$ and X is finite dimensional, then $M_n^{(\alpha)}(A) = O(n^{-\alpha/d})$ with $d = \dim X$. When X is infinite dimensional we obtain that $(M_n^{(\alpha)}(A))_n$ is bounded. Finally, with the results of Section 2 and Section 3 we construct subsets of X^* which are in $\mathcal{S}$ but not in $\mathcal{S}_\alpha$.

In the last section we find a sufficient condition to be in $\mathcal{S}_1$ when X is an infinite dimensional separable Banach space (Theorem 4.1). In particular we show that if there exists a $C^{1,1}$-smooth bump on X, then any convex open bounded subset of X^* containing 0 belongs to $\mathcal{S}_1$.

2 The Set $\mathcal{S}$

First, if A is a subset of X^* we define, for x and y in A,

$$p_A(x,y) = \inf\Big\{ \operatorname{diam}(\gamma([0,1])) \,;\, \gamma\colon [0,1] \to A \text{ is continuous and } (\gamma(0),\gamma(1)) = (x,y) \Big\}.$$

Then p_A is a distance on A and if $0 < \alpha \leq 1$, $p_A(x,y) \leq d_A^{(\alpha)}(x,y)$. For $n \geq 1$ we denote

$$R_n(A) = \sup_{(y_1,\dots,y_n) \in A^n} \Big\{ \inf\{ p_A(y_i,y_j) \,;\, 1 \leq i < j \leq n \} \Big\}.$$

The condition $R_n(A) \to 0$ means that A equipped with the metric p_A is precompact. We notice that for all $n \in \mathbb{N}$ and $0 < \alpha \leq 1$, $R_n(A) \leq M_n^{(\alpha)}(A)$.

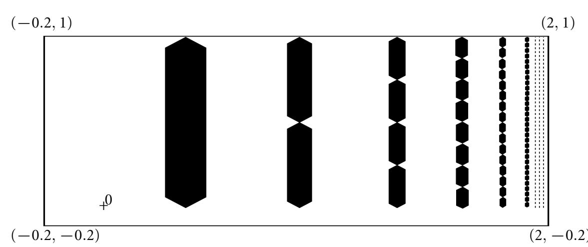
Theorem 2.1 *Let X be a finite dimensional Banach space and U be a connected open subset of X^* containing* 0*. We consider the following assertions:*

(i) $\lim_{n\to+\infty} R_n(U) = 0$.
(ii) $\overline{U} \in \mathcal{S}$.
(iii) $\lim_{n\to+\infty} R_n(\overline{U}) = 0$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof **Step 1:** $\lim_{n\to+\infty} R_n(U) = 0 \Rightarrow \overline{U} \in \mathcal{S}$. We fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $R_N(U) < \varepsilon$. Let $(y_i^*)_{i\in I}$ be a maximal set in U with the property that for all $i, j \in I$ with $i \neq j$, $p_U(y_i^*, y_j^*) \geq \varepsilon$. By the choice of N, $\operatorname{Card}(I) \leq N$. Then, by maximality, $(y_i^*)_{i\in I}$ is a finite ε-net in U for the distance p_U. For $i \in I$ we define $V_i = \{z^* \in U \ ; \ p_U(z^*, y_i^*) < \varepsilon\}$. Then $(V_i)_{i\in I}$ is a finite family of open connected subsets of U, covering U, each one with diameter less than ε. According to Theorem 2.4 of [2] this implies the existence of $b \colon X \to \mathbb{R}$ a C^1-smooth bump such that $b'(X) = \overline{U}$.
Step 2: $\overline{U} \in \mathcal{S} \Rightarrow \lim_{n\to+\infty} R_n(\overline{U}) = 0$. Let $b \colon X \to \mathbb{R}$ be a C^1-smooth bump with $b'(X) = \overline{U}$. We fix $\varepsilon > 0$. Since X is finite dimensional, b' is uniformly continuous on $\operatorname{Supp}(b)$ and hence we find $\delta > 0$ such that $\|b'(x) - b'(y)\| < \varepsilon$ if $\|x - y\| < \delta$. We take a finite δ-net in $\operatorname{Supp}(b)$ for the norm. Then its range by b' is a finite ε-net in $\overline{U}$ for the metric $p_{\overline{U}}$. We call N its cardinal; then $R_{N+1}(\overline{U}) < 2\varepsilon$. Since $(R_n(\overline{U}))_n$ is decreasing, this proves that $\lim_{n\to+\infty} R_n(\overline{U}) = 0$. ∎

The conditions are not equivalent since there exists an open subset A of $\mathbb{R}^2$ satisfying (iii) but not (i). Here is a representation of such a set:



A is the open rectangle without the black pieces. Clearly $\lim_{n\to+\infty} R_n(\overline{A}) = 0$ whereas $R_n(A) \geq 1$ for all $n \geq 1$. We do not know if A is in $\mathcal{S}$. In infinite dimensions we have:

Theorem 2.2 *Let X be a infinite dimensional Banach space with a separable dual and U be a connected open subset of X^* containing* 0*. Let us consider the following assertions:*

(i) *For all $y^* \in \overline{U}$, there exists a continuous path from* 0 *to y^* through points of U.*
(ii) $\overline{U} \in \mathcal{S}$.
(iii) *For all $y^* \in \overline{U}$, there exists a continuous path from* 0 *to y^* through points of $\overline{U}$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

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Proof The difficult implication (i) $\Rightarrow$ (ii) has been proved in [2, Theorem 2.3]. Let us prove that (ii) $\Rightarrow$ (iii). Let $y^* \in \overline{U} = b'(X)$. There exist x_0 and x in X such that $b'(x_0) = 0$ and $b'(x) = y^*$. Then the path γ defined by $\gamma(t) = b'(tx + (1-t)x_0)$, $t \in [0,1]$, is a continuous path from 0 to y^* through points of $b'(X) = \overline{U}$. ■

The previous example shows that (iii) and (i) are not equivalent. Indeed the point $(2,1)$ can be joined to 0 by a continuous path in the closure of A, but there is no continuous path from 0 to it through points of A.

We remark that, if $X = \mathbb{R}^d$ with $d \in \mathbb{N}$, the positive results deal only with subsets of $\mathbb{R}^d$ which are the closure of their interior. In fact, it is an open question if this condition is necessary. This question was written in [3] and was partially answered in [8] where it was proved that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^2-smooth bump, then $f'(\mathbb{R}^2)$ is the closure of its interior. J. Kolár and J. Kristensen have recently proved the same result with weaker assumptions on the regularity of $f\colon \mathbb{R}^2 \to \mathbb{R}$ (see [9]). On the other hand, L. Rifford has shown in [11] that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is a C^{d+1}-smooth bump, then $f'(\mathbb{R}^d)$ is the closure of its interior.

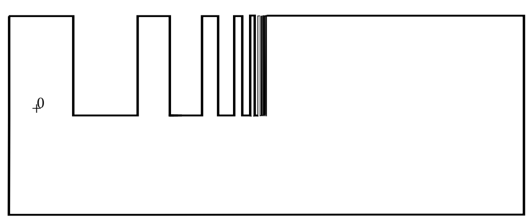
Now we give an example from [8] which illustrates the differences between the finite and the infinite dimensional cases. Let H be a separable Hilbert space and define

$$P_1 = (\,]-1,2[\, \times\,]-1,0[\,) \cup (\,]1,2[\, \times\,]-1,1[\,),$$

$$P_2 = \Big(\bigcup_{q\geq 1} \,]2^{-1} + \cdots + 2^{-q} - 8^{-q}, 2^{-1} + \cdots + 2^{-q} + 8^{-q}[\,\Big) \times [0,1[\quad \text{(comb's teeth)}$$

and

$$P = \big((-\tfrac{3}{2}, 0) + (P_1 \cup P_2)\big) \times \operatorname{int} B_H.$$



The comb in $\mathbb{R}^2$.

Then the comb P is an open subset of $X = \mathbb{R}^2 \times H$. If $\dim H = +\infty$ then $\overline{P} \in \mathcal{S}$ since P satisfies the condition (i) in Theorem 2.2. But if H is finite dimensional, $\overline{P} \notin \mathcal{S}$. Indeed $R_n(\overline{P}) \geq 1$ for all n and so P does not satisfy (iii) of Theorem 2.1.

3 Necessary Conditions to Be in $\mathcal{S}_\alpha$

Theorem 3.1 *Let X be a Banach space, A be a subset of X^* and $\alpha \in\,]0,1]$. If $A \in \mathcal{S}_\alpha$, then $\mathcal{F}_\alpha(A) < +\infty$.*

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Proof Let f be a $C^{1,\alpha}$-smooth bump such that $f'(X) = A$ and $\mathrm{Supp}(f) \subset B_X$. We are going to prove that

$$\mathcal{F}_\alpha(A) \leq 3\Big(\frac{4}{\alpha}\Big)^{\alpha}(1+\alpha)^{1+\alpha}\omega_\alpha(f').$$

We fix $f'(x_0) \in A$, $e_1 \in S_X$, $l > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} &B(f'(x_0), l) \cap \{y^* \in A\,;\, |\langle y^* - f'(x_0), e_1\rangle| = \varepsilon\} = \varnothing \\ \text{and}\quad &0 \notin B(f'(x_0), l) \cap \{y^* \in A\,;\, |\langle y^* - f'(x_0), e_1\rangle| \leq \varepsilon\}. \end{aligned} \tag{3.1}$$

We will show that

$$\omega_\alpha(f') \geq 3^{-1}\Big(\frac{\alpha}{4}\Big)^{\alpha}(1+\alpha)^{-(1+\alpha)} l^{1+\alpha}\varepsilon^{-\alpha}. \tag{3.2}$$

In the following we write $\omega = \omega_\alpha(f')$. We take $\delta \in (0, 1)$ and we define

$$C = B(f'(x_0), \delta l) \cap \{y^* \in X^*\,;\, |\langle y^* - f'(x_0), e_1\rangle| \leq \varepsilon\} \text{ and } D = f'^{-1}(C).$$

Since $f'(x_0) \in C$, x_0 belongs to D. Now $0 \notin C$, hence D is a subset of $\mathrm{Supp}(f) \subset B_X$. So we can define $s = \sup\{t \geq 0\,;\, [x_0, x_0 + te_1] \subset \operatorname{int} D\}$ and notice that $s \leq 2$. We denote

$$x_1 = x_0 + se_1.$$

Then $x_1 \in \partial D = \partial(f'^{-1}(C))$, and it follows with the continuity of f' that

$$\begin{aligned} f'(x_1) \in \partial C = &\Big(\partial B(f'(x_0), \delta l) \cap \{y^* \in X^*\,;\, |\langle y^* - f'(x_0), e_1\rangle| \leq \varepsilon\}\Big) \\ &\bigcup \Big(B(f'(x_0), \delta l) \cap \{y^* \in X^*\,;\, |\langle y^* - f'(x_0), e_1\rangle| = \varepsilon\}\Big). \end{aligned}$$

Now $f'(x_1) \in A$ and $A \cap \big(B(f'(x_0), \delta l) \cap \{y^* \in X^*\,;\, |\langle y^* - f'(x_0), e_1\rangle| = \varepsilon\}\big) = \varnothing$, because of (3.1). Therefore $f'(x_1) \in \partial B(f'(x_0), \delta l)$ and so

$$\|f'(x_1) - f'(x_0)\| \geq \delta l.$$

Now we fix $\beta \in (0, \delta)$ and $e_2 \in S_X$ with $\langle f'(x_1) - f'(x_0), e_2\rangle = \beta l$. We put $c = (\gamma l/\omega)^{1/\alpha}$, where γ is taken in $(0, 1 - \delta)$. The numbers δ, β and γ will be optimized at the end of the proof. We denote

$$x_2 = x_1 + ce_2 \ \text{ and } \ x_3 = x_0 + ce_2 = x_2 - se_1,$$
$$z_1 = \langle f'(x_0), e_1\rangle \ \text{ and } \ z_2 = \langle f'(x_0), e_2\rangle.$$

Then we have

$$\begin{aligned} |f(x_2) - f(x_1) - z_2 c| \leq\ &|f(x_2) - f(x_3) - z_1 s| + |f(x_3) - f(x_0) - z_2 c| \\ &+ |f(x_0) - f(x_1) + z_1 s|. \end{aligned} \tag{3.3}$$

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We are going to apply the mean value theorem on each side of the parallelogram (x_0, x_1, x_2, x_3). This will give an estimation of each member in the inequality (3.3), and then we will prove (3.2).

First we apply the mean value theorem to the function $g_1(t) = f(x_1 + te_2) - z_2 t$, $t \in [0, c]$ and we obtain t_0 in this interval such that

$$f(x_2) - f(x_1) - z_2 c = g_1'(t_0)c = \langle f'(x_1 + t_0 e_2) - f'(x_0), e_2 \rangle c.$$

Now, if $x \in [x_1, x_2]$, we have

$$\begin{aligned} |\langle f'(x) - f'(x_0), e_2 \rangle| &\geq |\langle f'(x_1) - f'(x_0), e_2 \rangle| - \| f'(x_1) - f'(x) \| \\ &\geq \beta l - \omega \|x_2 - x_1\|^\alpha \geq (\beta - \gamma) l. \end{aligned}$$

With this inequality we get

$$|f(x_2) - f(x_1) - z_2 c| \geq (\beta - \gamma) lc. \tag{3.4}$$

If $x \in [x_3, x_2]$ then $|\langle f'(x) - f'(x_0), e_1 \rangle| < \varepsilon$. Indeed let $x = x_3 + te_1 \in [x_3, x_2]$ with $t \in [0, s]$. If we put $w = x_0 + te_1 = x - ce_2$, then $w \in D$ hence $f'(w) \in C$. Moreover, for all $y \in [w, x]$, $\|f'(y) - f'(w)\| \leq \omega \|y - w\|^\alpha \leq \omega c^\alpha \leq \gamma l$. Therefore $f'([w, x]) \subset B(f'(x_0), \delta l + \gamma l) \subset B(f'(x_0), l)$, since $\gamma \in (0, 1 - \delta)$. Then $f'([w, x]) \cap \{y^* \in A \,;\, |\langle y^* - f'(x_0), e_1 \rangle| = \varepsilon\} = \varnothing$. Recall that f' is continuous, thus $f'([w, x])$ is connected. Since $f'(w) \in C \cap A \subset \{y^* \in A \,;\, |\langle y^* - f'(x_0), e_1 \rangle| < \varepsilon\}$, $f'([w, x])$ is also included in $\{y^* \in A \,;\, |\langle y^* - f'(x_0), e_1 \rangle| < \varepsilon\}$. Thus $|\langle f'(x) - f'(x_0), e_1 \rangle| < \varepsilon$. With the mean value theorem applied to the function $g_2(t) = f(x_3 + te_1) - z_1 t$, $t \in [0, s]$, we obtain that

$$|f(x_2) - f(x_3) - z_1 s| \leq \varepsilon s. \tag{3.5}$$

For all $x \in [x_0, x_3]$, $|\langle f'(x) - f'(x_0), e_2 \rangle| \leq \omega \|x_3 - x_0\|^\alpha \leq \gamma l$. Then the mean value theorem gives that

$$|f(x_3) - f(x_0) - z_2 c| \leq \gamma lc. \tag{3.6}$$

We now apply the mean value theorem to the function $g_3(t) = z_1 t - f(x_0 + te_1)$, $t \in [0, s]$. It gives $t_1 \in [0, s]$ such that $g_3(s) - g_3(0) = g_3'(t_1)s$. Consequently

$$\begin{aligned} |f(x_0) - f(x_1) + z_1 s| = |g_3'(t_1)s| &= |\langle f'(x_0), e_1 \rangle - \langle f'(x_0 + t_1 e_1), e_1 \rangle| s \\ &= |\langle f'(x_0 + t_1 e_1) - f'(x_0), e_1 \rangle| s. \end{aligned}$$

But $x_0 + t_1 e_1 \in D$, hence $f'(x_0 + t_1 e_1) \in C$. Thus we obtain that

$$|f(x_0) - f(x_1) + z_1 s| \leq \varepsilon s. \tag{3.7}$$

Now we use the inequalities (3.4), (3.5), (3.6) and (3.7) in (3.3) and we get

$$(\beta - \gamma) lc \leq \varepsilon s + \gamma lc + \varepsilon s \leq 4\varepsilon + \gamma lc$$

and hence $(\beta-2\gamma)l(\gamma l/\omega)^{\frac{1}{\alpha}} \leq 4\varepsilon$. We choose $\delta \in (0,1)$, $\beta \in (0,\delta)$ and $\gamma \in (0, 1-\delta)$ to maximize $\gamma(\beta - 2\gamma)^{\alpha}$. For this we put

$$\delta = 1 - \frac{1}{3(1+\alpha)}, \beta \to \delta \text{ and } \gamma \to 1-\delta$$

and we obtain

$$\omega \geq l^{1+\alpha}(4\varepsilon)^{-\alpha}3^{-1}\alpha^{\alpha}(1+\alpha)^{-(1+\alpha)}.$$

This gives (3.2) and this proves that

$$\mathcal{F}_{\alpha}(A) \leq 3\Big(\frac{4}{\alpha}\Big)^{\alpha}(1+\alpha)^{1+\alpha}\omega.$$

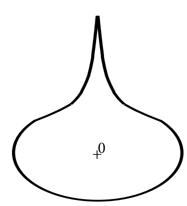
In particular $\mathcal{F}_{\alpha}(A) < +\infty$. ■

This theorem allows us to build simple examples of subsets of X^* which are in the set $\mathcal{S}$ but not in $\mathcal{S}_{\alpha}$.

Example 1: The drop too flat. We fix $\alpha \in]0,1]$ and a Hilbert space H and we build a subset of $\mathbb{R}^2 \times H$ which is in $\mathcal{S}$ but not in $\bigcup_{\beta\in]\alpha,1]} \mathcal{S}_{\beta}$. We put

$$D_{\alpha} = \Big(B_{\mathbb{R}^2} \cup \big\{(x,y)\ ;\ y \in \big[\frac{1}{2}, 2\big]\,, |x| \leq C_{\alpha}(2-y)^{1+1/\alpha}\big\}\Big) \times B_H$$

with $C_{\alpha} = 2^{1/\alpha}3^{-(1+1/\alpha)}$. Here is a representation of the drop D_{α} in the two-dimensional case.



If $\beta \in]\alpha, 1]$, $\mathcal{F}_{\beta}(D_{\alpha}) = +\infty$, since the quotient $y^{1+\beta}/(y^{1+1/\alpha})^{\beta} = y^{1-\beta/\alpha}$ goes to $+\infty$ when y goes to 0. Therefore, for all $\beta \in]\alpha, 1]$, $D_{\alpha} \notin \mathcal{S}_{\beta}$. However $D_{\alpha} \in \mathcal{S}$ since D_{α} satisfies the conditions (i) in Theorems 2.1 and 2.2.

Example 2: The comb with flat broken teeth. We construct a comb in $\mathbb{R}^2 \times H$ which is in $\mathcal{S}$ but not in any $\mathcal{S}_{\alpha}$ because its teeth are too flat. For $n \geq 1$ we denote

$$D_n = \Big[-1 + \sum_{k=1}^{n-1} 2^{1-k}, -1 + \sum_{k=1}^{n-1} 2^{1-k} + 2^{-n}\Big] \times [4^{-1}, 4^{-1} + n^{-2}],$$

$$C = \Big(([-1,1] \times [-4^{-1}, 4^{-1}]) \bigcup \big(\bigcup_{n\geq 1} D_n\big)\Big) \times B_H.$$

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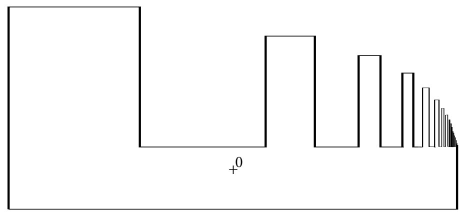

Then, for all $\alpha \in]0,1]$, $\mathcal{F}_\alpha(C) = +\infty$, since the quotient $(n^{-2})^{1+\alpha}/(2^{-n})^\alpha$ goes to infinity when $n \to +\infty$. Now C satisfies the conditions (i) of Theorems 2.1 and 2.2. Consequently, $C \in \mathcal{S}$ but $C \notin \bigcup_{\alpha\in]0,1]} \mathcal{S}_\alpha$.

We now establish the second necessary condition which is an adaptation of the condition (iii) in Theorem 2.1 in the case of Hölder derivatives.

Theorem 3.2 *Let X be a Banach space, A be a subset of X^* and $\alpha \in]0,1]$. If $A \in \mathcal{S}_\alpha$ and $\dim X = d < +\infty$, then*

$$M_n^{(\alpha)}(A) = O(n^{-\alpha/d}).$$

If $A \in S_\alpha$ and X is infinite dimensional, then the sequence $(M_n^{(\alpha)}(A))_n$ is bounded.

Proof Let $b\colon X \to \mathbb{R}$ be a $C^{1,\alpha}$-smooth bump such that $b'(X) = A$. We can suppose that $\mathrm{Supp}(b) \subset B_X$. We fix $n \geq 1$, we take $(y_1^*, \dots, y_n^*)$ in A^n and we write

$$M = \inf\{d_A^{(\alpha)}(y_i^*, y_j^*)\ ;\ 1 \leq i < j \leq n\}.$$

For all $i \in \{1, \dots, n\}$, there exists $x_i \in B_X$ with $b'(x_i) = y_i^*$. We fix i and j and we denote by $\gamma_{i,j}$ the path defined by $\gamma_{i,j}(t) = b'((1-t)x_i + tx_j)$, $t \in [0,1]$. Then

$$\begin{aligned}
l^{(\alpha)}(\gamma_{i,j}) &\leq \sup\Big\{ \Big(\sum_{k=1}^{n} \|b'((1-t_k)x_i + t_k x_j) - b'((1-t_{k-1})x_i + t_{k-1}x_j)\|^{\frac{1}{\alpha}}\Big)^{\alpha}\ ; \\
&\qquad\qquad n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1\Big\} \\
&\leq \sup\Big\{ \Big(\sum_{k=1}^{n} (\omega_\alpha(b')\|(t_k - t_{k-1})(x_i - x_j)\|^{\alpha})^{\frac{1}{\alpha}}\Big)^{\alpha}\ ; \\
&\qquad\qquad n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1\Big\} \\
&\leq \sup\Big\{ \Big(\omega_\alpha(b')^{\frac{1}{\alpha}}\|x_i - x_j\| \sum_{k=1}^{n} (t_k - t_{k-1})\Big)^{\alpha}\ ; \\
&\qquad\qquad n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1\Big\} \\
&\leq \omega_\alpha(b')\|x_i - x_j\|^{\alpha}.
\end{aligned}$$

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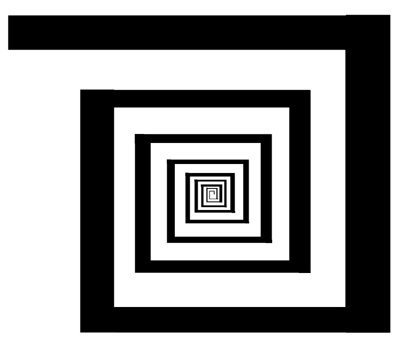
Thus
$$M \le \omega_\alpha(b') \inf\{\|x_i - x_j\|^\alpha ; 1 \le i < j \le n\}.$$
We first assume $d = \dim X < +\infty$ and we put $\beta = \inf\{\|x_i - x_j\|^\alpha ; 1 \le i < j \le n\}$. Then the disjoint union of the $B(x_i, 2^{-1}\beta^{\frac{1}{\alpha}})$, $1 \le i \le n$, is included in $(1+2^{-1}\beta^{\frac{1}{\alpha}})B_X$, and then $n(2^{-1}\beta^{\frac{1}{\alpha}})^d \le (1 + 2^{-1}\beta^{\frac{1}{\alpha}})^d$. It follows that
$$\beta^{\frac{1}{\alpha}} \le \frac{2}{n^{\frac{1}{d}} - 1}$$
and hence $M \le \omega_\alpha(b')(2/n^{\frac{1}{d}} - 1)^\alpha$. Finally, $M_n^{(\alpha)}(A) = O(n^{-\alpha/d})$. Now, if $d = +\infty$, $\inf\{\|x_i - x_j\| ; 1 \le i < j \le n\} \le 2$ and hence $M \le 2^\alpha \omega_\alpha(b')$. Thus the sequence $(M_n^{(\alpha)}(A))_n$ is bounded. ∎

Example 3: The spiral with infinite α-length. For all $\alpha \in]0, 1]$ there exists a set V_α with a finite α-flatness such that $M_n^{(\alpha)}(V_\alpha) = +\infty$ for all $n \in \mathbb{N}$. For example we can take
$$T_\alpha = (-\frac{1}{2}, 0) + \overline{\bigcup_{n\ge 0}(B_n \cup C_n)} \quad \text{where}$$
$$B_n = [a_n, a_{n+1}] \times [-a_n - \varepsilon_{n+1}, -a_n + \varepsilon_{n+1}],$$
$$C_n = [a_{n+1} - \varepsilon_{n+1}, a_{n+1} + \varepsilon_{n+1}] \times [-a_n, -a_{n+1}],$$
$$a_0 = 0, \quad a_n = \sum_{k=1}^{n} (-1)^{k-1} k^{-\alpha} \text{ and } \varepsilon_n = \frac{\alpha}{20} n^{-1-\alpha} \text{ for } n \ge 1.$$
Then T_α is a spiral in $\mathbb{R}^2$ which contains 0 and has an infinite α-length, since
$$\sum_{n\ge 1} |a_{n+1} - a_n|^{\frac{1}{\alpha}} = +\infty.$$



If H is a Hilbert space we define $V_\alpha = T_\alpha \times B_H$. Since the distance $d_{V_\alpha}^{(\alpha)}$ is unbounded, we have $M_n^{(\alpha)}(V_\alpha) = +\infty$ for all $n \in \mathbb{N}$ and hence $V_\alpha \notin \mathcal{S}_\alpha$. On the other hand,

$\mathcal{F}_\alpha(V_\alpha) < +\infty$ because the quotients $|a_{n+1} - a_n|^{1+\alpha}/\varepsilon_n^\alpha$ are bounded by the constant $(\frac{20}{\alpha})^\alpha$. Now we claim that $V_\alpha \in \mathcal{S}$. Indeed $\lim_{n\to+\infty} R_n(\operatorname{int} V_\alpha) = 0$ and then Theorem 2.1 gives the conclusion if H is finite dimensional. If H is infinite dimensional, for all $y^* \in V_\alpha$ there exists a continuous path from 0 to y^* through points of int V_α. So Theorem 2.2 proves that $V_\alpha \in \mathcal{S}$.

We remark that none of the two necessary conditions implies the other. Indeed, let $\alpha \in\,]0, 1]$. Then Example 3 shows a set V_α with a finite α-flatness such that $M_n^{(\alpha)}(V_\alpha) = +\infty$ for all n. On the other hand, if $\beta \in\,]\alpha, 1]$, the drop D_α in Example 1 has an infinite β-flatness but clearly $M_n^{(\beta)}(D_\alpha) = O(n^{-\beta/d})$ where d is the dimension.

4 Sufficient Conditions to Be in $\mathcal{S}_1$

We have shown that a set A of $\mathcal{S}_1$ satisfies two conditions: It must have a finite flatness and it cannot have too many points far away from each other for the distance $d_A^{(1)} = d_A$. We now find a sufficient geometrical condition on a subset A of X^* so that A belongs to $\mathcal{S}_1$.

Theorem 4.1 *Let X be an infinite dimensional separable Banach space with $b\colon X \to \mathbb{R}$ a $C^{1,1}$-smooth bump. There exists a constant $K > 1$ so that if U is an open subset of X^* satisfying*

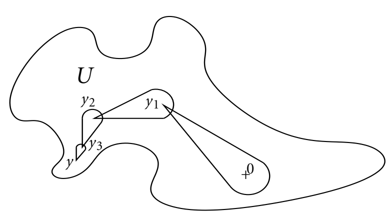
($\mathcal{J}$) *There exist $a \in (0, 1)$ and $C > 0$ such that for all $y^* \in U$, there are $n \in \mathbb{N}$, and $(y_0^*, y_1^*, \dots, y_n^*) \in U^{n+1}$ where $y_0^* = 0$ and $y_n^* = y^*$ with*

$$\operatorname{co}(B(y_{i-1}^*, a\|y_i^* - y_{i-1}^*\|) \cup \{y_i^*\}) \subset U$$

and $\|y_i^* - y_{i-1}^*\| < C(\frac{a}{K})^i$ for all $i \in \{1, \dots, n\}$.

Then $U \in \mathcal{S}_1$.

We notice that the existence of a C^1-smooth bump on X and the separability of X imply that X^* is separable ([6], page 58). The condition ($\mathcal{J}$) means that any point in U can be joined to 0 by a "good" path, that is a finite union of drops which are not too flat, as it is shown in the following picture:



This condition is stable by finite superpositions. Indeed if F_1, F_2 satisfy ($\mathcal{J}$) and $y_1^* \in F_1$, then $F_1 \cup (y_1^* + F_2)$ also satisfies ($\mathcal{J}$). We give examples of subsets satisfying this condition.

Definition 4.2 Let U be a bounded open subset of X^*. We say that U is *uniformly star-shaped* if there exists $a > 0$ such that $\operatorname{co}(aB_{X^*} \cup \{y^*\}) \subset U$ for all $y^* \in U$.

For example, convex open bounded subsets of X^* containing 0 are uniformly star-shaped. Clearly uniformly star shaped sets satisfy condition ($\mathcal{J}$), so Theorem 4.1 yields the following result.

Theorem 4.3 *Let X be an infinite dimensional separable Banach space with $b\colon X \to \mathbb{R}$ a $C^{1,1}$-smooth bump. Let U be a bounded open subset of X^*. If U is uniformly star-shaped, then $U \in \mathcal{S}_1$.*

The star-shaped condition must be uniform. Indeed let us consider the set

$$D = \Big(\operatorname{int}(B_{\mathbb{R}^2}) \cup \{(x, y)\ ;\ y \in (\frac{1}{2}, 2), |x| < \frac{4}{27}(2 - y)^3\}\Big) \times \operatorname{int} B_H$$

where H is an infinite dimensional Hilbert space. This drop was introduced in Example 1 of Section 3. Clearly, for all $y^* \in D$, there is $a > 0$ (which depends on y^*) such that $\operatorname{co}(aB_{X^*} \cup \{y^*\}) \subset D$. Nevertheless $D \notin \mathcal{S}_1$ because D has an infinite 1-flatness (see Theorem 3.1).

We are now going to prove Theorem 4.1. First we need the

Lemma 4.4 *Let X be an infinite dimensional separable Banach space with $b\colon X \to \mathbb{R}$ a $C^{1,1}$-smooth bump. There exists $K_1 > 1$ such that for all $y^* \in X^*$ and $\varepsilon \in (0, \|y^*\|)$, there exists a $C^{1,1}$-smooth bump $f\colon X \to \mathbb{R}$ such that*

(i) $f'(X) \subset co(\varepsilon B_{X^*} \cup \{y^*\})$,
(ii) $f'(x) = y^*$ *for all* $x \in (K_1\|y^*\|)^{-1}\varepsilon B_X$,
(iii) $\operatorname{Supp}(f) \subset B_X$ *and* f' *is* $(K_1\|y^*\|^2\varepsilon^{-1})$*–Lipschitzian.*

This lemma is a variant of a lemma from [4]. We give its proof for the sake of completeness.

Proof We take $b_0\colon X \to \mathbb{R}$ a $C^{1,1}$-smooth bump. Without loss of generality we may assume that $b_0 \geq 0$ and $b_0(0) = 1$. There is $M > 3$ such that $b_0'(X) \subset MB_{X^*}$, $\operatorname{Supp}(b_0) \subset MB_X$, $\operatorname{Lip}(b_0') \leq M$ and $b_0(X) \subset [0, M]$. The function defined by

$$b(x) = M^{-2}\varepsilon b_0(Mx)$$

satisfies $b'(X) \subset \varepsilon B_{X^*}$, $\operatorname{Supp}(b) \subset B_X$, $\operatorname{Lip}(b') \leq M\varepsilon$, $b(X) \subset [0, M^{-1}\varepsilon]$ and $b(0) = M^{-2}\varepsilon$. We fix

$$r = 6^{-1}b(0) = 6^{-1}M^{-2}\varepsilon.$$

Clearly there exists a C^∞-smooth function $\varphi\colon \mathbb{R} \to [r, +\infty[$ such that $\varphi'(\mathbb{R}) \subset [0, 1]$, $\varphi''(\mathbb{R}) \subset [-r^{-1}, r^{-1}]$ and $\varphi(t) = t$ if $t \geq 2r$. There exists also $g\colon \mathbb{R}^2 \to \mathbb{R}$ a C^∞-smooth function such that $\|g''(t, s)\| \leq 2r^{-1}$ for all $(t, s) \in \mathbb{R}^2$, $g'(\mathbb{R}^2) = \{(t, 1 - t)\ ;\ t \in [0, 1]\}$ and

$$g(t, s) = \begin{cases} t & \text{if } s \geq t + r, \\ s & \text{if } s \leq t - r. \end{cases}$$

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The construction of g is written in [4]. We define

$$f(x) = g\big(b(x), \varphi(\langle y^*, x\rangle + 3r)\big), x \in X.$$

Let us check that f satisfies the required properties. Clearly f is C^1-smooth. If $b(x) = 0$, then $\varphi(\langle y^*, x\rangle + 3r) \geq b(x) + r$ and hence $f(x) = b(x) = 0$. So f is a bump and $\mathrm{Supp}(f) \subset \mathrm{Supp}(b) \subset B_X$.

Let $x \in r\|y^*\|^{-1}B_X$. Then $\langle y^*, x\rangle + 3r \in [2r, 4r]$ so $f(x) = g\big(b(x), \langle y^*, x\rangle + 3r\big)$. With the mean value theorem,

$$b(x) \geq b(0) - \varepsilon\|x\| \geq 6r - r \geq 5r.$$

Thus $\langle y^*, x\rangle + 3r \leq b(x) - r$ and hence $f(x) = \langle y^*, x\rangle + 3r$. Consequently,

$$f'(x) = y^* \text{ for all } x \in r\|y^*\|^{-1}B_X = 6^{-1}M^{-2}\varepsilon\|y^*\|^{-1}B_X.$$

Let $x \in X$. There exists $t(x) \in [0, 1]$ so that $g'\big(b(x), \varphi(\langle y^*, x\rangle + 3r)\big) = (t(x), 1 - t(x))$. Thus

$$\begin{aligned} f'(x) &= g'\big(b(x), \varphi(\langle y^*, x\rangle + 3r)\big)\big(b'(x), \varphi'(\langle y^*, x\rangle + 3r)y^*\big) \\ &= t(x)b'(x) + (1 - t(x))\varphi'(\langle y^*, x\rangle + 3r)y^* \\ &= t(x)b'(x) + (1 - t(x))\alpha(x)y^* \text{ with } \alpha(x) \in [0, 1]. \end{aligned}$$

Then $f'(x) \in \mathrm{co}(b'(X) \cup \{\alpha(x)y^*\}) \subset \mathrm{co}(\varepsilon B_{X^*} \cup \{y^*\})$. Therefore

$$f'(X) \subset \mathrm{co}(\varepsilon B_{X^*} \cup \{y^*\}).$$

We are going to prove that

$$f' \text{ is } K_1\|y^*\|^2\varepsilon^{-1} \text{ Lipschitzian with } K_1 = 62M^2. \tag{4.1}$$

We take x_1 and x_2 in $\mathrm{Supp}(f) \subset B_X$. We write $a(x) = \langle y^*, x\rangle + 3r$. Then

$$\begin{aligned} f'(x_2) - f'(x_1) = g'\big(&b(x_2), \varphi(a(x_2))\big) \\ &\times \big(b'(x_2) - b'(x_1), \big(\varphi'(a(x_2)) - \varphi'(a(x_1))\big)\, y^*\big) \\ &- \big(g'\,(b(x_1), \varphi(a(x_1))) - g'(b(x_2), \varphi(a(x_2)))\big) \\ &\times \big(b'(x_1), \varphi'(a(x_1))y^*\big). \end{aligned}$$

Using this and the mean value theorem we obtain

$$\begin{aligned} \|f'(x_2) - f'(x_1)\| \leq \|g'\|_\infty\big(&\,\mathrm{Lip}(b') + \|\varphi''\|_\infty\|y^*\|^2\big)\,\|x_2 - x_1\| \\ &+ \|g''\|_\infty(\|b'\|_\infty + \|\varphi'\|_\infty\|y^*\|)^2\|x_2 - x_1\|. \end{aligned}$$

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With the hypotheses on g, φ and b this gives

$$\|f'(x_2) - f'(x_1)\| \leq 2\big(M\varepsilon + r^{-1}\|y^*\|^2 + r^{-1}(\varepsilon + \|y^*\|)^2\big)\,\|x_2 - x_1\|$$

and hence f' is Lipschitzian. Recall that $\varepsilon < \|y^*\|$, thus

$$\operatorname{Lip}(f') \leq 2(M + 6M^2 + 24M^2)\|y^*\|^2\varepsilon^{-1} \leq K_1\|y^*\|^2\varepsilon^{-1}.$$

Consequently (4.1) is proved and the proof of the lemma is complete. ■

We now put $K = 6K_1$ where K_1 is the constant given by Lemma 4.4.

Proof of Theorem 4.1 Let U be as in the theorem. For $i \geq 0$ and $y^* \in U$ we define

$$T_i(y^*) = \big\{ z^* \in U \,;\, \operatorname{co}(B(y^*, a\|z^* - y^*\|) \cup \{z^*\}) \subset U \text{ and } \|z^* - y^*\| < C(\frac{a}{K})^{i+1} \big\}.$$

The condition ($\mathcal{J}$) is clearly open. It means that if D is a dense countable subset of U, then for all $y^* \in U$ there are $n \in \mathbb{N}$, $(y_0^* = 0, \dots, y_{n-1}^*, y_n^* = y^*) \in D^n \times \{y^*\}$ such that for all $i \in \{1, \dots, n\}$, $y_i^* \in T_{i-1}(y_{i-1}^*)$. We now fix a dense subset D of U and $q \geq 1$. We define

$$U_q = \Big\{ y^* \in U \,;\, \text{there exist } n \geq 1, (y_0^* = 0, \dots, y_n^* = y^*) \in D^n \times \{y^*\} \\ \text{such that for all } i \in \{1, \dots, n\}, y_i^* \in T_{i-1}(y_{i-1}^*) \\ \text{and } \operatorname{dist}\big(\bigcup_{i=1}^{n} [y_{i-1}^*, y_i^*], \partial U\big) > q^{-1} \Big\}.$$

Step 1: We code U_q with multiindices. We define a mapping φ on $\mathbb{N}^{<\mathbb{N}}$ by induction. We first put

$$\{\varphi(s) \,;\, s \in \mathbb{N}^{<\mathbb{N}} \text{ and } |s| = 1\} = D \cap T_0(0) \cap U_q.$$

Then, if $\varphi(s)$ is defined for $s \in \mathbb{N}^{<\mathbb{N}}$, we denote

$$\{\varphi(s\hat{\ }j) \,;\, j \in \mathbb{N}\} = D \cap T_{|s|}(\varphi(s)) \cap U_q.$$

Now, if $\sigma \in \mathbb{N}^{\mathbb{N}}$, $(\varphi(\sigma|k))_k$ is clearly convergent. Moreover,

$$U_q \subset \big\{ \lim_k(\varphi(\sigma|k)) \,;\, \sigma \in \mathbb{N}^{\mathbb{N}} \big\}. \tag{4.2}$$

Indeed we let $y^* \in U_q$, $n \geq 1$ and $(y_0^* = 0, \dots, y_n^* = y^*) \in D^n \times \{y^*\}$ such that for all $i \in \{1, \dots, n\}$, $y_i^* \in T_{i-1}(y_{i-1}^*)$ and $\operatorname{dist}(\bigcup_{i=1}^n [y_{i-1}^*, y_i^*], \partial U) > q^{-1}$. Then there exists $s = (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$ such that for all $i \in \{1, \dots, n-1\}$, $y_i^* = \varphi(s|i)$. Since $y^* \in T_{n-1}(y_{n-1}^*) \cap U_q$ we can find $s_n \in \mathbb{N}$ with $\|y^* - \varphi(s\hat{\ }s_n)\|$ small enough to have $y^* \in T_n(\varphi(s\hat{\ }s_n))$. By induction, for all $k \geq n$, there is $s_k \in \mathbb{N}$ such that $y^* \in T_k(\varphi((s_1, \dots, s_k)))$ and hence

$$\|y^* - \varphi(s_1, \dots, s_k)\| < C(\frac{a}{K})^{k+1}.$$

Then $y^* = \lim_k \varphi((s_1, \dots, s_k))$ and (4.2) is proved.

In the following, if $|s| = 1$, we will denote $\varphi(s_-) = 0$ and $x_{s_-} = 0$. We remark that, by construction, for all $s \in \mathbb{N}^{<\mathbb{N}}$ we have

$$\|\varphi(s) - \varphi(s_-)\| \leq C(\frac{a}{K})^{|s|}. \tag{4.3}$$

We are now going to construct the required bump. First, since X is infinite dimensional, for a given $x \in X$ and $\delta > 0$, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $B(x, \frac{5\delta}{6})$ such that $\|w_k - w_q\| > \frac{\delta}{3}$ if $k \neq q$. We write $w_k = w_k(x, \delta)$. We will proceed by induction over $k := |s|$. We define $\beta = aK^{-1} = a(6K_1)^{-1}$ and remark that $\beta < 6^{-1}$.

For $k \in \mathbb{N}$, denote by $\mathcal{P}(k)$ the following statement: For all $s \in \mathbb{N}^k$, there are $x_s \in B_X$ and a $C^{1,1}$-smooth bump $h_s\colon X \to \mathbb{R}$ such that

(i) $h_s'(x) = \varphi(s) - \varphi(s_-)$ for all $x \in B(x_s, \beta^{|s|})$.
(ii) $\operatorname{Supp}(h_s) \subset B(x_{s_-}, \beta^{|s|-1}) \subset B_X$.
(iii) If $|r| = |s|$ and $r \neq s$, then $\operatorname{Supp}(h_r) \cap \operatorname{Supp}(h_s) = \varnothing$.
(iv) $\operatorname{Lip}(h_s') \leq C$.
(v) $\varphi(s_-) + h_s'(X) \subset \operatorname{co}(B(\varphi(s_-), a\|\varphi(s) - \varphi(s_-)\|) \cup \{\varphi(s)\}) \subset U_q$.

Step 2: $\mathcal{P}(k)$ holds for all $k \geq 1$. We first show that $\mathcal{P}(1)$ holds. Let $s \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = 1$. We obtain with Lemma 4.4 a $C^{1,1}$-smooth bump $g_s\colon X \to \mathbb{R}$ such that $g_s'(X) \subset \operatorname{co}(B(0, a\|\varphi(s)\|) \cup \{\varphi(s)\}) \subset U_q$, $\operatorname{Supp}(g_s) \subset B_X$, $g_s'(x) = \varphi(s)$ if $\|x\| \leq aK_1^{-1}$ and $\operatorname{Lip}(g_s') \leq K_1 a^{-1}\|\varphi(s)\| \leq 6^{-1}C$ since $\|\varphi(s)\| \leq C\frac{a}{K}$ (see (4.3)). We define

$$h_s(x) = 6^{-1} g_s\big(6(x - w_{s(1)}(0, 1))\big).$$

Then $\operatorname{Supp}(h_s) \subset B(w_{s(1)}(0, 1), 6^{-1}) \subset B_X$. Furthermore there exists $x_s \in B_X$ so that $h_s'(x) = \varphi(s)$ for all $x \in B(x_s, \beta)$. If $s \neq r$ and $|s| = |r| = 1$, then $\operatorname{Supp}(h_s) \cap \operatorname{Supp}(h_r) \subset B(w_{s(1)}(0, 1), 6^{-1}) \cap B(w_{r(1)}(0, 1), 6^{-1}) = \varnothing$. Finally,

$$\operatorname{Lip}(h_s') \leq 6 \operatorname{Lip}(g_s') \leq C$$

and hence $\mathcal{P}(1)$ holds.

We now fix $k \geq 1$ and assume that $\mathcal{P}(k)$ holds. Let $s \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = k + 1$. We apply Lemma 4.4 and obtain a $C^{1,1}$-smooth bump $g_s\colon X \to \mathbb{R}$ such that $\varphi(s_-) + g_s'(X) \subset \operatorname{co}(B(\varphi(s_-), a\|\varphi(s) - \varphi(s_-)\|) \cup \{\varphi(s)\}) \subset U_q$, $\operatorname{Supp}(g_s) \subset B_X$, $g_s'(x) = \varphi(s) - \varphi(s_-)$ if $\|x\| \leq aK_1^{-1}$ and $\operatorname{Lip}(g_s') \leq K_1 a^{-1}\|\varphi(s) - \varphi(s_-)\|$. We define $x_s = w_{s(k+1)}(x_{s_-}, \beta^{|s|-1})$ and

$$h_s(x) = 6^{-1}\beta^{|s|-1} g_s\big(6\beta^{1-|s|}(x - x_s)\big).$$

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Then $\operatorname{Supp}(h_s) \subset B(x_s, 6^{-1}\beta^{|s|-1}) \subset B(x_{s_-}, \beta^{|s|-1}) \subset B_X$. For all $x \in B(x_s, \beta^{|s|})$, $\|6\beta^{1-|s|}(x - x_s)\| \leq 6\beta \leq aK_1^{-1}$ and hence $h_s'(x) = \varphi(s) - \varphi(s_-)$. Clearly, if $s \neq r$ and $|s| = |r| = k+1$, then $\operatorname{Supp}(h_s) \cap \operatorname{Supp}(h_r) = \varnothing$. Finally, with (4.3),

$$\operatorname{Lip}(h_s') \leq 6\beta^{1-|s|} \operatorname{Lip}(g_s') \leq \|\varphi(s) - \varphi(s_-)\|\beta^{-|s|} \leq C.$$

So $\mathcal{P}(k+1)$ holds.

Step 3: The function $F_q = \sum_{k\geq 1} \sum_{|s|=k} h_s$ is a $C^{1,1}$-smooth bump. For $k \geq 1$ we put $H_k(x) = \sum_{|s|=k} h_s(x)$. Then H_k is C^1- smooth since it is the sum of C^1-smooth functions with disjoint supports. For all $x \in X$,

$$\|H_k'(x)\| \leq \sup\{\|h_s{}'(x)\| \,;\, |s| = k\} \leq C\beta^{k-1}$$

since, for all $s \in \mathbb{N}^k$, $h_s{}'$ is C-Lipschitzian and has its support in $B(x_{s_-}, \beta^{k-1})$. By the mean value theorem, and using $\operatorname{Supp}(H_k) \subset B_X$, we get

$$|H_k(x)| \leq 2C\beta^{k-1}.$$

Therefore F_q is a C^1-smooth bump. Moreover

$$\operatorname{Lip}(F_q') \leq \sup\{\operatorname{Lip}(h_s') \,;\, s \in \mathbb{N}^{<\mathbb{N}}\} \leq C.$$

Step 4: $U_q \subset F_q'(X) \subset U$. It is clear that $F_q'(X) \subset \overline{U_q} \subset U$. Now let $G_k(x) = \sum_{1\leq j\leq k} H_j(x)$. For all $s \in \mathbb{N}^{<\mathbb{N}}$, $B(x_s, \beta^{|s|}) \subset B(x_{s_-}, \beta^{|s|-1})$. Thus, if $k \geq 1$ and $|s| = k$, $H_j'(x_s) = \varphi(s|j) - \varphi(s|j-1)$ for all $1 \leq j \leq k$ and hence $G_k'(x_s) = \varphi(s)$.

We fix $y^* \in U_q$. By (4.2) there exists $\sigma \in \mathbb{N}^{\mathbb{N}}$ with $y^* = \lim_k \varphi(\sigma_{|k})$. We take x in $\bigcap_{k\geq 1} B(x_{\sigma|k}, \beta^k)$. Then $(x_{\sigma|k})_k$ converges to x and since $(G_k')_k$ is uniformly convergent, we have

$$F_q'(x) = \lim_k G_k'(x_{\sigma|k}) = \lim_k \varphi(\sigma|k) = y^*.$$

Step 5: The sum of the F_q is the desired bump. We consider a 3-separated sequence $(u_q)_{q\geq 1}$ in $7B_X$ and we denote

$$F(x) = \sum_{q\geq 1} F_q(x - u_q), x \in X.$$

Then F is a $C^{1,1}$-smooth bump and $\bigcup_{q\geq 1} U_q \subset F'(X) \subset U$, hence $F'(X) = U$. ■

In the finite dimensional case, there exist some partial results obtained with finite constructions. For example, any compact convex polyhedron P in $\mathbb{R}^2$, with $0 \in \operatorname{int} P$, is the range of the derivative of a C^∞-smooth bump $f\colon \mathbb{R}^2 \to \mathbb{R}$, and hence is in $\mathcal{S}_1$ (see [3]). We can ask the following question: "Does a uniformly star-shaped compact subset of $\mathbb{R}^d$ belong to $\mathcal{S}_1$?"

References

[1] D. Azagra and R. Deville, *James' theorem fails for starlike bodies*. J. Funct. Anal. **180**(2001), 328–346.

[2] D. Azagra, M. Fabian and M. Jimenez-Sevilla, *Exact filling of figures with the derivatives of smooth mappings*. Preprint (2001).

[3] J. M. Borwein, M. Fabian, I. Kortezov and P. D. Loewen, *The range of the gradient of a continuously differentiable bump*. J. Nonlinear Convex Anal. **2**(2001), 1–19.

[4] J. M. Borwein, M. Fabian and P. D. Loewen, *The range of the gradient of a Lipschitz C^1-smooth bump in infinite dimensions*. Israel J. Math. **132**(2002), 239–251.

[5] J. Dieudonné, *Fundations of Modern Analysis.* Academic Press, New York, 1969.

[6] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*. Pitman Monographs and Surveys in Pure and Applied Mathematic 64, Wiley, New York, 1993.

[7] M. Fabian, O. Kalenda and J. Kolář, *Filling analytic sets by the derivatives of C^1-smooth bumps*. Preprint (2002).

[8] T. Gaspari, *On the range of the derivative of a real valued function with bounded support*. Studia Math. **153**(2002), 81–99.

[9] J. Kolář, J. Kristensen, *The set of gradients of a bump*. Preprint (2002).

[10] J. Malý, *The Darboux property for gradients*. Real Anal. Exchange **22**(1996–1997), 167–173.

[11] L. Rifford, *Range of the gradient of a smooth bump function in finite dimensions*. Proc. Amer. Math. Soc. **131**(2003), 3063–3066.

[12] J. Saint-Raymond, *Local inversion for differentiable functions and Darboux property*. Preprint (2001).

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