

## COUNTABLE LENGTH EVERYWHERE CLUB UNIFORMIZATION

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**Abstract.** Assume ZF + AD and all sets of reals are Suslin. Let  $\Gamma$  be a pointclass closed under  $\wedge$ ,  $\vee$ ,  $\forall^{\mathbb{R}}$ , continuous substitution, and has the scale property. Let  $\kappa = \delta(\Gamma)$  be the supremum of the length of prewellorderings on  $\mathbb{R}$  which belong to  $\Delta = \Gamma \cap \check{\Gamma}$ . Let club denote the collection of club subsets of  $\kappa$ . Then the countable length everywhere club uniformization holds for  $\kappa$ : For every relation  $R \subseteq {}^{<\omega_1}\kappa \times \text{club}$  with the property that for all  $\ell \in {}^{<\omega_1}\kappa$  and clubs  $C \subseteq D \subseteq \kappa$ ,  $R(\ell, D)$  implies  $R(\ell, C)$ , there is a uniformization function  $\Lambda : \text{dom}(R) \rightarrow \text{club}$  with the property that for all  $\ell \in \text{dom}(R)$ ,  $R(\ell, \Lambda(\ell))$ . In particular, under these assumptions, for all  $n \in \omega$ ,  $\delta_{2n+1}^1$  satisfies the countable length everywhere club uniformization.

**§1. Introduction.** Club uniformization is a selection principle for club subsets of certain cardinals. These uniformization principles are useful in the study of combinatorics of partition measures under determinacy axioms.

If  $X \subseteq \kappa$  and  $\varepsilon \leq \kappa$ , then  $[X]_\varepsilon^*$  denote the set of increasing functions  $f : \varepsilon \rightarrow X$  which have the correct type (everywhere discontinuous and has uniform cofinality  $\omega$ ). The (correct type) partition relation  $\kappa \rightarrow_* (\kappa)_2^\varepsilon$  asserts that for all  $P : [\kappa]_\varepsilon^* \rightarrow 2$ , there exists an  $i \in 2$  and a club subset  $C \subseteq \kappa$  so that for all  $f \in [C]_\varepsilon^*$ ,  $P(f) = i$ . (The correct type partition relation is essentially equivalent to the ordinary partition relation.)

Martin showed under AD that the partition relation  $\omega_1 \rightarrow_* (\omega_1)_2^{\omega_1}$  holds. This implies that for each  $\varepsilon \leq \omega_1$ , the filter  $\mu_\varepsilon$  defined on  $[\omega_1]_\varepsilon^*$  by  $X \in \mu_\varepsilon$  if and only if there is a club  $C \subseteq \omega_1$  so that  $[C]_\varepsilon^* \subseteq X$  is a countably complete ultrafilter. (See [1] for a survey of partition relations on  $\omega_1$ .) The study of the combinatorics of the partition measures  $\mu_\varepsilon$  frequently requires the selection of clubs that witness  $\mu_\varepsilon$ -largeness or are homogeneous for partitions.

The most challenging partition measure on  $\omega_1$  is the strong partition measure  $\mu_{\omega_1}$ . There are several interesting combinatorial questions surrounding the strong partition measure. For instance, is every function  $\Phi : [\omega_1]_\varepsilon^{\omega_1} \rightarrow \omega_1$  continuous  $\mu_{\omega_1}$ -almost everywhere. Another class of questions involve the stable theory of the strong partition measure. Since for each  $\varepsilon \leq \omega_1$ ,  $\mu_\varepsilon$  is an ultrafilter, for any sentence  $\varphi$  in the language  $\{\dot{\in}, \dot{E}\}$  (where  $\dot{\in}$  is a binary relation symbol and  $\dot{E}$  is a unary relation symbol), either  $\mu_\varepsilon$ -almost all  $f$  satisfies  $L[f] \models \varphi$  or  $\mu_\varepsilon$ -almost all  $f$  satisfies  $L[f] \models \neg\varphi$ . The  $\varepsilon$ -stable theory, denoted  $\mathfrak{T}_\varepsilon$ , is the collection of sentences  $\varphi$  so that  $\mu_\varepsilon$ -almost all  $f$  satisfies  $L[f] \models \varphi$ . One can naturally ask whether important

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statements of set theory, such as GCH, belong to the stable theory of the strong partition measure  $\mathfrak{T}_{\omega_1}$ .

To answer these types of questions concerning the strong partition measure, [3, Theorem 3.10] proved under AD the almost everywhere short length club uniformization at  $\omega_1$ : Let club denote the collection of club subsets of  $\omega_1$ . For every relation  $R \subseteq [\omega_1]_*^{<\omega_1} \times \text{club}$  which is  $\subseteq$ -downward closed in the club-coordinate (which means for all  $\sigma \in [\omega_1]_*^{<\omega_1}$  and clubs  $D \subseteq E$ ,  $R(\sigma, E)$  implies  $R(\sigma, D)$ ), then there is a club  $C \subseteq \omega_1$  and a function  $\Lambda : ([C]_*^{<\omega_1} \cap \text{dom}(R)) \rightarrow \text{club}$  so that for all  $\sigma \in [C]_*^{<\omega_1} \cap \text{dom}(R)$ ,  $R(\sigma, \Lambda(\sigma))$ .

To illustrate a typical application, [3, Theorem 4.5] showed that under AD, every function  $\Phi : [\omega_1]_*^{\omega_1} \rightarrow \omega_1$  is continuous  $\mu_{\omega_1}$ -almost everywhere, which means there is a club  $C \subseteq \omega_1$  with the property that for all  $f \in [C]_*^{\omega_1}$ , there is an  $\alpha < \omega_1$  so that for all  $g \in [C]_*^{\omega_1}$ , if  $f \upharpoonright \alpha = g \upharpoonright \alpha$ , then  $\Phi(f) = \Phi(g)$ . Define a partition  $P : [\omega_1]_*^{\omega_1} \rightarrow 2$  by  $P(f) = 0$  if and only if there exists an  $\alpha < \omega_1$  so that for all clubs  $D \subseteq \omega_1$ , there exists a  $g \in [D]_*^{\omega_1}$  with  $\sup(f \upharpoonright \alpha) < g(0)$  and  $\Phi((f \upharpoonright \alpha) \hat{\ } g) < g(0)$ . Since  $\omega_1 \rightarrow_* (\omega_1)_2^{\omega_1}$ , there is a club  $C$  homogeneous for  $P$ . The most important step is to show that  $C$  is homogeneous for  $P$  taking value 0. Suppose otherwise. Define a relation  $R \subseteq [C]_*^{<\omega_1} \times \text{club}$  by  $R(\sigma, D)$  if and only if for all  $g \in [D]_*^{\omega_1}$ ,  $\Phi(\sigma \hat{\ } g) \geq g(0)$ .  $C$  being homogeneous for  $P$  taking value 1 implies  $\text{dom}(R) = [C]_*^{<\omega_1}$ . Applying the almost everywhere club uniformization to  $R$ , there is a club  $E \subseteq C$  and a function  $\Lambda : [E]_*^{<\omega_1} \rightarrow \text{club}$  so that for all  $\sigma \in [E]_*^{<\omega_1}$ ,  $R(\sigma, \Lambda(\sigma))$ . Using  $\Lambda$ , one can recursively construct a function  $h \in [E]_*^{\omega_1}$  so that for all  $\alpha < \omega_1$ ,  $R(h \upharpoonright \alpha, \Lambda(h \upharpoonright \alpha))$ . By definition of  $R$ , this means that for all  $\alpha < \omega_1$ ,  $\Phi(h) \geq h(\alpha)$ . Since  $h$  is an increasing function, this implies  $\Phi(h) \geq \omega_1$  which is impossible since  $\Phi$  takes values in  $\omega_1$ . Thus  $C$  must be homogeneous for  $P$  taking value 0 and this will eventually lead to the  $\mu_{\omega_1}$ -almost everywhere continuity of  $\Phi$ . Following this template, in forthcoming work by the authors, it is shown that many familiar statements of set theory like GCH belong to the stable theory  $\mathfrak{T}_\varepsilon$  for all  $\varepsilon \leq \omega_1$ . It is also shown that for  $\mu_\varepsilon$ -almost all  $f$ , there is a sequence of normal measures  $\bar{\nu}_f$  with a discontinuous sequence of critical points  $\bar{\kappa}$  so that  $f$  is a generic over  $L[\bar{\nu}_f]$  for a generalized Prikry forcing  $\bar{\mathbb{P}}_{\bar{\nu}_f}$  considered by Fuchs [4].

The argument in [3] to prove the almost everywhere short length club uniformization at  $\omega_1$  (although it uses just AD) appears peculiar and inefficient in that it passes first through an everywhere club uniformization principle whose argument requires generic coding, category arguments, and uniformization for certain relations on  $\mathbb{R} \times \mathbb{R}$ . [3, Theorem 3.7] shows that if  $R \subseteq [\omega_1]_*^{<\omega_1} \times \text{club}$  is a  $\subseteq$ -downward closed relation so that its coded version  $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$  has a uniformization, then there is a uniformization function  $\Lambda : \text{dom}(R) \rightarrow \text{club}$  for  $R$ .

$\text{AD}_{\mathbb{R}}$  is the determinacy of all games on  $\mathbb{R}$ .  $\text{AD}_{\frac{1}{2}\mathbb{R}}$  is the determinacy of games on  $\mathbb{R}$  where one of the two players must always make moves from  $\omega$ . Kechris [6] showed  $\text{AD}_{\frac{1}{2}\mathbb{R}}$  is equivalent to AD and all relations on  $\mathbb{R} \times \mathbb{R}$  can be uniformized. The relationship between  $\text{AD}_{\frac{1}{2}\mathbb{R}}$ ,  $\text{AD}_{\mathbb{R}}$ , and all sets of reals being Suslin is open. However, Woodin has shown that if AD and DC holds, then all three notions are equivalent.

Thus assuming  $\text{AD}_{\frac{1}{2}\mathbb{R}}$ , every  $\subseteq$ -downward closed relation  $R \subseteq [\omega_1]_*^{<\omega_1} \times \text{club}$  can be uniformized everywhere on its domain. Using the Moschovakis coding lemma,

a Martin good coding system for  ${}^{\omega_1}\omega_1$ , and the almost everywhere good code uniformization [1, Theorem 3.8], it can be shown that there is a club  $C \subseteq \omega_1$  so that  $R \cap ([C]_*^{<\omega_1} \times \text{club})$  has a coded version  $\bar{R}$  which is projective and hence uniformizable under AD. The prior result [3, Theorem 3.7] is then used to uniformize  $R \cap ([C]_*^{<\omega_1} \times \text{club})$ . Moreover, everywhere short length club uniformization is not provable under AD as it fails in  $L(\mathbb{R})$  by [3, Fact 3.9]. Thus the almost everywhere version is the best possible under AD.

Naturally one would like to study these properties at strong partition cardinals larger than  $\omega_1$  such as the next strong partition cardinal  $\delta_3^1$  (or more generally the odd projective ordinals  $\delta_{2n+1}^1$ ) or the  $\Sigma_1$ -stable ordinals  $\delta_A$  for  $L(A, \mathbb{R})$  where  $A \subseteq \mathbb{R}$ . As in [3], one would like to first prove the everywhere short length club uniformization at a strong partition cardinal  $\delta > \omega_1$ . Numerous issues with generalization quickly arise. First, more general generic coding functions exist for many cardinals beyond  $\omega_1$ ; however, these require that relevant sets possess scales. The stable ordinals  $\delta_A$  generally are not associated with pointclasses with scales. The odd projective ordinals, however, still have generic coding functions. These generic coding functions are more technical than the simple generic coding function on  $\omega_1$ , but a more substantial issue is that the generic coding function acts on  ${}^\omega\delta$ . Thus category and generic coding arguments of [3] would at best give an everywhere club uniformization for families indexed by countable sequences (which will be verified in this paper).

To obtain almost everywhere short length club uniformization at strong partition cardinals  $\delta$  greater than  $\omega_1$  under AD (or AD + DC $_{\mathbb{R}}$ ), one would need to find scale-free arguments. [2] defines a notion of a good coding family for  $\delta$  which augments a good coding system for  ${}^\delta\delta$  with a coding of the short functions on  $\delta$  which interact under strict definability conditions. Moreover, this good coding family has a continuous function which merges a code for a short function and a good code for a full function and returns a good code so the short function overrides an initial segment of the original full function. [2] shows that  $\omega_1$ , (and more generally for all  $n \in \omega$ )  $\delta_{2n+1}^1$ , and the stable ordinals  $\delta_A$  all possess very good coding families. It is then shown that a cardinal  $\delta$  that possesses a very good coding family is a strong partition cardinal which also satisfies the almost everywhere short length club uniformization at  $\delta$ .

The goal of this paper is to verify under suitable conditions that the everywhere countable length club uniformization holds at certain cardinals  $\kappa$ . That is, for every relation  $R \subseteq {}^{<\omega_1}\kappa \times \text{club}$  which is  $\subseteq$ -downward closed (where club refers to the collection of club subsets of  $\kappa$ ), there is a uniformization function  $\Lambda : \text{dom}(R) \rightarrow \text{club}$ . As mentioned above, this seems to be the best everywhere club uniformization result obtainable by the method of generic coding. In this general setting, one will encounter ordinal games so Suslin representations will be necessary to conclude the determinacy of such games. Moreover, one will need to find winning strategies uniformly which will require the ideas of the third periodicity theorem of Moschovakis. The main theorem is the following.

**THEOREM 3.8.** *Assume ZF + AD and all sets of reals are Suslin. Let  $\Gamma$  be a pointclass closed under  $\wedge, \vee$ , and  $\forall^{\mathbb{R}}$  with the scale property. Then the countable length everywhere club uniformization holds for  $\delta(\Gamma)$ . In particular, for all  $n \in \omega$ , the countable length everywhere club uniformization holds for  $\delta_{2n+1}^1$ .*

**§2. Basics.** This section will collect some standard facts concerning games, Suslin representations, and generic codings. The reader familiar with this material can skim this section for some definitions and notations and refer back as necessary.

**DEFINITION 2.1.** Let  $X$  be a set and  $\kappa$  be an ordinal.  ${}^\kappa X$  is the set of all functions from  $\kappa$  into  $X$ .  ${}^{<\kappa} X$  is the set of all functions  $\ell : \varepsilon \rightarrow X$  where  $\varepsilon < \kappa$ . If  $\ell : \varepsilon \rightarrow X$ , then let  $|\ell| = \varepsilon$  which is the length of  $\ell$ . If  $s \in {}^{<\kappa} X$  and  $t \in {}^{<\kappa} X \cup {}^\kappa X$ , one write  $s \subseteq t$  to indicate that  $t$  is an extension of  $s$ .

Let  $X$  be given the discrete topology and  ${}^\omega X$  be given the product of the discrete topology. If  $s \in {}^{<\omega} X$ , then let  $N_s^X = \{f \in {}^\omega X : s \subseteq f\}$ . The topology on  ${}^\omega X$  is equivalent to the topology generated by  $\{N_s^X : s \in {}^{<\omega} X\}$  as a basis.

A tree on  $X$  is a set  $T \subseteq {}^{<\omega} X$  which is closed under the substring relation  $\subseteq$ . If  $T$  is a tree, let  $[T] = \{f \in {}^\omega X : (\forall n)(f \upharpoonright n \in T)\}$ . A set  $A \subseteq {}^\omega X$  is closed if and only if there is a tree  $T$  so that  $A = [T]$ .

As common in descriptive set theory,  $\mathbb{R}$  may be used to denote either  ${}^\omega \omega$  or  ${}^\omega 2$ .

**DEFINITION 2.2.** A strategy on a set  $X$  is a function  $\rho : {}^{<\omega} X \rightarrow X$ . The run of a strategy  $\rho_1$  against a strategy  $\rho_2$  is denoted  $\rho_1 * \rho_2 \in {}^\omega X$  and it is defined recursively as follows: Suppose  $\rho_1 * \rho_2 \upharpoonright n$  has been defined, if  $n$  is even, then  $(\rho_1 * \rho_2)(n) = \rho_1(\rho_1 * \rho_2 \upharpoonright n)$  and if  $n$  is odd, then  $(\rho_1 * \rho_2)(n) = \rho_2(\rho_1 * \rho_2 \upharpoonright n)$ .

If  $A \subseteq {}^\omega X$ , then one says that  $A$  (or the game on  $X$  with payoff set  $A$ ) is determined if either there is a strategy  $\rho_1$  so that for all strategies  $\rho_2$ ,  $\rho_1 * \rho_2 \in A$  or there is a strategy  $\rho_2$  so that for all strategies  $\rho_1$ ,  $\rho_1 * \rho_2 \notin A$ . Intuitively, the game  $G_A^X$  consists of Player 1 and Player 2 taking turns playing elements of  $X$  where Player 1 wins if and only if the joint infinite run belongs to  $A$ . Thus the determinacy of  $A$  is the existence of a winning strategy for one of the two players in this game.

$AD_X$  is the statement that for all  $A \subseteq {}^\omega X$ ,  $A$  is determined (as a game on  $X$ ). The common determinacy axioms are  $AD_\omega$  (which is denoted simply  $AD$ ) and  $AD_{\mathbb{R}}$ .  $AD_{\frac{1}{2}\mathbb{R}}$  is the determinacy of games on  $\mathbb{R}$  where one player is required to play only elements of  $\omega$ .

If  $x \in {}^\omega X$ , let  $\rho_x$  be the strategy such that if  $s \in {}^{<\omega} \omega$  has length  $2n$  or  $2n + 1$ , then  $\rho_x(s) = x(n)$ . That is,  $\rho_x$  can be used as either a Player 1 or Player 2 strategy which simply outputs the bits of  $x$  on each turn.

If  $x \in {}^\omega X$ , let  $x_{\text{even}}, x_{\text{odd}} \in {}^\omega X$  be defined by  $x_{\text{even}}(k) = x(2k)$  and  $x_{\text{odd}}(k) = x(2k + 1)$ .

Let  $\rho$  be a strategy. Define  $\Sigma_\rho^1, \Sigma_\rho^2 : {}^\omega X \rightarrow {}^\omega X$  by  $\Sigma_\rho^1(z) = \rho * \rho_z$  and  $\Sigma_\rho^2(z) = \rho_z * \rho$ . Define  $\Xi_\rho^1 : {}^\omega X \rightarrow {}^\omega X$  by  $\Xi_\rho^1(z) = (\Sigma_\rho^1(z))_{\text{even}} = (\rho * \rho_z)_{\text{even}}$ .  $\Xi_\rho^1$  is a Lipschitz function which simply collects the moves of  $\rho$  (used as a Player 1 strategy) when played against  $\rho_z$ . Similar,  $\Xi_\rho^2 : {}^\omega X \rightarrow {}^\omega X$  is defined by  $\Xi_\rho^2(z) = (\Sigma_\rho^2(z))_{\text{odd}} = (\rho_z * \rho)_{\text{odd}}$ .

Observe that if  $\Xi : {}^\omega X \rightarrow {}^\omega X$  is a Lipschitz continuous function, then there is a strategy  $\rho$  so that  $\Xi = \Xi_\rho^1$ .

The article will work implicitly under  $ZF + AD$  and additional assumptions will be made explicit.

Next, one will review the necessary concepts concerning prewellordering and scales. See [7], [10, Chapters 2, 4, and 6], and [5, Section 2].

DEFINITION 2.3. Pointclasses are collections of subsets of  ${}^j\omega \times {}^k\mathbb{R}$  (for various  $j, k \in \omega$ ) which are closed under continuous substitution. If  $\Gamma$  is a pointclass, then  $\check{\Gamma}$  denotes the dual pointclass and  $\Delta = \Gamma \cap \check{\Gamma}$ .

DEFINITION 2.4. A norm on a set  $A$  is a map  $\phi : A \rightarrow \text{ON}$ . The associated prewellordering on  $A$  is  $\preceq_\phi \subseteq A \times A$  defined by  $x \preceq_\phi y$  if and only if  $\phi(x) \leq \phi(y)$ . (The term norm and prewellordering will be used interchangeably.) The length of the prewellordering  $\phi$  is the ordertype of  $\phi[A]$ .

Now suppose  $X$  is a set. Let  $P \subseteq {}^\omega X$  and  $\phi : P \rightarrow \text{ON}$  be a norm on  $P$ . Define  $\leq_\phi^* \subseteq {}^\omega X \times {}^\omega X$  by  $f \leq_\phi^* g$  if and only if  $f \in P \wedge (g \notin P \vee \phi(f) \leq \phi(g))$  and  $<_\phi^* \subseteq {}^\omega X \times {}^\omega X$  by  $f <_\phi^* g$  if and only if  $f \in P \wedge (g \notin P \vee \phi(f) < \phi(g))$ .

Let  $\Gamma$  be a pointclass closed under  $\wedge$  and  $\vee$ . Suppose  $P \subseteq {}^\omega\omega$  and  $\phi : P \rightarrow \text{ON}$  is a prewellordering.  $\phi$  is a  $\Gamma$ -norm if and only if  $P \in \Gamma$  and  $\leq_\phi^*, <_\phi^* \in \Gamma$ .

Let  $\delta(\Gamma)$  be the supremum of the length of all prewellorderings  $\phi$  on  ${}^\omega\omega$  such that  $\preceq_\phi \in \Delta$ .  $\delta(\Gamma)$  is called the prewellordering ordinal of  $\Gamma$ . Let  $\Theta$  be the supremum of the length of all prewellorderings on  $\mathbb{R}$ . (Every ordinal considered in this article will be below  $\Theta$ .)

DEFINITION 2.5. Fix a recursive bijection pair  $: \omega \times \omega \rightarrow \omega$ . If  $x \in {}^\omega\omega$  and  $n \in \omega$ , let  $x^{[n]} \in {}^\omega\omega$  be defined by  $x^{[n]}(k) = x(\text{pair}(n, k))$ .  $x^{[n]}$  is the  $n$ th-section of  $x$ .

If  $x \in {}^\omega 2$ , let  $\mathcal{R}_x \subseteq \omega \times \omega$  be defined by  $\mathcal{R}_x(a, b)$  if and only if  $x(\text{pair}(a, b)) = 1$ .

Let  $\text{WO}$  be the  $\Pi_1^1$ -complete set of  $w \in \mathbb{R}$  so that  $\mathcal{R}_x$  is a wellordering. Let  $\text{ot} : \text{WO} \rightarrow \omega_1$  be the ordertype function.  $\text{ot}$  is a  $\Pi_1^1$ -norm on  $\text{WO}$ . If  $w \in \text{WO}$  and  $n \in \text{field}(\mathcal{R}_w)$ , then let  $\text{ot}(w, n)$  denote the ordertype of  $n$  in  $\mathcal{R}_w$ . If  $w \in \text{WO}$  and  $\alpha < \text{ot}(w)$ , then let  $\text{num}(w, \alpha)$  be the unique element of  $\omega$  with ordertype  $\alpha$  according to  $\mathcal{R}_w$ .

DEFINITION 2.6. Let  $X$  be a set. A set  $A \subseteq {}^\omega X$  is Suslin if and only if there is an ordinal  $\delta$  and a tree  $T$  on  $X \times \delta$  so that  $A = \pi_1[[T]]$ , where  $\pi_1 : {}^\omega X \times {}^\omega\delta \rightarrow {}^\omega X$  is the projection onto the first coordinate.  $T$  is called a Suslin representation for  $A$ . A set  $A$  is co-Suslin if and only if  ${}^\omega X \setminus A$  is Suslin.

Let  $X$  be a set and  $A \subseteq {}^\omega X$ . A sequence of norms on  $A$ ,  $\bar{\phi} = \langle \phi_n : n \in \omega \rangle$ , is a semiscale if and only if for all  $f \in {}^\omega X$  and sequence  $\bar{f} = \langle f_n : n \in \omega \rangle$  of elements in  $A$  so that

- (1)  $f = \lim_{n \in \omega} f_n$  (in the natural topology on  ${}^\omega X$ ) and
- (2) for all  $n \in \omega$ , there is a  $\lambda_n \in \text{ON}$  so that  $\lim_{i \in \omega} \phi_n(f_i) = \lambda_n$  (i.e., is eventually constant taking value  $\lambda_n$ ),

one has that  $f \in A$ .

A semiscale  $\bar{\phi}$  on  $A$  is good if and only if for any sequence  $\bar{f}$  which satisfies just (2) above, there is an  $f \in {}^\omega X$  so that  $f = \lim_{n \in \omega} f_n$ . A semiscale is very good if and only if it is good and for all  $x, y \in A$  and  $n \in \omega$ ,  $\phi_n(x) \leq \phi_n(y)$  implies that for all  $m \leq n$ ,  $\phi_m(x) \leq \phi_m(y)$ .

A semiscale  $\phi$  on  $A$  is a scale if and only if it satisfies the lower semicontinuity property: Using the notations of (1) and (2) above, for all  $n \in \omega$ ,  $\phi_n(f) \leq \lambda_n$ .

Every semiscale  $\bar{\phi}$  on  $A$  yields a Suslin representation for  $A$ . Suppose a tree  $T$  on  $X \times \delta$  is a Suslin representation for  $A$ . If  $f \in A$ , then the tree  $T_f = \{u \in {}^{<\omega}\delta : (f \upharpoonright |u|, u) \in T\}$  has an infinite path so let  $L_f^T$  denote the leftmost

path of  $T_f$ . The canonical semiscale for  $A$  derived from  $T$  is  $\bar{\varphi}^T = \langle \varphi_n^T : n \in \omega \rangle$  defined by  $\varphi_n^T(f) = L_f^T(n)$ . (See [10, Theorem 2B.1].) Also every semiscale on  $A$  can be converted into a very good semiscale by a standard procedure (see [10, Lemma 4E.2]).

**DEFINITION 2.7.** Let  $R \subseteq \mathbb{R} \times \mathbb{R}$ . A uniformization for  $R$  is a function  $\Phi : \text{dom}(R) \rightarrow \mathbb{R}$  so that for all  $x \in \text{dom}(R)$ ,  $R(x, \Phi(x))$ . Let Uniformization be the statement that every relation  $R \subseteq \mathbb{R} \times \mathbb{R}$  has a uniformization.

By a game argument,  $\text{AD}_{\frac{1}{2}\mathbb{R}}$  implies Uniformization. Since  $\text{AD}_{\mathbb{R}}$  implies  $\text{AD}_{\frac{1}{2}\mathbb{R}}$ ,  $\text{AD}_{\mathbb{R}}$  also proves Uniformization. Kechris [6] showed that over  $\text{AD}$ ,  $\text{AD}_{\frac{1}{2}\mathbb{R}}$  and Uniformization are equivalent. If  $R \subseteq \mathbb{R} \times \mathbb{R}$  has a Suslin representation, then  $R$  has a uniformization. Thus every set having a Suslin representation implies Uniformization.  $\text{AD}$  alone cannot establish any of these principles since  $L(\mathbb{R}) \models \neg \text{AD}_{\frac{1}{2}\mathbb{R}}$  because the relation  $S(x, y)$  if and only if  $y$  is not  $\text{OD}_x$  cannot be uniformized in  $L(\mathbb{R})$ . It is open whether these principles are equivalent over  $\text{AD}$ ; however, Woodin has shown they are equivalent over  $\text{AD} + \text{DC}$ .

In this article, one will be concerned about the determinacy of certain games on ordinals. Generally, the determinacy of all games on uncountable ordinals is not consistent. However, the following result states that games with Suslin and co-Suslin payoff sets are determined.

**FACT 2.8** [9, Theorem 2.8]. *Suppose  $\kappa < \Theta$ . Let  $A \subseteq {}^\omega \kappa$  and suppose that  $A$  is Suslin and co-Suslin. Then the game on  $\kappa$  with payoff set  $A$  is determined.*

To apply Fact 2.8, one will need to show some relevant ordinal games have Suslin and co-Suslin payoff sets. Moreover, it will be very important in certain instances to have that the Suslin representations are obtained uniformly from certain objects. Next, one will give some closure properties of Suslin representations with a particular focus on uniformity.

**FACT 2.9.** *The class of Suslin subsets of  ${}^\omega \kappa$  contains the open and closed subsets of  ${}^\omega \kappa$  and is closed under countable unions, countable intersection, and projections. The Cartesian product of two Suslin subset of  ${}^\omega \kappa$  is a Suslin subset of  ${}^\omega \kappa \times {}^\omega \kappa$ .*

**FACT 2.10.** *Let  $\kappa$  be a cardinal. Let  $\varepsilon < \omega_1$ ,  $w \in \text{WO}$  with  $\text{ot}(w) = \varepsilon$ ,  $v < \kappa$ , and  $\ell : \varepsilon \rightarrow v$ . Define a relation  $R_\ell^v \subseteq {}^\omega v$  by  $R_\ell^v(g)$  if and only if  $\text{rang}(\ell) \subseteq \text{rang}(g)$ . Then  $R_\ell^v$  is Suslin and co-Suslin uniformly in  $\ell$ ,  $v$ , and  $w$ . The term “uniformly” means there are functions  $\mathfrak{T}$  and  $\mathfrak{U}$  so that whenever  $\ell$ ,  $w$ , and  $v$  have the above property,  $\mathfrak{T}(\ell, w, v)$  and  $\mathfrak{U}(\ell, w, v)$  are trees on  $v \times \kappa$ ,  $R_\ell^v = \pi_1[[\mathfrak{T}(\ell, w, v)]]$ , and  ${}^\omega v \setminus R_\ell^v = \pi_1[[\mathfrak{U}(\ell, w, v)]]$  where  $\pi_1 : {}^\omega v \times {}^\omega \kappa \rightarrow {}^\omega v$  is the projection onto the first coordinate.*

**PROOF.** Fix a bijection  $\Upsilon : \kappa \rightarrow <^\omega \kappa$  with the property that for all  $s, t \in <^\omega \kappa$ , if  $s \subseteq t$ , then  $\Upsilon^{-1}(s) \leq \Upsilon^{-1}(t)$ . If  $s, t \in <^\omega v$ , then say that  $s$  is compatible with  $t$  if  $s \subseteq t$  or  $t \subseteq s$ . Recall from Definition 2.5, if  $n \in \text{field}(\mathcal{R}_w)$  where  $\mathcal{R}_w$  is the wellordering coded by  $w \in \text{WO}$ , then  $\text{ot}(w, n)$  is the rank of  $n$  in  $\mathcal{R}_w$ .

Let  $S_\ell^v = {}^\omega v \setminus R_\ell^v$ . Note that  $S_\ell^v$  is a countable union of closed sets in the topology of  ${}^\omega v$  and thus has the following simple Suslin representation. Let  $U$  be a tree on  ${}^\omega v \times \omega$  defined by  $(s, u) \in U$  if and only if  $|s| = |u| = 0$  or there exists an  $m \in \text{field}(\mathcal{R}_w)$  so that  $u$  is the constant sequence taking value  $m$  and  $\ell(\text{ot}(w, m))$

$\notin \text{rang}(s)$ . If  $m \in \omega$ , let  $\bar{m} \in {}^\omega\omega$  be the constant infinite sequence taking value  $m$ . Note that if  $g \in \pi_1[[U]]$  then there exists an  $m \in \omega$  so that  $(g, \bar{m}) \in [U]$ . So for all  $n \in \omega$ ,  $(g \upharpoonright n, \bar{m} \upharpoonright n) \in U$  which implies that for all  $n \in \omega$ ,  $\ell(\text{ot}(w, m)) \notin \text{rang}(g \upharpoonright n)$ . Hence  $\ell(\text{ot}(w, m)) \notin \text{rang}(g)$  and  $\neg(\text{rang}(\ell) \subseteq \text{rang}(g))$ . Thus  $S_\ell^v(g)$ . Conversely, suppose  $S_\ell^v(g)$  which means  $\neg(\text{rang}(\ell) \subseteq \text{rang}(g))$ . Since  $\text{ot}(w) = \varepsilon = |\ell|$ , there is some  $m \in \text{field}(\mathcal{R}_w)$  so that  $\ell(\text{ot}(w, m)) \notin \text{rang}(g)$ . Then  $(g, \bar{m}) \in [U]$  and thus  $g \in \pi_1[[U]]$ . This shows that  $S_\ell^v$  is Suslin and therefore  $R_\ell^v$  is co-Suslin.

Define a tree  $T$  on  $v \times \kappa$  by  $(s, u) \in T$  if and only if for all  $k < |s|$ , the following holds:

- $\Upsilon(u(k)) \in <{}^\omega v$ .
- If  $k \notin \text{field}(\mathcal{R}_w)$ , then  $\Upsilon(u(k)) = \emptyset$ .
- If  $k \in \text{field}(\mathcal{R}_w)$ , then  $\ell(\text{ot}(w, k)) \in \text{rang}(\Upsilon(u(k)))$ .
- $\Upsilon(u(k))$  is compatible with  $s$ .

Suppose  $g \in \pi_1[[T]]$ . Then there exists an  $h \in {}^\omega\kappa$  so that  $(g, h) \in [T]$ . Thus for all  $k \in \omega$ ,  $\Upsilon(h(k)) \subseteq g$  and  $\ell(\text{ot}(w, k)) \in \text{rang}(\Upsilon(h(k)))$ . Hence  $\text{rang}(\ell) \subseteq \text{rang}(g)$  which is equivalent to  $R_\ell^v(g)$ . Conversely, suppose  $R_\ell^v(g)$ . Then  $\text{rang}(\ell) \subseteq \text{rang}(g)$ . If  $n \in \text{field}(\mathcal{R}_w)$ , let  $k_n$  be least  $k \in \omega$  with  $k > 0$  so that  $g(k-1) = \ell(\text{ot}(w, n))$ . If  $n \notin \text{field}(\mathcal{R}_w)$ , then let  $k_n = 0$ . Let  $h(n) = \Upsilon^{-1}(g \upharpoonright k_n)$ . Then  $(g, h) \in [T]$ . Thus  $g \in \pi_1[[T]]$ . Observe that this explicit  $h$  is actually the leftmost branch,  $L_g^T$ , of  $T$  corresponding to  $g$ . It has been shown that  $g \in \pi_1[[T]]$  if and only if  $R_\ell^v(g)$ , so  $R_\ell^v$  is Suslin.

Observe that both trees  $U$  and  $T$  are produced uniformly from  $\ell$ ,  $v$ , and  $w$ . Let  $\mathfrak{U}(\ell, w, v) = U$  and  $\mathfrak{T}(\ell, w, v) = T$ . ⊖

**FACT 2.11.** *Assume the setting of Fact 2.10. Let  $\bar{\varphi}^{\ell, w, v} = \langle \varphi_n^{\ell, w, v} : n \in \omega \rangle$  be the canonical semiscale derived from  $\mathfrak{T}(\ell, w, v)$  (using the leftmost branch as in Definition 2.6) and  $\bar{\zeta}^{\ell, w, v} = \langle \zeta_n^{\ell, w, v} : n \in \omega \rangle$  be the canonical semiscale derived from  $\mathfrak{U}(\ell, w, v)$ . For all  $n \in \omega$ , the norm relations  $\leq_{\varphi_n^{\ell, w, v}}^*$ ,  $<_{\varphi_n^{\ell, w, v}}^*$ ,  $\leq_{\zeta_n^{\ell, w, v}}$ , and  $<_{\zeta_n^{\ell, w, v}}^*$  are Suslin and co-Suslin.*

**PROOF.** The notation from the statement and proof of Fact 2.10 will be used. Note that from the definition of  $\mathfrak{U}(\ell, w, v)$  from Fact 2.10, the leftmost branch  $L_g^{\mathfrak{U}(\ell, w, v)}$  is simply  $\bar{m}$  where  $m$  is least so that  $\ell(\text{ot}(w, n)) \notin \text{rang}(g)$ .

For each  $m \in \text{field}(\mathcal{R}_w)$ , let  $E_m = \{g \in {}^\omega v : (\forall k < m)(k \in \text{field}(\mathcal{R}_w) \Rightarrow \ell(\text{ot}(w, k)) \in \text{rang}(g)) \wedge \ell(\text{ot}(w, m)) \notin \text{rang}(g)\}$ . If  $m \notin \text{field}(\mathcal{R}_w)$ , then let  $E_m = \emptyset$ . One can check that  $E_m$  is Suslin and co-Suslin using arguments similar to Fact 2.10. Observe that for any  $k \in \omega$

$$\leq_{\zeta_k^{\ell, w, v}}^* = \left( \bigcup_{m \leq n} E_m \times E_n \right) \cup \left( \bigcup_{m \in \omega} E_m \times R_\ell^v \right)$$

and

$$<_{\zeta_k^{\ell, w, v}}^* = \left( \bigcup_{m < n} E_m \times E_n \right) \cup \left( \bigcup_{m \in \omega} E_m \times R_\ell^v \right).$$

These norm relations are Suslin and co-Suslin by Fact 2.9, Fact 2.10, and the earlier observations.

In the proof of Fact 2.10, one showed that if  $f \in \pi_1[[\mathfrak{I}(\ell, w, v)]]$ , then the leftmost branch  $L_f^{\mathfrak{I}(\ell, w, v)}$  is explicitly given by the following: For each  $n \in \omega$ , let  $k_n$  be the least  $k \in \omega$  such that  $f(k - 1) = \ell(\text{ot}(w, n))$  if  $n \in \text{field}(\mathcal{R}_w)$  and  $k_n = 0$  if  $n \notin \text{field}(\mathcal{R}_w)$ . Then  $L_f^{\mathfrak{I}(\ell, w, v)}(n) = \Upsilon^{-1}(f \upharpoonright k_n)$ .

For each  $n \in \text{field}(\mathcal{R}_w)$ , let  $A_n \subseteq \kappa$  be the collection of  $\gamma$  so that  $|\Upsilon(\gamma)| > 0$ ,  $\Upsilon(\gamma)(|\Upsilon(\gamma)| - 1) = \ell(\text{ot}(w, n))$ , and for all  $i < |\Upsilon(\gamma)| - 1$ ,  $\Upsilon(\gamma)(i) \neq \ell(\text{ot}(w, n))$ . Define a tree  $K_n$  on  $v \times v \times \kappa \times \kappa$  by  $(s, t, u, v) \in K_n$  if and only if  $u(0) \leq v(0)$ ,  $u(0), v(0) \in A_n$ ,  $\Upsilon(u(0))$  is compatible with  $s$ , and  $\Upsilon(v(0))$  is compatible with  $t$ . Define a tree  $J_n$  similarly with  $u(0) \leq v(0)$  replaced with  $u(0) < v(0)$ .

Note that if  $f, g \in R_\ell^v$ , then  $\varphi_n^{\ell, w, v}(f) \leq \varphi_n^{\ell, w, v}(g)$  if and only if  $L_f^{\mathfrak{I}(\ell, w, v)}(n) \leq L_g^{\mathfrak{I}(\ell, w, v)}(n)$  if and only if  $(\exists x, y)((f, g, x, y) \in [K_n])$ . Similarly, if  $f, g \in R_\ell^v$ , then  $\varphi_n^{\ell, w, v}(f) < \varphi_n^{\ell, w, v}(g)$  if and only if  $L_f^{\mathfrak{I}(\ell, w, v)}(n) < L_g^{\mathfrak{I}(\ell, w, v)}(n)$  if and only if  $(\exists x, y)((f, g, x, y) \in [J_n])$ .

Note that  $f \leq_{\varphi_n^{\ell, w, v}}^* g$  if and only if

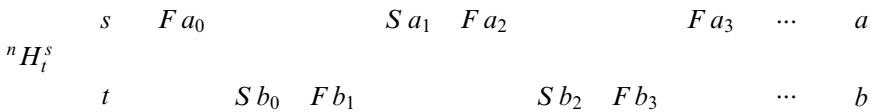
$$f \in R_\ell^v \wedge (g \notin R_\ell^v \vee (\exists x, y)((f, g, x, y) \in [K_n])).$$

Also  $\neg(f \leq_{\varphi_n^{\ell, w, v}}^* g)$  if and only if

$$f \notin R_\ell^v \vee (g \in R_\ell^v \wedge (\exists x, y)((g, f, x, y) \in [J_n])).$$

This shows that  $\leq_{\varphi_n^{\ell, w, v}}^*$  is Suslin and co-Suslin by Fact 2.9 and Fact 2.10. A similar argument shows that  $<_{\varphi_n^{\ell, w, v}}^*$  is also Suslin and co-Suslin. ⊥

FACT 2.12. [10, 6E]; *Moschovakis Third Periodicity Theorem*) Assume  $\text{AD} + \text{DC}_{\mathbb{R}}$ . Let  $v < \Theta$ . Let  $A \subseteq {}^\omega v$  be Suslin and  $\bar{\varphi} = \langle \varphi_n : n \in \omega \rangle$  be a very good semiscale on  $A$ . For each  $n \in \omega$ , odd  $m \in \omega$ , and  $s, t \in {}^m v$ , define the game  ${}^n H_t^s$  on  $v$  as in the following diagram.



The game has two players called the first player and second player making moves in  $v$  as indicated in the diagram. Say that the second player wins if and only if  $\hat{s} a \leq_{\varphi_n}^* \hat{t} b$ .

Assume that Player 1 has a winning strategy in the game  $G_A^v$  on  $v$  with payoff set  $A$ . Assume for all  $n \in \omega$ , odd  $m \in \omega$ , and  $s, t \in {}^m v$ , the games  ${}^n H_t^s$  are determined. Then uniformly from  $v$ ,  $A$ , and the very good semiscale  $\bar{\varphi}$ , one can obtain a strategy  $\sigma$  for Player 1 in  $G_A^v$ . (This means there is a function  $\Phi$  so that whenever  $v$ ,  $A$ , and  $\bar{\varphi}$  have the above property,  $\Phi(v, A, \bar{\varphi})$  is a Player 1 winning strategy for  $G_A^v$ .)

PROOF. This result is essentially a coarse form of the Moschovakis third periodicity theorem for ordinal value games using the idea of the “best” strategy. (The definability estimates for the strategy will not be relevant here.) The uniformity statement will be essential so an explicit definition of the Player 1 winning strategy will be provided. The reader can see [10, 6D and 6E] or [5, Section 2] for the details.



For each odd  $m$ , let  $W_m$  be the set of  $s \in {}^m v$  so that Player 1 can win in the game  $G_A^v$  when  $s$  is an initial partial run ( $m$  being odd implies that Player 2 is the next player to respond). Since Player 1 is assumed to have a winning strategy, for all odd  $m \in \omega$ ,  $W_m$  is nonempty. For  $s, t \in W_m$ , define  $s \preceq_n^m t$  if and only if the second player has a winning strategy in  ${}^n H_t^s$ . It can be shown that  $\preceq_n^m$  is a prewellordering on  $W_m$ .

If  $k \in \omega$  and  $u \in {}^{2k} v$ , then define  $\sigma(u)$  to be the least  $\gamma < v$  so that  $u \hat{\gamma} \in W_{2k+1}$  and for all  $\eta$  with  $u \hat{\eta} \in W_{2k+1}$ ,  $u \hat{\gamma} \preceq_k^{2k+1} u \hat{\eta}$ . (If  $u$  is odd length, then let  $\sigma(u) = 0$  as this case is irrelevant because  $\sigma$  is intended to be used as a Player 1’s strategy.) It can be shown that  $\sigma$  is a Player 1’s winning strategy in  $G_A^v$  and is produced uniformly from  $v, A$ , and  $\check{\varphi}$ . ⊣

**FACT 2.13.** *Let  $v < \Theta$ . Let  $\Xi : {}^\omega v \rightarrow {}^\omega \omega$  be a Lipschitz continuous function. Suppose  $\Gamma$  is a pointclass and  $\Delta = \Gamma \cap \check{\Gamma}$ . Suppose there is a norm  $\varphi : W \rightarrow v$  so that  $W \in \Delta$  and the associated prewellordering  $\preceq_\varphi$  on  $W$  is also in  $\Delta$ . Then  $\Xi[{}^\omega v]$  is  $\exists^{\mathbb{R}} \Delta$ .*

**PROOF.** First, a simple coding of  ${}^{<\omega} v$  by reals will be developed. Let  $\text{finS}$  consist of reals  $z$  so that  $(\forall i < z^{[0]}(0))(z^{[i+1]} \in W)$ . Let  $\text{finseq} : \text{finS} \rightarrow {}^{<\omega} v$  by  $\text{finseq}(z)$  be a sequence of length  $z^{[0]}(0)$  and for all  $i < z^{[0]}(0)$ ,  $\text{finseq}(z)(i) = \varphi(z^{[i+1]})$ . Note that  $\text{finseq}$  is a surjection of  $\text{finS}$  onto  ${}^{<\omega} v$ . The set  $\text{finS}$  is  $\Delta$ . The expression “ $u \in \text{finS}$ ,  $i < |\text{finseq}(u)|$ ,  $w \in W$ , and  $\text{finseq}(u)(i) = \varphi(w)$ ” is  $\Delta$  as a relation in the variables  $u, i$ , and  $w$ .

Fix a bijection  $\Upsilon : v \rightarrow {}^{<\omega} v$ . Next one will show that  $\Upsilon$  has a coded version which is  $\exists^{\mathbb{R}} \Delta$ . Define  $Z \subseteq W \times \text{finS}$  by  $Z(w, u)$  if and only if  $\Upsilon(\varphi(w)) = \text{finseq}(u)$ . By the Moschovakis coding lemma ([10, Section 7D] or [5, Theorem 2.12]), there is a  $\check{Z} \in \exists^{\mathbb{R}} \Delta$  so that  $\check{Z} \subseteq Z$  and for all  $\alpha < v$ ,  $\check{Z} \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$  if and only if  $Z \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$ .

Since  $\Xi$  is Lipschitz, there is a function  $\tau : {}^{<\omega} v \rightarrow \omega$  so that  $\Xi = \Xi_\tau^2$  using the notation of Definition 2.2. Define  $Y \subseteq W \times \omega$  by  $Y(w, n)$  if and only if  $\tau(\Upsilon(\varphi(w))) = n$ . By the Moschovakis coding lemma, there is a  $\check{Y} \in \exists^{\mathbb{R}} \Delta$  so that  $\check{Y} \subseteq Y$  and for all  $\alpha < v$ ,  $\check{Y} \cap (\varphi^{-1}[\{\alpha\}] \times \omega) \neq \emptyset$  if and only if  $Y \cap (\varphi^{-1}[\{\alpha\}] \times \omega) \neq \emptyset$ .

Now observe that  $x \in \Xi[{}^\omega v]$  if and only if there is an  $f \in {}^\omega v$  so that  $x = \Xi_\tau^2(f)$  if and only if there is a  $y \in \mathbb{R}$  coding  $f$  in the sense that each  $y^{[n]}$  codes  $f(n)$  and  $\Xi_\tau^2(f) = x$  as expressed by the coding through  $\check{Y}$  and  $\check{Z}$ . Formally,  $x \in \Xi[{}^\omega v]$  if and only if the conjunction of the following holds.

- (1) For all  $n \in \omega$ ,  $y^{[n]} \in W$ .
- (2) For all  $n \in \omega$ , there exist  $v, w \in W$  and there exists a  $u \in \text{finS}$  so that  $\varphi(v) = \varphi(w)$ ,  $|\text{finseq}(u)| = n$ , and for all  $i < |\text{finseq}(u)|$ ,  $\text{finseq}(u)(i) = \varphi(y^{[i]})$ ,  $\check{Z}(v, u)$ , and  $\check{Y}(w, x(n))$ .

The above expression is  $\exists^{\mathbb{R}} \Delta$  by using the above observations. ⊣

**FACT 2.14.** (*Boundedness Principle*) *Suppose  $\Gamma$  is a pointclass closed under  $\forall^{\mathbb{R}}$ . Suppose  $W \in \Gamma$  is a  $\Gamma$ -complete set with a surjective  $\Gamma$ -norm  $\varphi : W \rightarrow \kappa$ . If  $A \subseteq W$  and  $A \in \check{\Gamma}$ , then there is a  $\delta < \kappa$  so that  $\varphi[A] \subseteq \delta$ .*

**FACT 2.15.** (*Moschovakis [5, Theorem 2.6 and Lemma 2.13]*) *Let  $\Gamma$  be a pointclass closed under  $\wedge, \vee$ , and  $\forall^{\mathbb{R}}$  with a  $\Gamma$ -complete set  $W$  and a  $\Gamma$ -norm  $\varphi$  on  $W$ . Then the length of  $\varphi$  is  $\delta(\Gamma)$  and  $\delta(\Gamma)$  is a regular cardinal.*

Recall that under AD, Wadge’s lemma implies that every nonselfdual pointclass has complete sets. Solovay club coding techniques for  $\omega_1$  under AD can be extended to  $\delta(\Gamma)$  when  $\Gamma$  is a nonselfdual pointclass closed under  $\forall^{\mathbb{R}}$ .

FACT 2.16. *Let  $\Gamma$  be a nonselfdual pointclass closed under  $\forall^{\mathbb{R}}$ . Let  $\kappa = \delta(\Gamma)$ . Let  $W$  be a  $\Gamma$ -complete set with surjective  $\Gamma$ -norm  $\varphi : W \rightarrow \kappa$ . Let  $\text{clubcode} \subseteq \mathbb{R}$  consist of the strategies  $\tau$  with the property*

$$(\forall w)(w \in W \Rightarrow (\exists \tau^2(w) \in W \wedge \varphi(\exists \tau^2(w)) > \varphi(w))).$$

If  $\tau \in \text{clubcode}$ , then let

$$\mathfrak{C}_\tau = \{\eta < \kappa : (\forall w \in W)(\varphi(w) < \eta \Rightarrow \varphi(\exists \tau^2(w)) < \eta)\}.$$

$\mathfrak{C}_\tau$  is a club. If  $C \subseteq \kappa$  is club, then there is a  $\tau \in \text{clubcode}$  so that  $\mathfrak{C}_\tau \subseteq C$ .

If  $A \subseteq \text{clubcode}$  is  $\check{\Gamma}$ , then uniformly in  $A$ , one can produce a club  $C$  so that for all  $\tau \in A$ ,  $C \subseteq \mathfrak{C}_\tau$ . (Uniformly here means there is a function  $\Upsilon$  so that whenever  $A \subseteq \text{clubcode}$  is  $\check{\Gamma}$ ,  $\Upsilon(A)$  is club with the property that for all  $\tau \in A$ ,  $\Upsilon(A) \subseteq \mathfrak{C}_\tau$ .)

PROOF. These are proved using the boundedness principle (Fact 2.14). See [1, Fact 4.7] for a similar argument. ⊣

DEFINITION 2.17. Let  $\alpha \in \omega_1$ . If  $s \in <^\omega \alpha$ , let  $N_s^\alpha = \{f \in {}^\omega \alpha : s \subseteq f\}$ . The topology on  ${}^\omega \alpha$  generated by  $\{N_s^\alpha : s \in <^\omega \alpha\}$  as a basis is homeomorphic to  ${}^\omega \omega$ . Thus the familiar category notion can be formulated for  ${}^\omega \alpha$  in this topology. Let  $\text{surj}_\alpha$  be the collection of  $f \in {}^\omega \alpha$  such that  $f[\omega] = \alpha$ , i.e.,  $f$  is a surjection onto  $\alpha$ .  $\text{surj}_\alpha$  is comeager in  ${}^\omega \alpha$ .

Recall that under AD, the category ideal has full wellordered additivity. That is, if  $\delta$  is an ordinal and  $\langle X_\alpha : \alpha < \delta \rangle$  is a collection of meager subsets of  $\mathbb{R}$ , then  $\bigcup_{\alpha < \delta} X_\alpha$  is a meager subset of  $\mathbb{R}$ . Thus the meager ideal on  ${}^\omega \alpha$  also has the full wellordered additivity.

The following is the simplest example of the Kechris–Woodin generic coding function occurs at  $\omega_1$ .

FACT 2.18. *There is a function  $\mathfrak{G} : {}^\omega \omega_1 \rightarrow \text{WO}$  so that for all  $\alpha < \omega_1$ , if  $f \in \text{surj}_\alpha$ , then  $\text{ot}(\mathfrak{G}(f)) = \alpha$ .*

PROOF. Let  $A_f = \{n \in \omega : (\forall m)(m < n \Rightarrow f(m) \neq f(n))\}$ . Define  $\mathfrak{G}(f) \in \mathbb{R}$  so that  $\mathcal{R}_{\mathfrak{G}(f)}(a, b) = 1 \Leftrightarrow a, b \in A_f \wedge f(a) < f(b)$ . Note that the domain of  $\mathcal{R}_{\mathfrak{G}(f)}$  is  $A_f$  and  $\mathfrak{G}(f) \in \text{WO}$ . If  $f \in \text{surj}_\alpha$ , then  $(A_f, \mathcal{R}_{\mathfrak{G}(f)})$  is order-isomorphic to  $\alpha$ . ⊣

The following results are generalizations of the category boundedness arguments found in the proof of the main theorems in [3].

FACT 2.19. *Let  $\nu < \omega_1$  and  $\kappa$  be a cardinal with  $\text{cof}(\kappa) > \omega$ . Suppose  $A \subseteq {}^\omega \nu$  is comeager in  ${}^\omega \nu$  and  $\Phi : A \rightarrow \kappa$ . Then there is a  $\delta < \kappa$  and a comeager  $B \subseteq A$  so that  $\Phi[B] \subseteq \delta$ .*

PROOF. For each  $\alpha < \kappa$ , let  $A_\alpha = \{f \in A : \Phi(f) = \alpha\}$ . Note that  $A = \bigcup_{\alpha < \kappa} A_\alpha$ . Let  $T = \{\alpha < \kappa : A_\alpha \text{ is nonmeager}\}$ . AD implies that a wellordered union of meager sets in  ${}^\omega \nu$  is meager and since  $A$  is not meager,  $T \neq \emptyset$ . Since AD implies that all sets of reals have the Baire property and there are no uncountable sets of disjoint open

subsets of  ${}^\omega v$ , one has that  $T$  must be countable. Since  $\text{cof}(\kappa) > \omega$ ,  $\text{sup}(T) < \kappa$ . Let  $\delta = \text{sup}(T) + 1 < \kappa$ . Note that  $B = \bigcup_{\alpha < \delta} A_\alpha$  is comeager. By definition of  $B$ ,  $\Phi[B] \subseteq \delta$ . ⊣

**FACT 2.20.** *Let  $v < \omega_1$  and  $\kappa$  be cardinal with  $\text{cof}(\kappa) > \omega$ . Let club be the set of club subsets of  $\kappa$ . Suppose  $A \subseteq {}^\omega v$  is comeager and  $\Phi : A \rightarrow \text{club}$ . Then uniformly from  $\Phi$ , there is a club  $C \subseteq \kappa$  and a comeager set  $B \subseteq A$  so that for all  $f \in B$ ,  $C \subseteq \Phi(f)$ .*

**PROOF.** If  $X \subseteq \kappa$  and  $|X| = \kappa$ , then let  $\text{enum}_X : \kappa \rightarrow X$  be the increasing enumeration of  $X$ . For  $\alpha < \kappa$ , let  $E_\alpha : A \rightarrow \kappa$  be defined by  $E_\alpha(f) = \text{enum}_{\Phi(f)}(\alpha)$ . For  $\alpha < \delta < \kappa$ , let  $Y_\alpha^\delta = \{f \in A : E_\alpha(f) < \delta\}$ . Define  $K : \kappa \rightarrow \kappa$  by letting  $K(\alpha)$  be the least  $\delta$  so that  $Y_\alpha^\delta$  is comeager. Note that for each  $\alpha < \kappa$ ,  $K(\alpha)$  is well defined by Fact 2.19 applied to the function  $E_\alpha$ . Since for any  $f \in A$ ,  $E_\alpha(f) \geq \alpha$ , one has that  $K(\alpha) > \alpha$ . Also note that for any  $\alpha_0 \leq \alpha_1$  and  $f \in A$ ,  $E_{\alpha_0}(f) \leq E_{\alpha_1}(f)$  and thus  $K(\alpha_0) \leq K(\alpha_1)$ .

Let  $C = \{\eta < \kappa : (\forall \xi < \eta)(K(\xi) < \eta)\}$ . (Note that  $C$  is produced uniformly from  $\Phi$ .) First, to show  $C$  is unbounded. Let  $\alpha < \kappa$ . Let  $\alpha_0 = \alpha$ . If  $\alpha_n$  has been defined, then let  $\alpha_{n+1} = K(\alpha_n)$ . By the property of  $K$  mentioned above,  $\langle \alpha_n : n \in \omega \rangle$  is a strictly increasing sequence in  $\kappa$ . Let  $\eta = \text{sup}\{\alpha_n : n \in \omega\}$  and note that  $\alpha < \eta < \kappa$  since  $\text{cof}(\kappa) > \omega$ . Let  $\xi < \eta$  be arbitrary. There is an  $n \in \omega$  so that  $\xi < \alpha_n$ . Since  $K(\xi) \leq K(\alpha_n) = \alpha_{n+1} < \eta$ , one has that  $K(\xi) < \eta$ . Since  $\xi < \eta$  was arbitrary,  $\eta \in C$ . Next to show  $C$  is closed. Suppose  $\eta$  is a limit point of  $C$ . Let  $\xi < \eta$  be arbitrary. Then there is an  $\eta' \in C$  with  $\xi < \eta' < \eta$ . Thus  $K(\xi) < \eta' < \eta$ . Since  $\xi < \eta$  was arbitrary, one has that  $\eta \in C$ . It has been established that  $C$  is a club subset of  $\kappa$ .

Fix  $\eta \in C$ . For all  $\xi < \eta$ ,  $K(\xi) < \eta$  so  $Y_\xi^\eta$  is comeager. Let  $Y^\eta = \bigcap_{\xi < \eta} Y_\xi^\eta$ . Since wellordered intersection of comeager sets are comeager under AD,  $Y^\eta$  is comeager. Note that for each  $f \in Y^\eta$  and  $\xi < \eta$ ,  $\xi \leq E_\xi(f) < \eta$ . Since  $\xi < \eta$  is arbitrary,  $E_\xi(f) = \text{enum}_{\Phi(f)}(\xi) \in \Phi(f)$ , and  $\Phi(f) \subseteq \kappa$  is a club, one must have that  $\eta \in \Phi(f)$ . Thus for all  $f \in Y^\eta$ ,  $\eta \in \Phi(f)$ . Now let  $Y = \bigcap_{\eta \in C} Y^\eta$ . Again since a wellordering intersection of comeager sets is comeager under AD,  $Y$  is comeager. Take any  $f \in Y$ . For any  $\eta \in C$ ,  $f \in Y^\eta$ . By the previous observation,  $\eta \in \Phi(f)$ . Since  $\eta \in C$  was arbitrary, one has shown that  $C \subseteq \Phi(f)$ . ⊣

Next, the more general notion of a reliable ordinal and its associated Kechris–Woodin generic coding function will be defined.

**DEFINITION 2.21.** [8] An ordinal  $\lambda$  is reliable if and only if there is a  $W \subseteq \mathbb{R}$  and a scale  $\bar{\varphi} = \langle \varphi_i : i \in \omega \rangle$  on  $W$  with the following properties.

- For all  $n \in \omega$ ,  $\varphi_n : W \rightarrow \lambda$  and  $\varphi_0 : W \rightarrow \lambda$  is surjective.
- The norm relations  $\leq_{\varphi_0}^*$  and  $<_{\varphi_0}^*$  are Suslin and co-Suslin.

$(W, \bar{\varphi})$  is called the witness to the reliability of  $\lambda$ .

For  $\xi < \lambda$ , say that  $S \subseteq \lambda$  is  $\xi$ -honest if and only if there is a  $w \in W$  such that  $\varphi_0(w) = \xi$  and for all  $n \in \omega$ ,  $\varphi_n(w) \in S$ . A set  $S \subseteq \lambda$  is honest if and only if for all  $\xi \in S$ ,  $S$  is  $\xi$ -honest. (Note that the notions of honest and  $\xi$ -honest depend on the witness to reliability.)

**FACT 2.22.** *Let  $\lambda$  be a regular reliable cardinal as witnessed by  $(W, \bar{\varphi})$ . Then for any  $\alpha < \lambda$ , there exists an  $\alpha'$  such that  $\alpha \leq \alpha' < \lambda$  and  $\alpha'$  is honest.*

PROOF. For each  $\xi < \lambda$ , there is a  $\xi' \geq \xi$  so that  $\xi'$  is  $\xi$ -honest. To see this, pick any  $w \in W$  so that  $\varphi_0(w) = \xi$ . Let  $\xi' = \sup\{\varphi_n(w) + 1 : n \in \omega\}$ .  $\xi'$  is  $\xi$ -honest. Since  $\text{cof}(\lambda) > \omega$ ,  $\xi' < \lambda$ . Let  $\Lambda : \lambda \rightarrow \lambda$  be defined by:  $\Lambda(\xi)$  is the least  $\xi'$  with  $\xi \leq \xi' < \lambda$  and  $\xi'$  is  $\xi$ -honest.

Let  $\alpha_0 = \alpha$ . If  $\alpha_n < \lambda$  has been defined, let  $\alpha_{n+1} = \sup \Lambda[\alpha_n]$ . Note that  $\alpha_{n+1} < \lambda$  since  $\lambda$  is regular. Let  $\alpha' = \sup\{\alpha_n : n \in \omega\}$  and note that  $\alpha' < \lambda$  since  $\lambda$  is regular. Now suppose  $\xi < \alpha'$ . There is some  $n \in \omega$  so that  $\xi < \alpha_n$ . Since  $\Lambda(\xi) \leq \alpha_{n+1}$ , one has that  $\alpha_{n+1}$  is  $\xi$ -honest. Since  $\alpha_{n+1} \subseteq \alpha'$ , one has that  $\alpha'$  is  $\xi$ -honest. Since  $\xi < \alpha'$  was arbitrary, this shows that  $\alpha'$  is honest. ⊖

FACT 2.23 ([8, Lemma 1.1]; Kechris–Woodin generic coding). *Let  $\lambda$  be a reliable ordinal with witness  $(W, \varphi)$ . Then there is a Lipschitz continuous function  $\mathfrak{F} : {}^\omega \lambda \rightarrow \mathbb{R}$  so that for all  $f \in {}^\omega \lambda$ , for all  $n \in \omega$ ,  $\mathfrak{F}(f)^{[n]} \in W$ , and if  $f[\omega]$  is honest, then for all  $n \in \omega$ ,  $\varphi_0(\mathfrak{F}(f)^{[n]}) = f(n)$ .*

DEFINITION 2.24. Let BS consist of the collection of  $z \in \mathbb{R}$  so that  $z^{[0]} \in \text{WO}$  and for all  $n \in \omega$ ,  $(z^{[1]})^{[n]} \in \text{WO}$ . If  $z \in \text{BS}$ , then let  $\text{seq}(z) \in {}^{<\omega_1} \omega_1$  be the sequence of length  $\text{ot}(z^{[0]})$  such that for all  $\alpha < \text{ot}(z^{[0]})$ ,  $\text{seq}(z)(\alpha) = \text{ot}((z^{[1]})^{[\text{num}(z^{[0]}, \alpha)]})$ . Note that for all  $\ell \in {}^{<\omega_1} \omega_1$ , there is a  $z \in \text{BS}$  so that  $\text{seq}(z) = \ell$ .

DEFINITION 2.25. Let  $\lambda$  be an ordinal,  $W \subseteq \mathbb{R}$ , and  $\varphi : W \rightarrow \lambda$  be a surjective norm. Let CS consist of those  $z \in {}^\omega \omega$  so that  $z^{[0]} \in \text{WO}$  and for all  $n \in \omega$ ,  $(z^{[1]})^{[n]} \in W$ .

For each  $z \in \text{CS}$ , let  $\text{cseq}(z)$  be the sequence in  $\lambda$  of length  $\text{ot}(z^{[0]})$  defined by  $\text{cseq}(z)(\alpha) = \varphi((z^{[1]})^{[\text{num}(z^{[0]}, \alpha)]})$ . Note that for all  $\ell \in {}^{<\omega_1} \lambda$ , there is a  $z \in \text{CS}$  so that  $\text{cseq}(z) = \ell$ .

### §3. Countable Length Everywhere Club Uniformization.

DEFINITION 3.1. If  $\kappa$  is a cardinal, let club denote the collection of club subsets of  $\kappa$ . A relation  $R \subseteq {}^{<\omega_1} \kappa \times \text{club}$  is  $\subseteq$ -downward closed in the club-coordinate if and only if for all  $\ell \in {}^{<\omega_1} \kappa$  and clubs  $C \subseteq D$ ,  $R(\ell, D)$  implies  $R(\ell, C)$ . Let  $\text{dom}(R) = \{\ell \in {}^{<\omega_1} \omega_1 : (\exists C \in \text{club}) R(\ell, C)\}$ . A uniformization for  $R$  is a function  $\Lambda : \text{dom}(R) \rightarrow \text{club}$  so that for all  $\ell \in \text{dom}(R)$ ,  $R(\ell, \Lambda(\ell))$ .

Countable length everywhere club uniformization for  $\kappa$  is the statement that for every  $R \subseteq {}^{<\omega_1} \kappa \times \text{club}$  which is  $\subseteq$ -downward closed in the club-coordinate, there is a uniformization for  $R$ .

FACT 3.2 ([3]; Countable Length Everywhere Club Uniformization for  $\omega_1$ ). *Assume ZF + AD. Let  $R \subseteq {}^{<\omega_1} \omega_1 \times \text{club}$  be  $\subseteq$ -downward closed in the club-coordinate. Let  $\tilde{R} \subseteq \text{BS} \times \text{clubcode}$  be the coded version of  $R$  defined by  $\tilde{R}(z, e)$  if and only if  $R(\text{seq}(z), \mathfrak{C}_e)$ . Assume  $\tilde{R}$  has a uniformization (i.e., a function  $\Phi : \text{dom}(\tilde{R}) \rightarrow \mathbb{R}$  so that for all  $z \in \text{dom}(\tilde{R})$ ,  $\tilde{R}(z, \Phi(e))$ ). Then  $R$  has a uniformization.*

Thus, under ZF + AD $_{\frac{1}{2}\mathbb{R}}$ , countable length everywhere club uniformization for  $\omega_1$  holds.

$L(\mathbb{R}) \models \neg \text{AD}_{\frac{1}{2}\mathbb{R}}$ . [3] gives an example to show that countable length everywhere club uniformization for  $\omega_1$  cannot hold in  $L(\mathbb{R})$  and thus it is not provable under AD alone.

The main result is a generalization of the countable length everywhere club uniformization for  $\omega_1$ .

**THEOREM 3.3.** *Assume  $ZF + AD + DC_{\mathbb{R}}$ . Let  $\Gamma$  be a nonselfdual pointclass closed under  $\wedge, \vee,$  and  $\forall^{\mathbb{R}}$ . Let  $\kappa = \delta(\Gamma)$  and assume that  $\kappa$  is reliable with witness  $(W, \bar{\varphi})$  such that  $W$  is  $\Gamma$ -complete and  $\varphi_0 : W \rightarrow \kappa$  is a surjective  $\Gamma$ -norm. Let  $CS$  and  $cseq$  be the codings of  $^{<\omega_1}\kappa$  from Definition 2.25 defined relative to  $\varphi_0$ . Let clubcode and  $\mathfrak{C}_e$  (for each  $e \in \text{clubcode}$ ) be the codings of club subsets of  $\kappa$  from Definition 2.16 relative to  $\Gamma$  and the  $\Gamma$ -norm  $\varphi_0$  on the  $\Gamma$ -complete set  $W$ .*

*Let  $R \subseteq [\kappa]^{<\omega_1} \times \text{club}$  be a  $\subseteq$ -downward closed relation in the club-coordinate. Let  $\tilde{R} \subseteq CS \times \text{clubcode}$  be the coded version of  $R$  defined by  $\tilde{R}(z, e)$  if and only if  $R(cseq(z), \mathfrak{C}_e)$ . Assume that  $\tilde{R}$  is Suslin and co-Suslin meaning there are trees  $T$  on  $\omega \times \omega \times \zeta_0$  and  $S$  on  $\omega \times \omega \times \zeta_1$  so that  $\tilde{R} = \{(z, e) : (\exists f \in {}^\omega\zeta_0)((z, e, f) \in [T])\}$  and  $\mathbb{R} \times \mathbb{R} \setminus \tilde{R} = \{(z, e) : (\exists g \in {}^\omega\zeta_1)((z, e, f) \in [S])\}$ . Let  $\bar{\phi} = \langle \phi_n : n \in \omega \rangle$  be the canonical semiscale on  $\tilde{R}$  derived from the Suslin representation  $T$  for  $\tilde{R}$  as in Definition 2.6. Assume each norm relation  $\leq_{\phi_n}^*$  is Suslin and co-Suslin. Then there is a  $\Lambda : \text{dom}(R) \rightarrow \text{club}$  so that for all  $\ell \in \text{dom}(R)$ ,  $R(\ell, \Lambda(\ell))$ .*

*Thus assuming  $ZF + AD$  and all sets of reals are Suslin, countable length everywhere club uniformization holds for  $\kappa$  with the above properties.*

**PROOF.** By the hypothesis, each norm relation  $\leq_{\phi_n}^*$  for  $\tilde{R}$  is assumed to be Suslin and co-Suslin so there are trees  $\mathcal{P}$  on  $\omega \times \omega \times \lambda_0$  and  $\mathcal{Q}$  on  $\omega \times \omega \times \lambda_1$  with  $\lambda_0, \lambda_1 < \Theta$  which project onto  $\leq_{\phi_n}^*$  and its complement, respectively. By the Moschovakis coding lemma and  $AC_{\omega}^{\mathbb{R}}$ , one may find sequences  $\langle \mathcal{P}_n : n \in \omega \rangle$  and  $\langle \mathcal{Q}_n : n \in \omega \rangle$  so that for each  $n \in \omega$ ,  $\pi_1[[\mathcal{P}_n]] = \leq_{\phi_n}^*$  and  $\pi_1[[\mathcal{Q}_n]] = \mathbb{R} \times \mathbb{R} \setminus \leq_{\phi_n}^*$ .

Let  $\ell \in \text{dom}(R)$ . Recall that by Fact 2.15,  $\kappa = \delta(\Gamma)$  is a regular cardinal. By Fact 2.22, let  $\nu_\ell$  be the least honest ordinal greater than  $\text{sup}(\ell)$ . Let  $R_\ell^{\nu_\ell} \subseteq {}^\omega\nu_\ell$  be defined by  $R_\ell^{\nu_\ell}(g)$  if and only if  $\text{rang}(\ell) \subseteq \text{rang}(g)$ .

Fix  $g \in {}^\omega\nu_\ell$  so that  $R_\ell^{\nu_\ell}(g)$ . Fix  $w \in \text{WO}$  with  $\text{ot}(w) = |\ell|$ . Let  $\mathfrak{F}$  be the function from Fact 2.23. Let  $\tau(\ell, g, w)$  be the unique real with the following properties.

- For all  $n \in \omega$ , if  $n \notin \text{field}(\mathcal{R}_w)$ , then  $\tau(\ell, g, w)^{[n]} = \mathfrak{F}(g)^{[n]}$ .
- For  $n \in \text{field}(\mathcal{R}_w)$ , let  $i_n^g$  be the least  $k$  so that  $g(k) = \ell(\text{ot}(w, n))$ . Then one has  $\tau(\ell, g, w)^{[n]} = \mathfrak{F}(g)^{[i_n^g]}$ .

Let  $\text{extract}(\ell, g, w)$  be the unique real  $z$  so that  $z^{[0]} = w$ ,  $z^{[1]} = \tau(\ell, g, w)$ , and for all  $n > 1$ ,  $z^{[n]} = 0$ , the constant 0 sequence.

**LEMMA 3.4.** *Let  $E^{\ell, w} : R_\ell^{\nu_\ell} \rightarrow \mathbb{R}$  be defined by  $E^{\ell, w}(g) = \text{extract}(\ell, g, w)$ . If  $\text{rang}(g)$  is honest then  $cseq(\text{extract}(\ell, g, w)) = \ell$ .*

**PROOF.** Assume  $\text{rang}(\ell) \subseteq \text{rang}(g)$  and  $g$  is honest. Let  $\alpha < |\ell|$  and  $n = \text{num}(w, \alpha)$  (where  $\text{num}$  comes from Definition 2.5). Since  $\text{rang}(\ell) \subseteq \text{rang}(g)$ ,  $i_n^g$  is defined with the property that  $g(i_n^g) = \ell(\text{ot}(w, n)) = \ell(\text{ot}(w, \text{num}(w, \alpha))) = \ell(\alpha)$ . Since  $g$  is honest, one has that for all  $n \in \omega$ ,  $g(n) = \varphi_0(\mathfrak{F}(g)^{[n]})$ . This implies that  $cseq(\text{extract}(\ell, g, w)) = \ell$ . ⊖

**LEMMA 3.5.** *Let  $\text{graph}(E^{\ell, w}) \subseteq R_\ell^{\nu_\ell} \times \mathbb{R}$  be defined as the graph of  $E^{\ell, w}$ .  $\text{graph}(E^{\ell, w})$  is Suslin and co-Suslin uniformly in  $\ell$  and  $w$ . Moreover, the canonical semiscale derived from this Suslin representation as in Definition 2.6 has associated norm relations which are Suslin and co-Suslin.*

PROOF. Note that although  $E^{\ell,w}$  is only defined on  $R_\ell^{v_\ell}$ , it is continuous on  $R_\ell^{v_\ell}$ . For instance, for  $n \in \text{field}(w)$  and  $g_0, g_1 \in {}^\omega \kappa$ , let  $i_n^{g_0}$  and  $i_n^{g_1}$  be the corresponding objects for  $g_0$  and  $g_1$ , respectively. If  $i_n^{g_0} = k$  and  $g_0 \upharpoonright k + 1 = g_1 \upharpoonright k + 1$ , then  $i_n^{g_0} = i_n^{g_1}$ . The continuity of  $E^{\ell,w}$  follows from this observation and the fact that the generic coding function  $\mathfrak{F}$  is Lipschitz.

The domain of  $E^{\ell,w}$  is  $R_\ell^{v_\ell}$ . Fact 2.10 and Fact 2.11 give an analogous result for  $R_\ell^{v_\ell}$ . The proof of the lemma is quite similar to the arguments of these two facts. The details are left to the reader.  $\dashv$

Consider the game  $G_\ell^w$  defined as follows.

$G_\ell^w$	I	$g(0), e(0)$	$g(2), e(1)$	$g(4), e(2)$	$\dots$	$e$
	II	$g(1)$	$g(3)$	$g(5)$		$g$

For all  $n \in \omega$ ,  $g(n) \in v_\ell$ . Player 1 plays  $g(2n)$  for all  $n \in \omega$ . Player 2 plays  $g(2n + 1)$  for all  $n \in \omega$ . Player 1 also plays  $e(n) \in \omega$  for all  $n \in \omega$ . After an infinite run, Player 1 and Player 2 together produce  $g \in {}^\omega v_\ell$  and Player 1 alone produces  $e \in \mathbb{R}$ . Player 1 wins  $G_\ell^w$  if and only if  $P_\ell^w(g)$ , where  $P_\ell^w(g)$  is defined by the conjunction of the following.

- (1)  $\text{rang}(\ell) \subseteq \text{rang}(g)$ .
- (2)  $\text{extract}(\ell, g, w) \in \text{dom}(\tilde{R})$ .
- (3)  $\tilde{R}(\text{extract}(\ell, g, w), e)$ .

LEMMA 3.6. *The payoff set  $P_\ell^w$  for the game  $G_\ell^w$  is Suslin and co-Suslin uniformly in  $\ell$  and  $w$ . Moreover, the semiscale derived from this Suslin representation as in Definition 2.6 is Suslin and co-Suslin.*

PROOF. (1) is Suslin and co-Suslin uniformly in  $\ell$  and  $w$  by Fact 2.10 (and note that  $v_\ell$  is defined uniformly from  $\ell$ ). (2) is Suslin and co-Suslin uniformly from  $\ell$  and  $w$  using this observation and Lemma 3.5. Similarly (3) is Suslin and co-Suslin uniformly from  $\ell$  and  $w$  using the tree  $\mathcal{T}$  and  $\mathcal{S}$  and Lemma 3.5. This establishes that the payoff set  $P_\ell^w$  is Suslin and co-Suslin uniformly in  $\ell$  and  $w$ . Using Fact 2.11 and Lemma 3.5, one can show each norm relation of the derived semiscale is Suslin and co-Suslin.  $\dashv$

The first part of Lemma 3.6 implies the ordinal game  $G_\ell^w$  is determined by Fact 2.8.

LEMMA 3.7. *Suppose  $\tau$  is a Player 2 strategy for  $G_\ell^w$  and  $e \in \mathbb{R}$ . Then there is an  $h \in {}^\omega v_\ell$  with the following properties.*

- $\text{rang}(\ell) \subseteq \text{rang}(h)$ .
- Let  $h \oplus e \in {}^\omega v_\ell$  be defined by  $(h \oplus e)(n) = \langle h(n), e(n) \rangle$ . Let  $(g, e) = \Sigma_\tau^2(h \oplus e)$ . (That is,  $(g, e)$  is the run of the game where Player 2 uses  $\tau$  against Player 1 using  $\rho_{h \oplus e}$ . See Definition 2.2 for the notations associated with strategies.) Then  $\text{rang}(g)$  is honest.

*Suppose  $\sigma$  is a Player 1 strategy for  $G_\ell^w$ . Then there is an  $h \in {}^\omega v_\ell$  with the following properties.*

- $\text{rang}(\ell) \subseteq \text{rang}(h)$ .
- Let  $(g, e) = \Sigma_\sigma^1(h)$ . Then  $\text{rang}(g)$  is honest.

PROOF. Fix a bijection  $B : \omega \rightarrow |\ell|$ . Using  $\text{AC}_\omega^\mathbb{R}$ , find a sequence  $\langle x_n : n \in \omega \rangle$  in  $W$  so that for all  $n \in \omega$ ,  $\varphi_0(x_n) = \ell(B(n))$ . Let  $\text{pair} : \omega^2 \rightarrow \omega$  be a bijection with the property that for all  $a, b, c \in \omega$ , if  $\text{pair}(a, b) = c$ , then  $a, b \leq c$  and for all  $a, b, c \in \omega$ , if  $b < c$ , then  $\text{pair}(a, b) < \text{pair}(a, c)$ . Let  $\pi_1 : v_\ell \rightarrow v_\ell$  be such that  $\pi_1(\langle \alpha, n \rangle) = \alpha$  for all  $\alpha \in v_\ell$  and  $n \in \omega$ . Now define a tree  $K$  on  $\mathbb{R}$  by  $s \in K$  if and only if the following holds.

- (1) For all  $k < |s|$ ,  $s(k) \in W$ .
- (2) Let  $n = |s|$ . Let  $p_s : 2n \rightarrow v_\ell$  be defined as follows.
  - (a) For  $k < n$ ,

$$p_s(2k) = \begin{cases} \langle \varphi_j(x_i), e(k) \rangle, & k = 2d \wedge d = \text{pair}(i, j), \\ \langle \varphi_j(s(i)), e(k) \rangle, & k = 2d + 1 \wedge d = \text{pair}(i, j). \end{cases}$$

- (b) For  $k < n$ ,  $p_s(2k + 1)$  is the result of applying  $\tau$  to the partial run  $\langle p_s(j) : j < 2k + 1 \rangle$  in  $G_\ell^w$ .
  - (c) For each  $k < n$ , if  $k$  is odd, then  $\varphi_0(s(k)) = p_s(k)$  and if  $k$  is even, then  $\varphi_0(s(k)) = \pi_1(p_s(k))$ .

Observe that  $p_s$  is continuous in the sense that if  $s \subseteq t$ , then  $p_t \upharpoonright 2|s| = p_s$ . The tree  $K$  is ordered by proper string extension  $\subsetneq$ . One can check every node  $s$  of  $K$  can be strictly extended by appending some  $y \in W$  so that  $\varphi_0(y) = p_s(|s|)$  if  $|s|$  is odd and  $\varphi_0(y) = \pi_1(p_s(|s|))$  if  $|s|$  is even. By  $\text{DC}_\mathbb{R}$ , there is an  $f \in [K]$ . Let  $q = \bigcup_{n \in \omega} p_{f \upharpoonright n}$ . Let  $h = \langle \pi_1(q(2n)) : n \in \omega \rangle$ . Let  $g$  be such that  $(g, e) = \Sigma_\tau^2(h \oplus e)$  and observe that  $\text{rang}(g)$  is honest. To see this, note that by (1) and (2b), for all  $k \in \omega$ ,  $f(k) \in W$  and  $\varphi_0(f(k)) = g(k)$ . Then the second case of (2a) will eventually include each  $\varphi_j(f(k))$  for all  $j \in \omega$  into  $\text{rang}(g)$ . Note that  $\text{rang}(\ell) \subseteq \text{rang}(h)$  by the first case of (2a).  $\Sigma_\tau^2(h \oplus e) = (g, e)$  by (2b). Thus  $h$  is the desired object.

The argument for the second statement is quite similar. ⊣

Next one seeks to show that Player 1 has the winning strategy for  $G_\ell^w$ . Suppose  $\tau$  is a Player 2 strategy. Since  $\ell \in \text{dom}(R)$ , there is a club  $C \subseteq \kappa$  so that  $R(\ell, C)$ . Pick any  $e$  so that  $\mathcal{C}_e \subseteq C$  which is possible by Fact 2.16. Now by Lemma 3.7 pick an  $h$  so that  $\text{rang}(\ell) \subseteq \text{rang}(h)$  and if  $(g, e) = \Sigma_\tau^2(h \oplus e)$  is the run according to  $\tau$  where Player 1 plays  $(h(n), e(n))$  for its  $n$ th-move, then  $\text{rang}(g)$  is honest. Note that (1) of  $P_\ell^w$  clearly holds. By Lemma 3.4, one has  $\text{cseq}(\text{extract}(\ell, g, w)) = \ell$ . Then  $\text{extract}(\ell, g, w) \in \text{dom}(\tilde{R})$  and hence (2) of  $P_\ell^w$  holds. Also (3) of  $P_\ell^w$  is true since  $\tilde{R}(\text{extract}(\ell, g, w), e)$  holds by choice of  $e$ . Thus Player 1 wins and hence  $\tau$  cannot be a winning strategy for Player 2.

This completes the argument that Player 2 cannot have a winning strategy in  $G_\ell^w$ . By the determinacy of  $G_\ell^w$ , Player 1 has a winning strategy. Next, one will need to show that a winning strategy for  $G_\ell^w$  can be found uniformly in  $w$  and  $\ell$ .

Lemma 3.6 implies that  $P_\ell^w$  has a Suslin representation whose derived semiscale has norm relations which are Suslin and co-Suslin. From these semiscales, one can construct a very good semiscale for  $P_\ell^w$  (uniformly in  $w$  and  $\ell$ ) whose associated norm relations are all Suslin and co-Suslin. In the notation of Fact 2.12, this can be used to show that the payoff set of each game  ${}^n H_t^s$  is Suslin and co-Suslin.  ${}^n H_t^s$  is

determined by Fact 2.8. Fact 2.12 can now be applied to find, uniformly in  $w$  and  $\ell$ , a Player 1 winning strategy  $\sigma_\ell^w$  in the game  $G_\ell^w$ .

Functions  $\varrho : {}^\omega v_\ell \rightarrow {}^\omega v_\ell$  and  $\vartheta : {}^\omega v_\ell \rightarrow {}^\omega \omega$  will be defined so that if  $(g, e)$  is the resulting run in the game  $G_\ell^w$  where Player 1 uses  $\sigma_\ell^w$  against Player 2 using  $\rho_h$ , then  $\varrho(h) = g$  and  $\vartheta(h) = e$ . Formally, let  $\pi_1 : v_\ell \times \omega \rightarrow v_\ell$  and  $\pi_2 : v_\ell \times \omega \rightarrow \omega$  be the projections onto the first and second coordinates, respectively. Define  $\varrho$  by

$$\varrho(h)(n) = \begin{cases} \pi_1(\Sigma_{\sigma_\ell^w}^1(h)(n)), & n \text{ is even,} \\ \Sigma_{\sigma_\ell^w}^1(h)(n), & n \text{ is odd.} \end{cases}$$

Define  $\vartheta$  by  $\vartheta(h)(n) = \pi_2(\Sigma_{\sigma_\ell^w}^1(h)(2n))$ . Both  $\varrho$  and  $\vartheta$  are Lipschitz continuous functions and are produced uniformly from  $w$  and  $\ell$  (since they depend only on  $\sigma_\ell^w$ ).

Since  $\sigma_\ell^w$  is a Player 1 winning strategy in  $G_\ell^w$ , one has by (1) of the payoff set  $P_\ell^w$  that for all  $g \in \varrho[{}^\omega v_\ell]$ ,  $\text{rang}(\ell) \subseteq \text{rang}(g)$  and thus  $\text{extract}(\ell, g, w)$  is well defined. Also since  $\sigma_\ell^w$  is a Player 1 winning strategy in  $G_\ell^w$ , one has that  $\vartheta[{}^\omega v_\ell] \subseteq \text{clubcode}$ . Let  $W_\ell = \varphi_0^{-1}[v_\ell]$  and let  $\varphi_\ell : W_\ell \rightarrow v_\ell$  be defined by  $\varphi_\ell = \varphi_0 \upharpoonright W_\ell$ . Since  $\varphi$  is a  $\Gamma$ -norm, one has that the associated prewellordering  $\preceq_{\varphi_\ell}$  belongs to  $\Delta$ . Fact 2.13 can now be applied to show  $\vartheta[{}^\omega v_\ell]$  is  $\exists^{\mathbb{R}}\Delta \subseteq \check{\Gamma}$  since  $\Gamma$  is closed under  $\forall^{\mathbb{R}}$ . By Fact 2.16, there is a club  $D \subseteq \kappa$  (produced uniformly from the set  $\vartheta[{}^\omega v_\ell]$ ) with the property that for all  $e \in \vartheta[{}^\omega v_\ell]$ ,  $D \subseteq \mathfrak{C}_e$ . By Lemma 3.7, one can find a sequence  $h^*$  so that  $\text{rang}(\ell) \subseteq \text{rang}(h^*)$  and if  $(g, e^*) = \Sigma_{\sigma_\ell^w}^1(h^*)$  is the run according to  $\sigma_\ell^w$  where Player 2 uses  $h^*$ , then  $\text{rang}(g)$  is honest. By Lemma 3.4,  $\text{extract}(\ell, g, w) = \ell$ . Thus since  $e^* = \vartheta(h^*)$ ,  $\check{R}(\ell, \vartheta(h^*))$  and hence  $R(\ell, \mathfrak{C}_{\vartheta(h^*)})$ . Since  $\vartheta(h^*) \in \vartheta[{}^\omega v_\ell]$ , one has that  $D \subseteq \mathfrak{C}_{\vartheta(h^*)}$ . Since  $R$  is  $\subseteq$ -downward closed, one has that  $R(\ell, D)$ . Finally, observe that  $D$  is produced uniformly from  $w$  and  $\ell \in \text{dom}(R)$ .

By the uniformity observation, it has been shown that there is a function  $\Psi$  so that whenever  $\ell \in \text{dom}(R)$  and  $w \in \text{WO}$  with  $\text{ot}(w) = |\ell|$ ,  $\Psi(\ell, w) \in \text{club}$  and  $R(\ell, \Psi(\ell, w))$ . One will need to remove the dependence on  $w$ .

Fix  $\ell \in \text{dom}(R)$ . Observe that if  $f \in \text{surj}_{|\ell|}$ , then  $\mathfrak{G}(f) \in \text{WO}_{|\ell|}$  where  $\mathfrak{G}$  is the simple generic coding function at  $\omega_1$  from Fact 2.18. Let  $\Phi_\ell : \text{surj}_{|\ell|} \rightarrow \text{club}$  be defined by  $\Phi_\ell(f) = \Psi(\ell, \mathfrak{G}(f))$ . Note that for all  $f \in \text{surj}_{|\ell|}$ ,  $R(\ell, \Phi_\ell(f))$ . Since  $\text{surj}_{|\ell|}$  is comeager in  ${}^\omega|\ell|$ , Fact 2.20 states that one can find uniformly from  $\Phi_\ell$  (which was constructed uniformly from  $\ell$ ), a comeager set  $B_\ell \subseteq \text{surj}_{|\ell|}$  and club  $D_\ell \subseteq \kappa$  so that for all  $f \in B_\ell$ ,  $D_\ell \subseteq \Phi_\ell(f)$ . Pick any  $f \in B_\ell$ . Since  $R(\ell, \Phi_\ell(f))$  and  $R$  is  $\subseteq$ -downward closed, one has that  $R(\ell, D_\ell)$ .

By the uniformity of the construction, one can define  $\Lambda : \text{dom}(R) \rightarrow \text{club}$  by  $\Lambda(\ell) = D_\ell$ . It has been shown that for all  $\ell \in \text{dom}(R)$ ,  $R(\ell, \Lambda(\ell))$ .  $\Lambda$  is the desired uniformization, and this completes the proof of the theorem.  $\dashv$

In particular, the following is a corollary of Theorem 3.3.

**THEOREM 3.8.** *Assume ZF + AD and all sets of reals are Suslin. Let  $\Gamma$  be a pointclass closed under  $\wedge, \vee$ , and  $\forall^{\mathbb{R}}$  with the scale property. Then the countable length everywhere club uniformization holds for  $\delta(\Gamma)$ . In particular, for all  $n \in \omega$ , the countable length everywhere club uniformization holds for  $\delta_{2n+1}^1$ .*



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