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p-ADIC ORDER BOUNDED GROUP VALUATIONS ON ABELIAN GROUPS

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Abstract. For a fixed integer e and prime p we construct the p-adic order bounded group valuations for a given abelian group G. These valuations give Hopf orders inside the group ring KG where K is an extension of $\mathbb{Q}_p$ with ramification index e. The orders are given explicitly when G is a p-group of order p or p^2. An example is given when G is not abelian.

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Let $p > 0$ be prime. Let R be a discrete valuation ring with uniformizing parameter π and with quotient field K, an extension of $\mathbb{Q}_p$. Let e be the absolute ramification index of $K/\mathbb{Q}_p$. Any finite group G gives rise to group rings RG and KG – these have the structures of an R-Hopf algebra and a K-Hopf algebra respectively. Clearly we have $RG \subset KG$.

One of the objectives in local Galois module theory is to find finitely generated projective R-Hopf algebras H such that $H \otimes_R K \cong KG$. Such Hopf algebras are called Hopf orders in KG (more precisely, R-Hopf orders in KG). There are several reasons why we might want to find such Hopf orders. For example, in the case where G is cyclic order n a classification of R-Hopf orders would yield a classification of group schemes over R with generic fibre μ_n. Additionally, if L is an extension of K with ring of integers S, and L/K is Galois with group G then S has a normal integral basis over R if and only if the associated order $\mathfrak{A} = \{\alpha \in KG \mid \alpha(S) \subseteq S\}$ is an R-Hopf order in KG [**3**].

While much work has been done in constructing Hopf orders in the case where G is cyclic of order p^n for $n \leq 3$ – see, for example, [**4**], [**6**], [**10**], and [**11**] – for many other groups the orders are unknown. In 1976 Larson [**8**] showed a correspondence between certain Hopf orders in KG and functions $G \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ satisfying certain properties, where $\mathbb{Z}^{\geq 0}$ is the set of nonnegative integers. These functions are called p-adic order bounded group valuations, and their corresponding orders are called Larson orders.

In general Larson orders do not exhaust all of the Hopf orders, yet they remain worthy of study for two reasons. First, they are the only class of Hopf orders constructed in the case where G is nonabelian. Second, they can be useful in constructing other orders – as an example of this the classification of orders in KG, where G is the cyclic group of order p^2 was started by Greither in [**6**] using extensions of Larson orders of cyclic groups of order p and the work was completed by Underwood in [**10**] by considering the duals of these "Greither orders."

In this work we shall focus on the case where G is an abelian p-group. It should be pointed out that in the abelian case there are other classes of orders which have been constructed – in addition to the Greither orders above Childs et al. have constructed triangular Hopf orders [**5**] and Hopf orders via formal groups [**4**]. We construct the p-adic order bounded group valuations on G, providing explicit calculations in the special cases where G is cyclic and where G is an elementary abelian group. We will also give the corresponding Larson order. While the applications are to local Galois module theory, the calculations are entirely group-theoretic: the approach starts with the construction of a sequence of nested subgroups of G satisfying certain relations. The results in the elementary abelian case will be needed in an upcoming work by the first author [**7**] and hopefully will be of use in the classification of all Hopf orders in KG for G an elementary abelian p-group.

The first section introduces the concept of a p-adic order bounded group valuation. Following this we investigate the case where G is a cyclic group of order p, providing a very easy (and well-known) classification of the corresponding Hopf orders. Then we turn our attention to arbitrary finite abelian groups. This is the point where we introduce the nested sequence of subgroups that a p-adic order bounded group valuation determines, and how we may start with certain nested sequences to construct valuations. Next, we focus on the two special cases mentioned above. Finally, we discuss the difficulties that arise when we try to extend these ideas to the nonabelian case, yet we provide an example in the case that p^2 divides the order of G and $|G| < p^3$.

Throughout this paper p will denote a fixed prime, K is an extension of $\mathbb{Q}_p$ with ramification index e, and we will set $e' = \lfloor e/(p-1) \rfloor$. Furthermore, v will denote the unique extension of the p-adic valuation on $\mathbb{Q}_p$ with the property that $v(e) = p$. While it is common to express the operation in an abelian group additively, we will always use multiplicative notation since it creates less confusion when working with group rings.

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1. Background. We start with the definition of a p-adic order bounded group valuation. As in the introduction, we use the symbol $\mathbb{Z}^{\geq 0}$ to denote the nonnegative integers.

DEFINITION 1.1. Let G be a finite group with identity 1. A *p-adic order bounded group valuation* is a function

$$\xi : G \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$$

such that, for all $g, h \in G$:

GV1. $\xi(1) = \infty$ and $\xi(g) < \infty$ if $g \neq 1$
GV2. $\xi(gh) \geq \min\{\xi(g), \xi(h)\}$
GV3. $\xi([g, h]) \geq \xi(g) + \xi(h)$ where $[g, h]$ is the commutator of g and h
GV4. $\xi(g) = 0$ if $|g|$ is not a power of p, and

$$\xi(g) \leq \frac{e}{\phi(|g|)}$$

for $|g| = p^s,\ s \geq 1$

GV5. $\xi(g^p) \geq p\xi(g)$.

Notice that this definition depends not only on p but on the field K – more precisely, the valuation v. It would be more precise to define the above as a "p-adic group valuation with order bounded by $e/\phi(|g|)$", however we will refer to such a function simply as a p-adic order bounded group valuation on G, or a "p-adic obgv" for short.

The motivation for studying p-adic obgv's is as follows. Let R be the ring of integers of K. Let π be a uniformizing parameter of R. We say an R-Hopf algebra H is a Hopf order in KG if H is finitely generated and projective as an R-module and $H \otimes_R K \cong KG$. Clearly RG is a simple example of a Hopf order, and in fact every Hopf order in KG contains RG [**1**, 5.2].

Given a p-adic order bounded group valuation ξ, it is easy to construct a Hopf order. Indeed, define H_ξ to be the R-algebra generated by $\{(g-1)\pi^{-\xi(g)}\}$, where g runs through all of the nontrivial elements of G. Then H_ξ has a Hopf algebra structure given by the restriction of the Hopf algebra structure maps on KG, i.e.

$$\begin{aligned}
\Delta\big((g-1)\pi^{-\xi(g)}\big) &= \frac{1}{\pi^{\xi(g)}}(\Delta(g)-\Delta(1)) \\
&= \frac{1}{\pi^{\xi(g)}}(g\otimes g - 1\otimes 1) \\
\varepsilon\big((g-1)\pi^{-\xi(g)}\big) &= \frac{1}{\pi^{\xi(g)}}(\varepsilon(g)-\varepsilon(1)) = \frac{1}{\pi^{\xi(g)}}(1-1)=0 \\
\lambda\big((g-1)\pi^{-\xi(g)}\big) &= \frac{1}{\pi^{\xi(g)}}(\lambda(g)-\lambda(1)) = \frac{1}{\pi^{\xi(g)}}(g^{-1}-1).
\end{aligned}$$

The reader can verify that $\Delta(H_\xi) \subset H_\xi \otimes H_\xi$ and that $\lambda(H_\xi) \subset H_\xi$ and hence H_ξ is an R-Hopf algebra. By construction $H_\xi \subset KG$, and since for all $g \in G$ we have

$$g = ((g-1)\pi^{-\xi(g)})\pi^{\xi(g)} + 1$$

and thus $RG \subset H_\xi$. It can be shown [**1**, 18.1] that H_ξ is a finitely generated R-module. The "finitely generated" part is a nontrivial argument: note that in what is presented above the only p-adic obgv property we use is that $\xi(1)=\infty$. Thus H_ξ is a Hopf order in KG. These Hopf orders were originally constructed by R. Larson in [**8**] and are called *Larson orders.*

EXAMPLE 1.2. For any finite group G we can define a map $\xi : G \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ by

$$\xi(g) = \begin{cases} \infty & g = 1 \\ 0 & g \neq 1 \end{cases}.$$

It is clear that this map is a p-adic order bounded group valuation. We shall call it the trivial valuation. In this case the corresponding Hopf algebra is generated by $\{(g-1) \mid g \in G\}$ and we can see that $H_\xi = RG$. In the case that $p \nmid |G|$ this is the only p-adic obgv.

Of course, if G is abelian then the commutator is trivial, hence any map $\xi : G \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ satisfies GV3. Furthermore, if G is an elementary abelian p-group, then $g^p = 1$ for all g and hence GV5 is also satisfied.

2. p-adic OBGV's on C_p. Let C_p denote the cyclic group of order p. Recall we are viewing this cyclic group multiplicatively; hence 1 will be used to denote the identity element. We will see it is easy to find all p-adic obgv's on this group. While the results here would also follow from the work in the next section, the simplicity of the $G = C_p$ case makes it a useful first example. This case is stated as an example in [**2**, p. 3] and is certainly well-known. We start with a necessary condition.

PROPOSITION 2.1. *Let ξ be a p-adic obgv on C_p. Then $\xi(g) = \xi(h)$ for all nontrivial $g, h \in C_p$.*

Proof. Let g, h be nontrivial elements in C_p. Then both g and h are generators of C_p, hence there exist integers $m, n \in \mathbb{Z}$ such that $g^m = h$ and $h^n = g$. By GV2 we have

$$\xi(h) = \xi(g^m) = \xi(g \cdot g \cdot \cdots \cdot g) \geq \min\{\xi(g), \xi(g), \ldots, \xi(g)\} = \xi(g)$$

as well as

$$\xi(g) = \xi(h^n) = \xi(h \cdot h \cdot \cdots \cdot h) \geq \min\{\xi(h), \xi(h), \ldots, \xi(h)\} = \xi(h)$$

and hence $\xi(g) = \xi(h)$. □

For any p-adic obgv ξ we let $\xi(g)$ denote the range of ξ and $|\xi(g)|$ will be the number of elements in $\xi(g)$. In other words, $\xi(g)$ is the number of distinct values achieved by ξ on G. The elements of $\xi(g)$ will frequently be referred to as "values." The above proposition shows that a p-adic obgv on C_p has at most one finite value, and so we get:

COROLLARY 2.2. *Let ξ be a p-adic obgv on C_p. Then $|\xi(C_p)| = 2$.*

Now let G be any group, and let $H \leq G$. Then any p-adic obgv on G restricts to a p-adic obgv on H. Thus the above result can give us some insight into p-adic obgv's on other groups.

COROLLARY 2.3. *Let G be a group, and let ξ be a p-adic obgv on G. Let H be a subgroup of order p. Then $|\xi(h)| = 2$. In particular, all of the nontrivial elements of H have the same valuation.*

The classification of p-adic obgv's on C_p reduces to a study of the possible values of $\xi(g)$, $g \neq 1$. By GV4 we must have $\xi(g) \leq e'$. Pick $0 \leq v \leq e'$ and set $\xi(g) = v$. Then GV4 is clearly satisfied, as is GV1. For $h \neq 1$ note that $\xi(h) = v$ and we have

$$\begin{aligned} \infty &= \xi(1 \cdot 1) \geq \min\{\xi(1), \xi(1)\} \\ v &= \xi(g) = \xi(g \cdot 1) \geq \min\{v, \infty\} = \min\{\xi(g), \xi(\infty)\} \\ \xi(gh) &\geq \min\{v, \xi(h)\} = \min\{v, v\} = v \end{aligned}$$

and hence GV2 is satisfied and ξ is a p-adic order bounded group valuation. We summarize.

THEOREM 2.4. *There are $e' + 1$ p-adic order bounded group valuations on C_p. Each p-adic obgv is uniquely determined by its value on a generator of C_p. In particular, $\xi_i(g) = i$ for $0 \leq i \leq e'$, where $g \neq 1$.*

Following the construction above, ξ_i determines the Hopf order

$$R\left[\frac{g-1}{\pi^i}\right] \subset KC_p$$

where $C_p = \langle g \rangle$.

REMARK 2.5. The construction above gives that the Hopf order is generated as an R-algebra by $\{(g^j - 1)\pi^{-i} \mid 0 \leq j \leq p-1\}$, however for $j > 1$ these generators can be expressed in terms of $(g-1)\pi^i$.

REMARK 2.6. These Hopf orders are well-known since a classification of all rank p Hopf algebras is given in [**9**], albeit in a different form. The Tate-Oort orders are typically parameterized by elements b which appear in a factorization $a^{p-1}b = w_p$ for some $a \in R$ where w_p is a certain element of R. Given such an element b

$$H_b = R\left[\frac{g-1}{\pi^{v(a)}}\right]$$

and hence corresponds to the p-adic obgv $\xi(g) = a$ for $g \neq 1$.

3. p-adic OBGV's on abelian groups. Now we address the main objective of this paper: to determine all p-adic obgv's on any abelian group. Let G be abelian. If p does not divide $|g|$ then by GV4 clearly the only p-adic obgv is the trivial one. If p does divide $|g|$ we can write $G \cong G' \times G''$, where G' is a p-group and p does not divide the order of G''.

The following lemma is easy to check.

LEMMA 3.1. *Let $G = G' \times G''$, where $p \mid |G'|$ and $p \nmid |G''|$. Then the p-adic obgv's on G are in one-to-one correspondence with the p-adic obgv's on G'. Specifically, if ξ is a p-adic obgv on G then $\xi\ |_{G'\times\{1\}}$ is a p-adic obgv on G'; and conversely any ξ' on G' extends to a p-adic obgv on G via*

$$\xi(g', g'') = \begin{cases} \xi'(g') & g'' = 1. \\ 0 & g'' \neq 1 \end{cases}$$

Thus it suffices only to consider the case where G is an abelian p-group.

LEMMA 3.2. *Let ξ be a p-adic obgv. Let $v_1 > v_2 > \cdots > v_r$ be the finite values of ξ, and let $v_0 = \infty$. For $1 \leq k \leq r$ let*

$$G_k = \{g \in G \mid \xi(g) \geq v_k\}.$$

Then for all $1 < k \leq r$ we have the following:

1. *$G_k \leq G$ and $G_r = G$*
2. *$G_{k-1} \leq G_k$*
3. *For $k < r$ we have G_k/G_{k-1} is elementary, and if $v_r \neq 0$ then G_r/G_{r-1} is also elementary.*
4. *Given G_k let x_k be the base-p log of the exponent of G_k. Then*
$$v_k \leq e'/p^{x_k - 1}.$$
5. *Given G_k with $k \neq r$ if $v_r = 0$, let l_k be the largest positive integer such that G_k/G_{k-l_k} is elementary. Then $pv_k \leq v_{k-l_k}$.*

Proof. Note that since it is clear by construction that $G_{k-1} \subset G_k$ we know **2** quickly follows from **1**. Let $g, h \in G_k$. Then $\xi(gh) \geq \min\{\xi(g), \xi(h)\} \geq v_k$ and so $gh \in G_k$. Furthermore, if $|g| = t$ we have

$$\xi(g^{-1}) = \xi(g^{t-1}) \geq \min\{\xi(g), \xi(g), \ldots, \xi(g)\} = \xi(g) = v_k$$

and so $g^{-1} \in G_k$ thus $G_k \leq G$.

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To prove **3**, let $g \in G_k$, $k > 0$. Notice that by GV5 it follows that $\xi(g^p) \geq p\xi(g) \geq pv_k$. Since $v_k > 0$ this means $\xi(g^p) > v_k$ and hence $g^p \in G_{k-1}$. Thus all nontrivial elements in G_k/G_{k-1} have order p hence this factor group is elementary.

Finally, since G_k/G_{k-l_k} is elementary we have $\xi(G_k) \subseteq \xi(G_{k-l_k})$ and the inequality in **4** (resp. **5**) follows by GV4 (resp.GV5). □

Let $G_0 = \{1\}$. Then the values of ξ determine a series of subgroups

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

and since $|g| = p^n$ we quickly obtain

COROLLARY 3.3. *The set of values $\xi(g)$ has at most $n+1$ elements.*

REMARK 3.4. Note that we did not use the fact G is abelian in the proof above, hence the two lemmas and the corollary above are also true in the nonabelian case. Of course a nonabelian G does not necessarily factor as $G' \times G''$, however for any group $|g| = p^n$ we have $|\xi(g)| \leq n+1$.

Now we will try to reverse the process. In other words, given an elementary abelian group G with series

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

of subgroups such that G_k/G_{k-1} is p-elementary abelian for each k, we would like to choose a (strictly) decreasing sequence of non-negative integers $v_1 > \cdots > v_r$ such that it gives rise to a p-adic obgv. We will attempt to do so as follows. Let l_k and x_k be as defined in the statement of the lemma. Put $x_0 = 0$, $v_0 = \infty$ and let $v_1, v_2, \ldots, v_r$ be a set of positive integers satisfying $v_{k-1} > v_k$, $v_1 \leq e'/p^{x_1-1}$ and $pv_k \leq v_{k-l_k}$ for all $k = 1, 2, \ldots, r$. For each $k \geq 1$, let $G'_k = G_k \setminus G_{k-1}$. Clearly the G'_k's are pairwise disjoint and their union is all of G. Set $\xi(1) = \infty$, and for $g_k \in G'_k$ we let $\xi(g_k) = v_k$. By construction we see that ξ satisfies GV1 and GV5. To show GV4 holds we need to have $v_k \leq e'/p^{x_k-1}$ for all k – this clearly holds for $k = 1$. Now suppose $v_t \leq e'/p^{x_t-1}$ is true for all $t < k$. Since $x_{k-l_k} \geq x_k - 1$ we have

$$v_k \leq \frac{1}{p} v_{k-l_k} \leq \frac{1}{p} \frac{e'}{p^{x_{k-l_k}-1}} \leq \frac{1}{p} \frac{e'}{p^{x_k-2}} = \frac{e'}{p^{x_k-1}}$$

and hence ξ satisfies GV4 as well. We now claim that ξ satisfies GV2. Let $g \in G'_k$, $h \in G'_l$, and assume $k \leq l$. If $k = l$ then $\xi(g) = \xi(h) = v_k$ and since $gh \in G_k$ we have $\xi(gh) \geq v_k = \min\{\xi(g), \xi(h)\}$. On the other hand, if $k < l$ then there is an integer i such that $k+1 \leq i \leq l$ with the property that $gh \in G'_i$, hence

$$v_k > v_{k+1} \geq v_i = \xi(gh) \geq v_l = \xi(h) = \min\{\xi(g), \xi(h)\}$$

thus GV2 is satisfied. Therefore, ξ is a p-adic obgv on G.

Given an elementary abelian group G and a series

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

of subgroups with the property that each G_k/G_{k-1} is p-elementary abelian, pick a decreasing sequence of nonnegative integers $v_1 > \cdots > v_r$ satisfying $v_1 \leq e'/p^{x_1-1}$ and $pv_k \leq v_{k-l_k}$. The series, together with the chosen values, form what we shall call a *valued*

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series for G. The above establishes that a valued series gives rise to a p-adic obgv. We summarize.

THEOREM 3.5. *The p-adic obgv's on an abelian group G are in one-to-one correspondence with its valued series.*

4. Two special cases. In practice, it is usually quite simple to construct all of the valued series for a given abelian group. We will illustrate this in the cases where G is an elementary abelian group and where G is cyclic.

Suppose G is an elementary abelian group. Then $x_k = 1$. Furthermore, $G_k/\{1\}$ is elementary and thus $l_k = k$. Thus

COROLLARY 4.1. *Let G be an elementary abelian group, and let*

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

be a series for G. Then any decreasing sequence $e' \geq v_1 > \cdots > v_r$ of nonnegative integers determines a p-adic obgv on G.

EXAMPLE 4.2. Let us find all p-adic obgv's on $G = C_p \times C_p$. Let ξ be a nontrivial p-adic obgv on G. Then the series for G that gives ξ has the form

$$1 \leq C_p \times C_p$$

or

$$1 \leq \langle g \rangle \leq C_p \times C_p$$

where $g \in G$ is not the identity. The two types above correspond to when $\xi(g)$ has either one or two finite values. Let us first consider the case where there is only one finite value, say v. Then for any $h \in G$ not the identity we have $\xi_v(h) = v$. There are $e' + 1$ different choices for v and hence $e' + 1$ different p-adic obgv's of this form. If we write $C_p \times C_p = \langle g_1 \rangle \times \langle g_2 \rangle$, then the corresponding Hopf order is

$$H_{\xi_v} = R\left[\frac{g_1 - 1}{\pi^v}, \frac{g_2 - 1}{\pi^v}\right].$$

Suppose that $\xi(g)$ has two finite values, say $v_1 > v_2$. Let $g \in C_p \times C_p$, $g \neq (1, 1)$. Then let

$$\xi_{v_1, v_2, g}(h) = \begin{cases} \infty & h = (1, 1) \\ v_1 & h \in \langle g \rangle,\ h \neq (1, 1) \\ v_2 & h \notin \langle g \rangle \end{cases}.$$

This is the p-adic obgv corresponding to the Larson order

$$H_{v_1, v_2, g} = R\left[\frac{g - 1}{\pi^{v_2}}, \frac{g' - 1}{\pi^{v_1}}\right],$$

where g' is any element not in $\langle g \rangle$. The resulting Larson order does not depend on the choice of g'. Notice that we can keep the same descending sequence of values but change the subgroup and obtain a new Larson order – if g, h are nontrivial elements of $C_p \times C_p$ such that $\langle g \rangle \neq \langle h \rangle$ then

$$H_{v_1,v_2,g} = R\left[\frac{g-1}{\pi^{v_2}}, \frac{h-1}{\pi^{v_1}}\right] \quad \text{and} \quad H_{v_1,v_2,h} = R\left[\frac{h-1}{\pi^{v_2}}, \frac{g-1}{\pi^{v_1}}\right]$$

which are clearly different since $v_1 > v_2$. A quick counting argument shows that there are $p+1$ different subgroups of order p in G and thus each descending sequence determines $p+1$ different Hopf orders. In summary, the total number of p-adic obgv's on $C_p \times C_p$ (and hence the total number of Larson orders in $K(C_p \times C_p)$) is

$$e' + (p+1)\frac{e'(e'+1)}{2} + 2.$$

This number is the sum of the numbers in the two cases plus the trivial p-adic obgv.

To generalize, for any $r \leq n$ let $f(n, r)$ be the number of subgroups of C_p^n of order p^r. It can be shown that

$$f(n, r) = \prod_{i=0}^{r-1} \frac{p^n - p^i}{p^r - p^i}.$$

Then we have

Corollary 4.3. *The number of Larson orders in KC_p^n, $n \geq 2$ is*

$$e' + 2 + \sum_{r=2}^{n}\left(\binom{e'+1}{r}\left(\sum_{0<n_1<\cdots<n_{r-1}<n} f(n, n_{r-1}) f(n_{r-1}, n_{r-2}) \cdots f(n_2, n_1)\right)\right).$$

Proof. We count by breaking up the collection of Larson orders in KC_p^n by the number r of nontrivial subgroups, in a given valued series. The first $e'+2$ in the above expression corresponds to the trivial valuation, i.e. the case $r = 0$, together with the case $r = 1$, in which case there are $e'+1$ choices for the valuation of the nontrivial elements of C_p^n. The binomial coefficient appears since there are $\binom{e'+1}{r}$ different choices for the values $e' \geq v_1 > v_2 > \cdots > v_r \geq 0$. For any given r we consider all sequences $0 < n_1 < \cdots < n_{r-1} < n$ – the corresponding series of subgroups will have order p^{n_i} for all i. The product $f(n, n_{r-1}) f(n_{r-1}, n_{r-2}) \cdots f(n_2, n_1)$ is readily seen to count the number of different series of subgroups for a given collection of $\{n_i\}$. □

Note that in the case $n = 2$ we get

$$\begin{aligned} e' + 2 + \binom{e'+1}{2} \sum_{0<n_1<2} f(n, n_1) \\ = e' + 2 + \binom{e'+1}{2} f(2, 1) = e' + 2 + \frac{e'(e'+1)}{2}(p+1), \end{aligned}$$

which agrees with the formula in Example 4.2.

We now turn our attention to the case where G is cyclic. If we write $G = \langle g \rangle = C_{p^n}$, then each proper subgroup is of the form $\langle g^{p^t} \rangle$ for some t. Let us first consider the case where 0 is not a value of ξ. In any valued series $\{1\} \leq G_0 \leq \cdots \leq G_r = G$ we must have

$G_k/G_{k-1} \cong C_p$ for all k in order to have each factor group be elementary. Thus the series must be of the form

$$1 \leq \left\langle g^{p^{n-1}} \right\rangle \leq \cdots \leq \langle g^p \rangle \leq \langle g \rangle = G.$$

Furthermore, by this construction it is clear that $x_k = k$ and $l_k = 1$. Thus the condition $pv_k \leq v_{k-l_k}$ is simply $pv_k \leq v_{k-1}$.

On the other hand, if 0 is a value of ξ then the series is of the form

$$\{1\} \leq \left\langle g^{p^{n-1}} \right\rangle \leq \cdots \leq \left\langle g^{p^{t+1}} \right\rangle \leq \left\langle g^{p^t} \right\rangle \leq \langle g \rangle = G$$

for some $1 \leq t < n$. We have the same bound on pv_k as above.

Example 4.4. Let us explicitly find the p-adic obgv's on C_{p^2}. Let g be a generator of C_{p^2}. Then the possible series are

$$\{1\} \leq \langle g \rangle = G$$
$$\{1\} \leq \langle g^p \rangle \leq \langle g \rangle = G.$$

The first series arises only in the case where the corresponding p-adic obgv is trivial. For the second series, pick, if possible, integers v_1, v_2 not both zero with $0 \leq pv_2 \leq v_1 \leq e'$. This will give a valued series.

5. Generalizations. As mentioned in the introduction, one of the reasons we study Larson orders is because they are the only tool at this point in the study of Hopf orders in KG when G is not abelian. In this section we will show how the results above can be used to give us a classification of Larson orders for a certain class of nonabelian groups.

We start with a few properties of p-adic obgv's. These properties are well-known, see, e.g., [**1**, 17.2–17.4]. First, note that given a p-adic obgv ξ on G, for all $g \in G$ we have $\xi(g^{-1}) = \xi(g)$: if g has order m then $g^{-1} = g^{m-1}$ and so

$$\xi(g^{-1}) = \xi(g^m) \geq \min\{\xi(g), \xi(g), \ldots, \xi(g)\} \geq \xi(g).$$

Replacing g by g^{-1} gives that $\xi(g^{-1}) = \xi(g)$.

Next, note that by GV3 we have, for all $g, h \in G$,

$$\xi(ghg^{-1}h^{-1}) \geq \xi(g) + \xi(h) \geq \xi(h)$$

and furthermore

$$\xi(ghg^{-1}) = \xi(ghg^{-1}h^{-1}h) \geq \min\{\xi(ghg^{-1}h^{-1}), \xi(h)\} = \xi(h)$$

and thus $\xi(ghg^{-1}) \geq \xi(h)$. Replacing g with g^{-1} gives $\xi(g^{-1}hg) \geq \xi(h)$. If we then replace h with ghg^{-1} we get

$$\xi(g^{-1}ghg^{-1}g) = \xi(h) \geq \xi(ghg^{-1})$$

and thus ξ is constant on conjugacy classes. As a result of this, if we let

$$G_+ = \{g \in G \,|\, \xi(g) > 0\}$$

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then G_+ is a normal subgroup of G, necessarily a p-group. Notice that from this we see that there are no nontrivial p-adic obgv's on a nonabelian simple group. Additionally for $n \geq 4$ the symmetric group S_n has no normal p-group and hence no p-adic obgv's.

PROPOSITION 5.1. *Let G be a group with a normal abelian Sylow p-subgroup H. Let ξ be a p-adic obgv on H. Then ξ extends uniquely to a p-adic obgv on G.*

Proof. Extend ξ to a map $\bar{\xi} : G \to \mathbb{Z}^{\geq 0} \cup \infty$ by $\bar{\xi}(g) = 0$ for all $g \notin H$. This clearly satisfies GV4 and GV5. Since $1 \in H$ we have $\bar{\xi}(1) = \xi(1) = \infty$, furthermore $\bar{\xi}(g) = 0 < \infty$ for $g \notin H$ and $\bar{\xi}(h) < \infty$ when $h \in H,\ h \neq 1$ since ξ is a p-adic obgv, thus GV1 holds. Clearly $\bar{\xi}(gh) \geq \min\{\bar{\xi}(g), \bar{\xi}(h)\}$ when $g, h \in H$ (since ξ is a p-adic obgv) as well as when $g \notin H$ (since the right-hand side of the inequality is zero) and hence GV2 holds as well. For GV3, if $g, h \in H$ there is nothing to check – the same is true if $g, h \notin H$ since $\bar{\xi}(g) + \bar{\xi}(h) = 0$. We examine the case $h \in H,\ g \notin H$. Since $H \lhd G$ we have $ghg^{-1} \in H$, hence

$$\begin{aligned}\bar{\xi}([g, h]) &= \xi((ghg^{-1})h^{-1}) \\ &\geq \xi(ghg^{-1}) + \xi(h^{-1}) \\ &\geq \bar{\xi}(g) + \bar{\xi}(h),\end{aligned}$$

the last inequality being true since $\bar{\xi}(g) = 0$ and $\xi(h^{-1}) = \xi(h)$. The case concerning $\bar{\xi}([h, g])$ for $h \in H,\ g \notin H$ is similar. Thus $\bar{\xi}$ is a p-adic obgv on G.

Uniqueness follows since each $g \notin H$ has order not a power of p, hence by GV4 any p-adic obgv maps all of the elements of $G \setminus H$ to zero. $\square$

REMARK 5.2. The construction above extends a ξ on any normal abelian p-subgroup H of G to a p-adic obgv on all of G. However, if H is not a Sylow p-subgroup then the extension may not be unique. This can be readily seen by considering examples such as $G = C_{p^2} = \langle g \rangle,\ H = \langle g^p \rangle$, and assume $e' \geq p$. Pick integers v_1, v_2 such that $0 < pv_2 \leq v_1 \leq e'$. The p-adic obgv ξ on H given by $\xi(g^p) = v_1$ extends to $\bar{\xi}(g) = 0$ above. Of course, it also extends to a p-adic obgv $\tilde{\xi}$ on G given by $\tilde{\xi}(g^p) = v_1$, $\tilde{\xi}(g) = v_2$.

Thus, if there are normal abelian p-subgroups we can construct a class of p-adic obgv's. In the case where G has a unique abelian p-Sylow subgroup P we can use the above proposition to find all p-adic obgv's. We illustrate this technique with a final example.

EXAMPLE 5.3. Let $|G| = p^2 t,\ t < p$. Then Sylow theory tells us that G has a unique Sylow p-subgroup P, which is necessarily abelian since $|P| = p^2$. Thus $P \cong C_{p^2}$ or $P \cong C_p \times C_p$. Examples 4.4 and 4.2 have found all the p-adic obgv's on P in both of these cases, hence we have found all p-adic obgv's on G.

REMARK 5.4. In general p-adic obgv's are trickier when the Sylow p-subgroup is not normal. The problem is that GV3 is not always satisfied if we try to extend a p-adic obgv from the Sylow subgroup to the group. Take, for example, the case where $p = 2$ and $G = S_3$. Let P be the Sylow 2-subgroup generated by (12). Let ξ be the p-adic obgv on $\langle (12) \rangle$ given by $\xi((12)) = 1$.Then $\bar{\xi}$ is trivial on 3-cycles. But then

$$\xi((12)(123)(12)(132)) = \xi(123) = 0 \not\geq 1 = \xi(123) + \xi(12).$$

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