Bull. Aust. Math. Soc. **97** (2018), 102–109
doi:10.1017/S0004972717000740

EXISTENCE OF TRAVELLING WAVES IN THE FRACTIONAL BURGERS EQUATION

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(Received 29 May 2017; accepted 8 June 2017; first published online 20 December 2017)

Abstract

We construct travelling waves in the Burgers equation with the fractional Laplacian $(D^2)^\alpha$, $\alpha \in (1/2, 1)$. This is done by first constructing odd solutions u_ε of $uu' = K_{\varepsilon_1} * u - k_{\varepsilon_1} u + \varepsilon_2 u''$, $u(-\infty) = u_c > 0$, with $K_{\varepsilon_1} * u - k_{\varepsilon_1} u$ nonsingular, and then passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$, to give $K_{\varepsilon_1} * u_\varepsilon - k_{\varepsilon_1} u_\varepsilon \to (D^2)^\alpha u_0$ pointwise. The proof relies on operator splitting.

2010 *Mathematics subject classification*: primary 45E10; secondary 35A01, 35C07, 35R11.
Keywords and phrases: singular integral equation, travelling waves.

1. Introduction

We study the equation

$$u_t + uu_x - (\partial_{xx})^\alpha u = 0, \tag{1.1}$$

in which the fractional power of the Laplacian in one dimension for $\alpha \in (0, 1]$ can be represented as

$$(D^2)^\alpha u(x) = -\frac{1}{2\Gamma(-2\alpha)\cos(\pi\alpha)}\, PV \int_{\mathbb{R}} \frac{u(y) - u(x)}{|x-y|^{1+2\alpha}}\, dy,$$

where PV denotes the Cauchy principal value. It is also a pseudodifferential operator of the symbol $-|\xi|^{2\alpha}$ given by

$$(D^2)^\alpha u = \mathcal{F}^{-1}(-|\xi|^{2\alpha}(\mathcal{F}u)) \quad \text{for all } u \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz class.

The simplest Cauchy problem for the classical Burgers equation

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = (1 - H(x))u_- + H(x)u_+, \end{cases} \tag{1.2}$$

where H is the Heaviside function, has two types of solutions. If $u_- > u_+$, the shock wave $u(x,t) = (1 - H(x - st))u_- + H(x - st)u_+$, with $s = \frac{1}{2}(u_- + u_+)$, is the unique weak solution of (1.2). If $u_- < u_+$, the rarefaction wave

$$u(x,t) = \begin{cases} u_- & \text{for } x < u_- t, \\ x/t & \text{for } u_- t \le x \le u_+ t, \\ u_+ & \text{for } x > u_+ t, \end{cases} \tag{1.3}$$

is the unique weak solution of (1.2) with the entropy condition.

Here we are interested in the first type of solution of (1.1), that is, travelling waves $U(x - st)$, such that $U(-\infty) = u_-$ and $U(\infty) = u_+$, where $u_- > u_+$. Thus U satisfies

$$U'(U - s) = (D^2)^\alpha U.$$

If we let $u = U - s$, then

$$uu' = (D^2)^\alpha u. \tag{1.4}$$

Integrating (1.4) over $\mathbb{R}$ shows that $u_c = u(-\infty) = -u(\infty)$, and thus $u_c = \frac{1}{2}(u_- - u_+)$ and $s = \frac{1}{2}(u_- + u_+)$ (the Rankine–Hugoniot condition). These solutions are expected to be globally stable, that is, the solution of the Cauchy problem with an initial value having tails asymptotic to u_- and u_+, where $u_- > u_+$, should converge to a translate of the travelling wave asymptotic to u_- and u_+.

The only result in the literature in this direction is the formal nonexistence of smooth travelling waves of (1.1) in the case $\alpha \in (0, 1/2]$ in [5, Proposition 5.1]. Solutions of the Cauchy problem with an initial value having tails asymptotic to u_- and u_+, where $u_- < u_+$, were shown to converge to the rarefaction wave (1.3) in the case $\alpha \in (1/2, 1)$ [11], to a self-similar solution in the case $\alpha = 1/2$ [4] and to the solution of the linear part of (1.1) with initial condition $(1 - H(x))u_- + H(x)u_+$ in the case $\alpha \in (0, 1/2)$ [4]. Solutions of the Cauchy problem always remain smooth in the cases $\alpha \in (1/2, 1)$ [10] and $\alpha = 1/2$ [12, 13], and may become discontinuous in the case $\alpha \in (0, 1/2)$ [3]. In [6], such a weak solution was shown to become smooth eventually for α a little less than $1/2$. Some other papers on the subject are [1, 2].

Existence of travelling waves was a longstanding problem for the nonlocal Burgers equation

$$u_t + uu_x - K * u + u = 0, \tag{1.5}$$

where K is nonsingular. It was solved in [8] and in more generality in [7]. See also the references in [7, 8] for the special case $K(x) = \frac{1}{2}e^{-|x|}$, which was solved earlier. The travelling wave can be a shock wave, that is, discontinuous, if u_c is large enough. Travelling waves for (1.5) are the starting point of our construction, which uses the idea from [9] of deriving travelling waves of (1.1) from an appropriate limit. In [9], we constructed travelling waves of

$$u_t - (\partial_{xx})^\alpha + f(u) = 0,$$

where f is bistable, by passing to the limit from travelling wave solutions of

$$u_t - b_\alpha(J_\varepsilon * u - j_\varepsilon u) + f(u) = 0, \tag{1.6}$$

where $b_\alpha = -1/(2\Gamma(-2\alpha)\cos(\pi\alpha))$,

$$J_\varepsilon(x) = \begin{cases} 1/|x|^{1+2\alpha} & \text{for } |x| \geq \varepsilon, \\ 1/\varepsilon^{1+2\alpha} & \text{for } |x| < \varepsilon, \end{cases} \tag{1.7}$$

and $j_\varepsilon = \int_{\mathbb{R}} J_\varepsilon = (\alpha^{-1} + 2)/\varepsilon^{2\alpha}$, so that, formally, $b_\alpha(J_\varepsilon * u - j_\varepsilon u) \to (D^2)^\alpha u$. Travelling waves of (1.6) are guaranteed to be smooth (not discontinuous) if j_ε is large enough. This should also be the case for (1.5) if the nonlocal operator is as in (1.6). However, it is not known how to show it. Since members of the limiting sequence need to be smooth, we overcome this difficulty by first using (1.5) to construct odd solutions of

$$uu' = b_\alpha(K_{\varepsilon_1} * u - k_{\varepsilon_1} u) + \varepsilon_2 u'', \tag{1.8}$$

with $u_c = u(-\infty)$ and $K_{\varepsilon_1} = J_{\varepsilon_1}$, as in (1.7). If $\varepsilon_2 > 0$ is appropriately chosen, we can then pass to the limit $\varepsilon_1, \varepsilon_2 \to 0$ to obtain the following result.

Theorem 1.1. *Let $\alpha \in (1/2, 1)$. There exists an odd and smooth solution of (1.4) such that $u(-\infty) = u_c$ and $u' < 0$.*

We prove Theorem 1.1 in Section 2. Getting the necessary estimates for the passage to the limit is harder than in [9], even though we use the same operator splitting trick.

2. Existence

The following proposition was proved in [8].

Proposition 2.1. *Suppose that $\int_{\mathbb{R}} K = 1$, K is even, $K \geq 0$, $K \in W^{1,1}(\mathbb{R})$, K is nonincreasing on $(0, \infty)$ and $K(y) = o(1/y^4)$ as $y \to \infty$. Then there exists an odd solution u of*

$$uu' = K * u - u,$$

such that $u' < 0$ and $u(-\infty) = u_c$. If $u_c > 2\int_{\mathbb{R}} |x|K(x)\,dx$, then u is discontinuous at zero.

It follows that there is such a solution u_δ of the equation

$$uu' = b_\alpha(K_{\varepsilon_1,\delta} * u - k_{\varepsilon_1,\delta} u) + \frac{1}{\delta^2}(L_\delta * u - u), \tag{2.1}$$

where $K_{\varepsilon_1,\delta} \nearrow K_{\varepsilon_1}$ and $\delta^{-2}(L_\delta * \phi - \phi) \to \varepsilon_2 \phi''$ for smooth enough ϕ as $\delta \to 0$. Here L is even, $L_\delta(x) = \delta^{-1} L(x\delta^{-1})$, $L \geq 0$, $\int_{\mathbb{R}} L = 1$ and $\varepsilon_2 = \frac{1}{2}\int_{\mathbb{R}} x^2 L(x)\,dx$. Since each u_δ is monotone, from Helly's theorem there is a subsequence of u_δ, denoted again by u_δ, such that $u_\delta \to u_0$ as $\delta \to 0$. We need to show that u_0 satisfies (1.8) and that $u_0(-\infty) = u_c$. For the first, we use the weak formulation, and for the second we use the strong. Let $S \geq 0$ be such that $\int_{\mathbb{R}} S = 1$, $S \in W^{2,1}(\mathbb{R})$ and $v_\delta = S * u_\delta$. Applying S to (2.1) and integrating from $-\infty$ to x,

$$S * \left(\frac{1}{2}u_\delta^2\right) - \frac{1}{2}u_c^2 = \int_{-\infty}^{x} \left[b_\alpha(K_{\varepsilon_1,\delta} * v_\delta - k_{\varepsilon_1,\delta} v_\delta) + \frac{1}{\delta^2}(L_\delta * v_\delta - v_\delta)\right].$$

https://doi.org/10.1017/S0004972717000740 Published online by Cambridge University Press

Passing to the limit $\delta \to 0$ and integrating from 0 to x,

$$\int_0^x \Big(S * \Big(\frac{1}{2}u_0^2\Big) - \frac{1}{2}u_c^2\Big) = \int_0^x \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1}(y) \int_0^1 v_0(s+ty)\,dt\,dy\,ds + \varepsilon_2 v_0(x), \tag{2.2}$$

where $v_0 = S * u_0$. This is derived from

$$\begin{aligned}\int_{-\infty}^x b_\alpha(K_{\varepsilon_1,\delta} * v_\delta - k_{\varepsilon_1,\delta}v_\delta) &= \lim_{r\to-\infty} \int_r^x \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1,\delta}(y) \int_0^1 v_\delta'(s+ty)\,dt\,dy\,ds \\ &= \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1,\delta}(y) \int_0^1 [v_\delta(x+ty) - u_c]\,dt\,dy \\ &\to \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1}(y) \int_0^1 v_0(x+ty)\,dt\,dy \quad \text{as } \delta \to 0,\end{aligned}$$

where we used Fubini's theorem and dominated convergence twice, and

$$\int_{-\infty}^x \frac{1}{\delta^2}(L_\delta * v_\delta - v_\delta) = \varepsilon_2 v_\delta' + \int_{\mathbb{R}} L(y) \int_0^y (y-t)[v_\delta'(x+\delta t) - v_\delta'(x)]\,dt\,dy,$$

where $v_\delta' \to v_0'$ from $\int_{\mathbb{R}} |S'| < \infty$ with dominated convergence and

$$\begin{aligned}\Big|\int_{\mathbb{R}} L(y) \int_0^y (y-t)[v_\delta'(x+\delta t) - v_\delta'(x)]\,dt\,dy\Big| &\le \delta \max |v_\delta''| \int_{\mathbb{R}} L(y) \Big|\int_0^y (y-t)t\,dt\Big|\,dy \\ &\to 0 \quad \text{as } \delta \to 0,\end{aligned}$$

using $\int_{\mathbb{R}} |S''| < \infty$ and an additional assumption that $\int_{\mathbb{R}} |y^3| L(y)\,dy < \infty$.

It is clear that $u_0 \not\equiv 0$. We would now like to take $S_\varepsilon(x) = \varepsilon^{-1}S(x\varepsilon^{-1})$ in (2.2) and pass to the limit $\varepsilon \to 0$. However, we do not know if u_0 is continuous, and if it is not continuous at x_{dc}, then $v_0(x_{dc}) \to \frac{1}{2}(u_0(x_{dc}-) + u_0(x_{dc}+))$ as $\varepsilon \to 0$. Before we return to (2.2), we use the weak formulation to show that u_0 is continuous.

Multiplying (2.1) by $\phi \in C_0^\infty$, integrating over $\mathbb{R}$ and passing to the limit $\delta \to 0$,

$$\int_{\mathbb{R}} \Big[\frac{1}{2}u_0^2\phi' + b_\alpha(K_{\varepsilon_1} * u_0 - k_{\varepsilon_1}u_0)\phi + \varepsilon_2 u_0 \phi''\Big] = 0. \tag{2.3}$$

For any finite a, b, by integration in (2.3) over (a, b),

$$\int_a^b f\phi'' = 0,$$

where

$$f(x) = -\int_0^x \frac{1}{2}u_0^2 + \int_0^x ds \int_0^s b_\alpha(K_{\varepsilon_1} * u_0 - k_{\varepsilon_1}u_0) + \varepsilon_2 u_0(x). \tag{2.4}$$

It is standard that

$$f(x) = c_1 + c_2 x \quad \text{almost everywhere},$$

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where c_1, c_2 satisfy the system $F_1(b) = 0$, $F_2(b) = 0$, with

$$F_1(x) = \int_a^x (f(s) - c_1 - c_2 s)\, ds, \quad F_2(x) = \int_a^x F_1.$$

Then u_0 in (2.4) is continuous since u_0 is L^∞ and K_{ε_1} is L^1. We can now differentiate (2.4) twice to show that u_0 is a solution of (1.8).

Replace S by S_ε in (2.2), pass to the limit $\varepsilon \to 0$ and differentiate to get

$$\frac{1}{2}u_0^2(x) - \frac{1}{2}u_c^2 = \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1}(y) \int_0^1 u_0(x+ty)\, dt\, dy + \varepsilon_2 u_0'(x).$$

On the other hand, we can integrate (1.8) with u_0 from $-\infty$ to x to get

$$\frac{1}{2}u_0^2(x) - \frac{1}{2}u_0^2(-\infty) = \int_{\mathbb{R}} y b_\alpha K_{\varepsilon_1}(y) \int_0^1 u_0(x+ty)\, dt\, dy + \varepsilon_2 u_0'(x). \tag{2.5}$$

Thus $u_0(-\infty) = u_c$. In the last line, we used $\int_{-\infty}^x u_0'' = u_0'(x)$, which can be obtained from $u_0'' \in L^1(\mathbb{R})$, since

$$\|K_{\varepsilon_1} * u_0 - k_{\varepsilon_1} u_0\|_{L^1} \le \int_{\mathbb{R}} |y| K_{\varepsilon_1}(y) \|u_0'\|_{L^1}$$

and $\|u_0 u_0'\|_{L^1} \le u_c \|u_0'\|_{L^1}$ and any $W^{1,1}(\mathbb{R})$ function tends to zero at infinity.

In passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$, if we can show that the first three derivatives of the solution $u_{\varepsilon_1,\varepsilon_2}$ of (1.8) are uniformly bounded, then, from the Arzelà–Ascoli theorem, there is a subsequence of $u_{\varepsilon_1,\varepsilon_2}$, also denoted by $u_{\varepsilon_1,\varepsilon_2}$, such that $u_{\varepsilon_1,\varepsilon_2} \to u_0$ as $\varepsilon_1, \varepsilon_2 \to 0$ pointwise on $\mathbb{R}$ and

$$b_\alpha (K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1} u_{\varepsilon_1,\varepsilon_2}) + \varepsilon_2 u''_{\varepsilon_1,\varepsilon_2} \to (D^2)^\alpha u_0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \to 0$$

pointwise on $\mathbb{R}$ (see, for example, [9]), so that u_0 satisfies (1.4). The idea is to split

$$b_\alpha K_{\varepsilon_1} = P_{\varepsilon_1} + R_{\varepsilon_1},$$

with $R_{\varepsilon_1} \ge 0$, $P_{\varepsilon_1} \in W^{1,1}(\mathbb{R})$ and $p_{\varepsilon_1} = \int_{\mathbb{R}} P_{\varepsilon_1} = 2|\min_{x\in\mathbb{R}} u'_{\varepsilon_1,\varepsilon_2}(x)|$, by taking

$$P_{\varepsilon_1}(x) = \begin{cases} b_\alpha K_{\varepsilon_1}(x) & \text{for } x \in \mathbb{R}\backslash[-e, e], \\ b_\alpha K_{\varepsilon_1}(-e) & \text{for } x \in (e, e), \end{cases} \tag{2.6}$$

where $e = p_{\varepsilon_1}^{-1/2\alpha}(2 + 1/\alpha)^{1/2\alpha}$. Note that the minimum is attained, since $R_{\varepsilon_1} \ge 0$ and $u'_{\varepsilon_1,\varepsilon_2}(x) \to 0$ as $x \to -\infty$ from (2.5). Let $r_{\varepsilon_1} = \int_{\mathbb{R}} R_{\varepsilon_1}$. After differentiating (1.8), at the minimum,

$$-\frac{p_{\varepsilon_1}^2}{4} = P'_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} + R_{\varepsilon_1} * u'_{\varepsilon_1,\varepsilon_2} - r_{\varepsilon_1} u'_{\varepsilon_1,\varepsilon_2} + \varepsilon_2 u'''_{\varepsilon_1,\varepsilon_2} \ge P'_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2}.$$

With (2.6), this becomes

$$\frac{p_{\varepsilon_1}^2}{4} \le \frac{2u_c b_\alpha}{(2 + 1/\alpha)^{(1+2\alpha)/2\alpha}} p_{\varepsilon_1}^{(1+2\alpha)/2\alpha}.$$

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Since $\alpha > 1/2$, p_{ε_1} is bounded. However, we need to show that such a splitting exists. Note that here we are adjusting it to the solution, whereas in [9] the splitting in (1.6) was adjusted to the nonlinearity, that is, we had $J_\varepsilon = K + S_\varepsilon$ with $b_\alpha k + f' > 0$. We show that $|\min_{x\in\mathbb{R}} u'_{\varepsilon_1,\varepsilon_2}(x)|$ is of order lower than $b_\alpha k_{\varepsilon_1}$. From (2.5),

$$\varepsilon_2 |u'| \le \frac{1}{2}u_c^2 + u_c b_\alpha k_{\varepsilon_1}^{1-1/2\alpha}\Big(2 + \frac{1}{\alpha}\Big)^{(1/2\alpha)-1} \frac{2\alpha+1}{2\alpha-1}.$$

It now suffices to take $\varepsilon_2 = 1/k_{\varepsilon_1}^{\beta}$, where $\beta < 1/2\alpha$.

To estimate $\max_{x\in\mathbb{R}}|u''_{\varepsilon_1,\varepsilon_2}| = \max_{x\in\mathbb{R}} u''_{\varepsilon_1,\varepsilon_2}$, first note that this maximum is attained, since $u''_{\varepsilon_1,\varepsilon_2} \to 0$ as $x \to -\infty$ from (1.8) and $u'_{\varepsilon_1,\varepsilon_2}(x) \to 0$ as $x \to -\infty$. Use another splitting

$$b_\alpha K_{\varepsilon_1} = P + R_{\varepsilon_1}. \tag{2.7}$$

After differentiating (1.8) twice, at the maximum,

$$(p + 3u'_{\varepsilon_1,\varepsilon_2})u''_{\varepsilon_1,\varepsilon_2} \le P' * u'_{\varepsilon_1,\varepsilon_2}.$$

Since $|u'_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded, so is $|u''_{\varepsilon_1,\varepsilon_2}|$ after taking P such that $p + 3u'_{\varepsilon_1,\varepsilon_2} > 0$.

To estimate $\max_{x\in\mathbb{R}}|u'''_{\varepsilon_1,\varepsilon_2}|$, note that

$$\max_{x\in\mathbb{R}} |u'''_{\varepsilon_1,\varepsilon_2}| = \max\Big(\max_{x\in\mathbb{R}} u'''_{\varepsilon_1,\varepsilon_2}, -\min_{x\in\mathbb{R}} u'''_{\varepsilon_1,\varepsilon_2}\Big)$$

and both the maximum and minimum are attained. Using another splitting as in (2.7), after differentiating (1.8) three times, at $\max_{x\in\mathbb{R}} u''_{\varepsilon_1,\varepsilon_2}$,

$$(p + 4u'_{\varepsilon_1,\varepsilon_2})u'''_{\varepsilon_1,\varepsilon_2} \le P' * u''_{\varepsilon_1,\varepsilon_2},$$

and at $-\min_{x\in\mathbb{R}} u'''_{\varepsilon_1,\varepsilon_2}$,

$$-(p + 4u'_{\varepsilon_1,\varepsilon_2})u'''_{\varepsilon_1,\varepsilon_2} \le -P' * u''_{\varepsilon_1,\varepsilon_2} + 3u''^2_{\varepsilon_1,\varepsilon_2},$$

so, as before, $|u'''_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded.

To show that $u_0(-\infty) = u_c$, we argue as before. Integrate (1.8) from $-\infty$ to x, pass to the limit $\varepsilon_1, \varepsilon_2 \to 0$, then integrate (1.4) from $-\infty$ to x and compare the two. The only difficulty is in showing that

$$\int_{-\infty}^{x} (K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1} u_{\varepsilon_1,\varepsilon_2}) \to \int_{-\infty}^{x} (D^2)^\alpha u_0 \ \text{ as } \ \varepsilon_1 \to 0.$$

To manage the singularity, it is standard to consider separately integration on $\mathbb{R}\backslash(-1,1)$ and $(-1,1)$, that is,

$$\int_{-\infty}^{x} (K_{\varepsilon_1} * u_{\varepsilon_1,\varepsilon_2} - k_{\varepsilon_1} u_{\varepsilon_1,\varepsilon_2}) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \lim_{r\to-\infty} \int_r^x \int_{\mathbb{R}\backslash(-1,1)} yK_{\varepsilon_1}(y) \int_0^1 u'_{\varepsilon_1,\varepsilon_2}(s+ty)\,dt\,dy\,ds \\ &= \int_{\mathbb{R}\backslash(-1,1)} yK_{\varepsilon_1}(y) \int_0^1 u_{\varepsilon_1,\varepsilon_2}(x+ty)\,dt\,dy \end{aligned}$$

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and

$$I_2 = \lim_{r\to-\infty}\int_r^x \int_{-1}^1 y^2 K_{\varepsilon_1}(y)\int_0^1 (1-t)u''_{\varepsilon_1,\varepsilon_2}(s+ty)\,dt\,dy\,ds$$
$$= \int_{-1}^1 y^2 K_{\varepsilon_1}(y)\int_0^1 (1-t)u'_{\varepsilon_1,\varepsilon_2}(x+ty)\,dt\,dy.$$

By passing to the limit $\varepsilon_1, \varepsilon_2 \to 0$ in I_1 and I_2 and doing the same integrations in $\int_{-\infty}^x (D^2)^\alpha u_0$,

$$I_1 + I_2 \to \int_{-\infty}^x (D^2)^\alpha u_0.$$

Note that we can show that I_1 is finite only for $\alpha > 1/2$.

To show that $u_0' < 0$, differentiate (1.4). If $u_0'(x_{\max}) = 0$ at a point $x_{\max}$, then also $u_0''(x_{\max}) = 0$. On the other hand, $((D^2)^\alpha u_0)(x_{\max}) < 0$, which is a contradiction. To justify $((D^2)^\alpha u_0)' = (D^2)^\alpha u_0'$, it suffices that $|u_0'''|$ is bounded. After, additionally, showing that $|u''''_{\varepsilon_1,\varepsilon_2}|$ is uniformly bounded, this follows in the same way as above.

Acknowledgement

The author thanks the referee for pointing out an error in the first version of this paper and for other helpful suggestions.

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https://doi.org/10.1017/S0004972717000740 Published online by Cambridge University Press