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Sturm-Liouville Problems Whose Leading Coefficient Function Changes Sign

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Abstract. For a given Sturm-Liouville equation whose leading coefficient function changes sign, we establish inequalities among the eigenvalues for any coupled self-adjoint boundary condition and those for two corresponding separated self-adjoint boundary conditions. By a recent result of Binding and Volkmer, the eigenvalues (unbounded from both below and above) for a separated self-adjoint boundary condition can be numbered in terms of the Prüfer angle; and our inequalities can then be used to index the eigenvalues for any coupled self-adjoint boundary condition. Under this indexing scheme, we determine the discontinuities of each eigenvalue as a function on the space of such Sturm-Liouville problems, and its range as a function on the space of self-adjoint boundary conditions. We also relate this indexing scheme to the number of zeros of eigenfunctions. In addition, we characterize the discontinuities of each eigenvalue under a different indexing scheme.

In this paper, we study self-adjoint Sturm-Liouville problems (SLP's) associated with regular Sturm-Liouville equations (SLE's)

$$-(py')' + qy = \lambda wy \quad \text{on } (a,b), \tag{0.1}$$

where

$$-\infty \le a < b \le \infty, \tag{0.2}$$

$$1/p, q, w \in \mathrm{L}\big((a,b),\mathbb{R}\big), \quad p \text{ changes sign on } (a,b), \quad w > 0 \text{ a.e. on } (a,b), \tag{0.3}$$

and $\lambda \in \mathbb{C}$ is the so-called spectral parameter. Here, for an interval $J \subseteq \mathbb{R}$, we denote by $\mathrm{L}(J,\mathbb{R})$ the space of Lebesgue integrable real functions on J.

As motivation, we first recall some results on other classes of SLP's. When the leading coefficient function p is positive, the following inequalities are well known:

$$\begin{aligned}\lambda_0^{\mathbf{N}} \le \lambda_0^{\mathbf{P}} < \lambda_0^{\mathrm{SP}} \le \boldsymbol{\{}\lambda_0^{\mathbf{D}}, \lambda_1^{\mathbf{N}}\boldsymbol{\}} \le \lambda_1^{\mathrm{SP}} < \lambda_1^{\mathbf{P}} \le \boldsymbol{\{}\lambda_1^{\mathbf{D}}, \lambda_2^{\mathbf{N}}\boldsymbol{\}} \\ \le \lambda_2^{\mathbf{P}} < \lambda_2^{\mathrm{SP}} \le \boldsymbol{\{}\lambda_2^{\mathbf{D}}, \lambda_3^{\mathbf{N}}\boldsymbol{\}} \le \lambda_3^{\mathrm{SP}} < \lambda_3^{\mathbf{P}} \le \boldsymbol{\{}\lambda_3^{\mathbf{D}}, \lambda_4^{\mathbf{N}}\boldsymbol{\}} \le \cdots,\end{aligned} \tag{0.4}$$

where $\{\lambda_n^{\mathbf{P}}\}_{n=0}^{+\infty}$, $\{\lambda_n^{\mathrm{SP}}\}_{n=0}^{+\infty}$, $\{\lambda_n^{\mathbf{D}}\}_{n=0}^{+\infty}$ and $\{\lambda_n^{\mathbf{N}}\}_{n=0}^{+\infty}$ are the eigenvalues for the periodic, semi-periodic, Dirichlet and Neumann boundary conditions (BC's), respectively. Here, for any two numbers c_1 and c_2, the notation $\boldsymbol{\{}c_1, c_2\boldsymbol{\}}$ with bold-faced braces means each of c_1 and c_2. The above inequalities have been extended to the case of an arbitrary coupled self-adjoint BC in [6] (see also [9]). A key point in the work [6] is the identification of two separated self-adjoint BC's corresponding to the

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given coupled self-adjoint BC that play, in these general inequalities, the role of the Dirichlet and Neumann BC's in the above classical inequalities.

Such inequalities have also been found for singular Sturm-Liouville problems with positive p and regular or limit circle non-oscillatory end-points [12], and for left-definite regular Sturm-Liouville problems with positive p and indefinite weight function w [14].

Now, we start discussing regular self-adjoint SLP's whose p changes sign. Recently, using the Prüfer transformation, Binding and Volkmer have established in [4] the existence and unboundedness, from both below and above, of the eigenvalues for any separated self-adjoint BC without using operator theory. By their work, the eigenvalues for a separated self-adjoint BC can be indexed in terms of the Prüfer angle just as in the case where p is positive. However, the relationship between the index of an eigenvalue and the number of zeros of its eigenfunctions is much more complicated. In this paper we first show the existence and unboundedness, from both below and above, of the eigenvalues for any coupled self-adjoint BC, also without using operator theory. Then, we obtain inequalities parallel to the general inequalities mentioned above. For example, if the eigenvalues $\{\lambda_n^{\mathbf{P}}\}_{n\in\mathbb{Z}}$ for the periodic BC and $\{\lambda_n^{\mathrm{SP}}\}_{n\in\mathbb{Z}}$ for the semi-periodic BC are indexed appropriately, then

$$
\begin{aligned}
(0.5)\qquad \cdots \le \lambda_{-2}^{\mathbf{P}} < \lambda_{-2}^{\mathrm{SP}} \le \{\lambda_{-2}^{\mathbf{D}}, \lambda_{-1}^{\mathbf{N}}\} \le \lambda_{-1}^{\mathrm{SP}} < \lambda_{-1}^{\mathbf{P}} \le \{\lambda_{-1}^{\mathbf{D}}, \lambda_{0}^{\mathbf{N}}\} \\
\le \lambda_{0}^{\mathbf{P}} < \lambda_{0}^{\mathrm{SP}} \le \{\lambda_{0}^{\mathbf{D}}, \lambda_{1}^{\mathbf{N}}\} \le \lambda_{1}^{\mathrm{SP}} < \lambda_{1}^{\mathbf{P}} \le \{\lambda_{1}^{\mathbf{D}}, \lambda_{2}^{\mathbf{N}}\} \le \cdots,
\end{aligned}
$$

where $\{\lambda_n^{\mathbf{D}}\}_{n\in\mathbb{Z}}$ and $\{\lambda_n^{\mathbf{N}}\}_{n\in\mathbb{Z}}$ are the eigenvalues (numbered in terms of the Prüfer angle) for the Dirichlet and Neumann BC's, respectively. Therefore, the eigenvalues for *any* self-adjoint BC can now be indexed in terms of the Prüfer angle. These inequalities also imply that an asymptotic formula for the eigenvalues for separated self-adjoint BC's (see [1] and [3]) holds for coupled self-adjoint BC's, too. Next, under this indexing scheme, we characterize the discontinuities of each eigenvalue as a function on the space of SLP's studied, and the range of each eigenvalue as a function on the space of self-adjoint BC's.

Even though the number of zeros of an eigenfunction can be any integer bigger than or equal to a certain minimum, one can determine the index of an eigenvalue for a separated self-adjoint BC via an appropriate count of the zeros of its eigenfunctions. In this count, the zeros are weighted by the sign of p. When the self-adjoint BC is a coupled one, the count gives the index with a possible error of ± 1.

For these SLP's, there is already a direct way to index the eigenvalues: the negative ones are numbered as $\ldots, \lambda_{-3}, \lambda_{-2}, \lambda_{-1}$, while the non-negative ones are indexed as $\lambda_0, \lambda_1, \lambda_2, \ldots$, all in non-decreasing order. In general, at a self-adjoint BC with 0 as an eigenvalue, these direct indices jump (*i.e.*, do not stay invariant when the self-adjoint BC varies continuously). The self-adjoint BC's having 0 as an eigenvalue have been characterized in [13]. Here we determine how the direct indices of the eigenvalues jump at such a BC. These results are important for numerical computations of the eigenvalues, since the problems (*i.e.*, the end points a, b of the interval, the coefficient functions p, q, w of the differential equation, and the coefficients of the self-adjoint BC) are usually approximated in these computations.

The organization of this paper is as follows. In Section 1, we introduce our notation and recall some basic results. The inequalities are established in Section 2, while Section 3 is devoted to the study of the discontinuities and the range of each indexed eigenvalue. In Section 4, we discuss the relationship between the index of an eigenvalue and the number of zeros of its eigenfunctions. Finally, in Section 5, we characterize the jumps in the direct indices of the eigenvalues.

1 Notation and Basic Results

Let $J \subseteq \mathbb{R}$ be an interval. A function $f \in \mathrm{L}(J, \mathbb{R})$ is said to *change sign* on J if both of the sets $\{x \in J\,;\, f(x) < 0\}$ and $\{x \in J\,;\, f(x) > 0\}$ have positive or infinite Lebesgue measures; otherwise, we say that f *does not change sign* on J.

By a solution of (0.1) we mean a function y on (a, b) such that y and py' are absolutely continuous on all compact subintervals of (a, b) and satisfy (0.1) a.e. The regularity conditions in (0.3) imply that every solution y and its quasi-derivative py' have finite limits at the both end-points a and b, and any initial-value problem for (0.1) on $[a, b]$ has a unique solution. We will abbreviate the SLE (0.1) as $(a, b, 1/p, q, w)$ and denote by $\boldsymbol{\Omega}$ the space of all such differential equations (DE's), *i.e.*,

$$\boldsymbol{\Omega} = \{(a, b, 1/p, q, w)\,;\, (0.2) \text{ and } (0.3) \text{ hold}\}. \tag{1.1}$$

Bold faced lower case Greek letters, such as $\boldsymbol{\omega}$, will be used to stand for elements of $\boldsymbol{\Omega}$. A natural topology on $\boldsymbol{\Omega}$ is the product topology induced from the usual topologies on $\mathbb{R}$ and on $\mathrm{L}(\mathbb{R}, \mathbb{R})$. More precisely, given $\epsilon > 0$, each $(a_0, b_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$ with finite a_0 and b_0 has a neighborhood in $\boldsymbol{\Omega}$ consisting of the elements $(a, b, 1/p, q, w)$ satisfying

$$|a - a_0| + |b - b_0| + \int_{\mathbb{R}} \bigl(|\widetilde{1/p} - \widetilde{1/p_0}| + |\widetilde{q} - \widetilde{q_0}| + |\widetilde{w} - \widetilde{w_0}|\bigr) < \epsilon, \tag{1.2}$$

where $\widetilde{1/p}$ is the extension of $1/p$ to $\mathbb{R}$ that equals 0 on $\mathbb{R} \setminus (a, b)$ and $\widetilde{1/p_0}$, $\widetilde{q}$, $\widetilde{q_0}$, $\widetilde{w}$, $\widetilde{w_0}$ have similar meanings; each $(-\infty, b_0, 1/p_0, q_0, w_0) \in \boldsymbol{\Omega}$ with finite b_0 has a neighborhood in $\boldsymbol{\Omega}$ formed by the elements $(a, b, 1/p, q, w)$ satisfying

$$a < -\frac{1}{\epsilon}, \quad |b - b_0| + \int_{\mathbb{R}} \bigl(|\widetilde{1/p} - \widetilde{1/p_0}| + |\widetilde{q} - \widetilde{q_0}| + |\widetilde{w} - \widetilde{w_0}|\bigr) < \epsilon\,; \tag{1.3}$$

etc. Such topologies have been used in [15], [10] and [11]. We note that $\boldsymbol{\Omega}$ is path-connected.

For any $m, n \in \mathbb{N}$, we use $\mathrm{M}_{m,n}(\mathbb{C})$ to denote the vector space of m by n complex matrices and $\mathrm{M}^*_{m,n}(\mathbb{C})$ its open subspace consisting of the elements with the maximum rank $\min\{m, n\}$, while $\mathrm{M}_{m,n}(\mathbb{R})$ and $\mathrm{M}^*_{m,n}(\mathbb{R})$ are the real analogs of $\mathrm{M}_{m,n}(\mathbb{C})$ and $\mathrm{M}^*_{m,n}(\mathbb{C})$, respectively. When a capital Latin letter other than Y stands for a matrix, the entries of the matrix will be denoted by the corresponding lower case letter with two indices. Let $\mathrm{GL}(2, \mathbb{C})$ be the set of invertible complex matrices in dimension 2, and $\mathrm{SL}(2, \mathbb{R})$ its subset consisting of the real elements having determinant 1.

For a complex matrix A, A^* stands for its complex conjugate transpose. For a solution y of (0.1), we set

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}. \tag{1.4}$$

The self-adjoint BC's are represented by linear algebraic systems of the form

$$AY(a) + BY(b) = 0, \tag{1.5}$$

where $(A \,|\, B) \in \mathrm{M}^*_{2,4}(\mathbb{C})$ satisfies

$$A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B^*. \tag{1.6}$$

Following [13], we take the quotient space

$$\mathrm{GL}(2,\mathbb{C}) \backslash \mathrm{M}^*_{2,4}(\mathbb{C}) = \big\{ \{(TA \,|\, TB) \;;\; T \in \mathrm{GL}(2,\mathbb{C})\} \;;\; (A \,|\, B) \in \mathrm{M}^*_{2,4}(\mathbb{C}) \big\} \tag{1.7}$$

as the space of BC's, *i.e.*, *each BC is an equivalence class of coefficient matrices of linear algebraic systems of the form* (1.5) *with* $(A \,|\, B) \in \mathrm{M}^*_{2,4}(\mathbb{C})$. The BC represented by (1.5) will be denoted by $[A \,|\, B]$. Note here that square brackets, not parentheses, are used. Usual bold-faced capital Latin letters, such as **A**, will also be used for BC's. The space $\mathcal{B}^{\mathbb{R}}$ of real self-adjoint BC's consists of the separated real BC's and the coupled BC's of the form $[K \,|\, -I]$ with $K \in \mathrm{SL}(2,\mathbb{R})$. The space $\mathcal{B}^{\mathbb{C}}$ of complex self-adjoint BC's is made of the real self-adjoint BC's and the non-real BC's of the form $[e^{i\omega} K \,|\, -I]$ with $\omega \in (0,\pi)$ and $K \in \mathrm{SL}(2,\mathbb{R})$. By [13], $\mathcal{B}^{\mathbb{C}}$ can be obtained by "gluing" its open sets

$$\mathcal{O}^{\mathbb{C}}_1 = \mathcal{O}^{\mathbb{C}}_6 = \{[e^{i\omega} K \,|\, -I] \;;\; \omega \in [0,\pi), K \in \mathrm{SL}(2,\mathbb{R})\}, \tag{1.8}$$

$$\mathcal{O}^{\mathbb{C}}_2 = \left\{ \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} ; a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.9}$$

$$\mathcal{O}^{\mathbb{C}}_3 = \left\{ \begin{bmatrix} 1 & a_{12} & -\bar{z} & 0 \\ 0 & z & b_{21} & -1 \end{bmatrix} ; a_{12} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\}, \tag{1.10}$$

$$\mathcal{O}^{\mathbb{C}}_4 = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & -\bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} ; a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{22} \in \mathbb{R} \right\}, \tag{1.11}$$

$$\mathcal{O}^{\mathbb{C}}_5 = \left\{ \begin{bmatrix} a_{11} & 1 & \bar{z} & 0 \\ z & 0 & b_{21} & -1 \end{bmatrix} ; a_{11} \in \mathbb{R}, z \in \mathbb{C}, b_{21} \in \mathbb{R} \right\} \tag{1.12}$$

via the coordinate transformations among these open sets. Note that the topology on the open set in (1.8) is the one induced from the usual topology on $\mathrm{M}_{2,2}(\mathbb{C})$, and each of the four open sets in (1.9)–(1.12) can be identified with $\mathbb{R}^4$. Open sets $\mathcal{O}^{\mathbb{R}}_i$, $i = 1, \dots, 6$, can be defined using (1.8)–(1.12) with $\omega = 0$ and $\mathbb{C}$ replaced by $\mathbb{R}$. Then, $\mathcal{B}^{\mathbb{R}}$ can be obtained by gluing these open sets via the coordinate transformations among them, and each of $\mathcal{O}^{\mathbb{R}}_2, \dots, \mathcal{O}^{\mathbb{R}}_5$ can be identified with $\mathbb{R}^3$.

For each $[A \,|\, B] \in \mathcal{B}^{\mathbb{C}}$ and every $K \in \mathrm{SL}(2, \mathbb{R})$, we set

$$[A \,|\, B]_{\bullet} K = [AK \,|\, B]. \tag{1.13}$$

Note that $[A \,|\, B]_{\bullet} K \in \mathcal{B}^{\mathbb{R}}$ if $[A \,|\, B] \in \mathcal{B}^{\mathbb{R}}$.

The space of SLP's studied in this paper is $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$. The following result is well known (see, for example, [5, Chapter 7, Theorem 2.1]).

Theorem 1.14 *The eigenvalues of any Sturm-Liouville problem in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ are all real.*

For each $\lambda \in \mathbb{C}$, let $\phi_{11}(\,\cdot\,, \lambda)$ and $\phi_{12}(\,\cdot\,, \lambda)$ be the solutions of (0.1) determined by the initial conditions

$$\phi_{11}(a, \lambda) = 1, \quad (p\phi_{11}')(a, \lambda) = 0\,; \qquad \phi_{12}(a, \lambda) = 0, \quad (p\phi_{12}')(a, \lambda) = 1. \tag{1.15}$$

We denote $p\phi_{11}'$ by ϕ_{21} and $p\phi_{12}'$ by ϕ_{22}. Set

$$\Phi(t, \lambda) = \begin{pmatrix} \phi_{11}(t, \lambda) & \phi_{12}(t, \lambda) \\ \phi_{21}(t, \lambda) & \phi_{22}(t, \lambda) \end{pmatrix}, \quad t \in [a, b], \lambda \in \mathbb{C}. \tag{1.16}$$

To indicate the dependence of Φ on the SLE $\boldsymbol{\omega}$, we sometimes write $\Phi_{\boldsymbol{\omega}}$. For each $t \in [a, b]$, $\Phi(t, \lambda)$ is an entire matrix function of λ. Moreover, $\Phi(t, \lambda) \in \mathrm{SL}(2, \mathbb{R})$ for $t \in [a, b]$ and $\lambda \in \mathbb{R}$. The following result is also well known (see, for example, [17, Lemma 4.2]).

Theorem 1.17 *A number $\lambda \in \mathbb{R}$ is an eigenvalue of the Sturm-Liouville problem consisting of (0.1) and (1.5) if and only if*

$$\Delta(\lambda) := \det\bigl(A + B\Phi(b, \lambda)\bigr) = 0. \tag{1.18}$$

We will call the entire function Δ, unique up to a non-zero constant multiple, the *characteristic function* of the SLP for its importance. Recall that the *algebraic multiplicity* (or just *multiplicity*) of an isolated eigenvalue is the order of the eigenvalue as a zero of Δ. An eigenvalue is said to be *simple* if it has multiplicity 1, while the eigenvalues of multiplicity 2 are called *double* eigenvalues. When we count the (isolated) eigenvalues of an SLP in a domain in $\mathbb{C}$, their multiplicities are taken into account. The linear space spanned by the eigenfunctions for an eigenvalue is called the *eigenspace* for the eigenvalue. The *geometric multiplicity* of an eigenvalue is defined to be the dimension of its eigenspace, which is either 1 or 2. The following result is from Theorem 5.5 in [13].

Theorem 1.19 *The algebraic and geometric multiplicities of an eigenvalue of any Sturm-Liouville problem in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ are equal.*

The next result is a slight generalization of Theorem 3.1 in [15] or Theorem 3.2 in [10], and can be proved using Rouché's Theorem from complex analysis.

Theorem 1.20 *Let $\mathcal{R} \subset \mathbb{R}$ be a bounded open set such that its boundary does not contain any eigenvalue of a given Sturm-Liouville problem in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$, and $n \geq 0$ the number of the problem's eigenvalues in $\mathcal{R}$. Then there exists a neighborhood $\mathcal{N}$ of the problem in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ such that any Sturm-Liouville problem in $\mathcal{N}$ also has exactly n eigenvalues in $\mathcal{R}$.*

Remark 1.21 Let λ_* be an eigenvalue of an SLP $(\boldsymbol{\omega}_0, \mathbf{A}_0) \in \boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ and n its multiplicity. Pick a small $\epsilon > 0$ such that $(\boldsymbol{\omega}_0, \mathbf{A}_0)$ has exactly n eigenvalues in the interval $[\lambda_* - \epsilon, \lambda_* + \epsilon]$. Then, by Theorem 1.20, there is a connected neighborhood $\mathcal{O}$ of $(\boldsymbol{\omega}_0, \mathbf{A}_0)$ in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ such that each SLP in $\mathcal{O}$ has exactly n eigenvalues in $(\lambda_* - \epsilon, \lambda_* + \epsilon)$. Thus, there are continuous functions $\Lambda_1, \dots, \Lambda_n \colon \mathcal{O} \to \mathbb{R}$ defined on $\mathcal{O}$ such that

(i) $\Lambda_1(\boldsymbol{\omega}_0, \mathbf{A}_0) = \cdots = \Lambda_n(\boldsymbol{\omega}_0, \mathbf{A}_0) = \lambda_*$;
(ii) $\Lambda_1(\boldsymbol{\omega}, \mathbf{A}) \leq \cdots \leq \Lambda_n(\boldsymbol{\omega}, \mathbf{A})$ for any $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}$;
(iii) for each $(\boldsymbol{\omega}, \mathbf{A}) \in \mathcal{O}$, $\Lambda_1(\boldsymbol{\omega}, \mathbf{A}), \dots$ and $\Lambda_n(\boldsymbol{\omega}, \mathbf{A})$ are eigenvalues of $(\boldsymbol{\omega}, \mathbf{A})$.

From Theorem 1.19 we see that $n \leq 2$. By Theorem 4.1 in [13], when $n = 2$, these are actually different functions on $\mathcal{O}$. In any case, locally, they are the only such functions and are called the *continuous eigenvalue branches* through λ_*.

We can apply the above ideas to any finite number of eigenvalues of an SLP to get similar conclusions.

Remark 1.22 We can restrict our attention to any connected subspace of $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$, such as $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{R}}$ or just a curve in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$, to obtain results similar to Theorem 1.20 and Remark 1.21. These results will be used in Section 2 to prove the existence of eigenvalues for coupled self-adjoint BC's without using operator theory.

Each separated self-adjoint BC can be written in the form

$$\mathbf{S}_{\alpha,\beta} := \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \end{bmatrix} \tag{1.23}$$

with $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$. For example, the Dirichlet BC $\mathbf{D} = \mathbf{S}_{0,\pi}$. Note that the space

$$\mathcal{T} = \{\mathbf{S}_{\alpha,\beta}\ ;\ \alpha \in [0, \pi), \beta \in (0, \pi]\} \tag{1.24}$$

of separated self-adjoint BC's is diffeomorphic to the torus.

For any non-trivial real solution of (0.1), there are two unique absolutely continuous functions ρ and θ on $[a, b]$ such that $\rho(t, \lambda) \neq 0$ for all $t \in [a, b]$, and

$$y = \rho \sin\theta, \quad py' = \rho\cos\theta, \quad 0 \leq \theta(a, \lambda) < \pi. \tag{1.25}$$

The function θ is called the *Prüfer angle* of the solution y. The zeros of y in $[a, b]$ are exactly the points of $[a, b]$ where θ attains an integer multiple of π. Note that y satisfies the self-adjoint BC (1.23) if and only if

$$\theta(a, \lambda) = \alpha, \quad \theta(b, \lambda) = \beta + n\pi \quad \text{for some } n \in \mathbb{Z}. \tag{1.26}$$

If an eigenvalue has geometric multiplicity 1, then all its eigenfunctions share the same Prüfer angle. The proof of the following result in [4] does not use operator theory.

Theorem 1.27 *Fix a Sturm-Liouville equation in $\boldsymbol{\Omega}$, and let $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. Then, for each $n \in \mathbb{Z}$, there is a unique eigenvalue $\lambda_n = \lambda_n(\mathbf{S}_{\alpha,\beta})$ for $\mathbf{S}_{\alpha,\beta}$ such that its Prüfer angle θ satisfies*

$$\theta(b, \lambda_n) = \beta + n\pi. \tag{1.28}$$

Moreover, $\lambda_n \to -\infty$ as $n \to -\infty$, and $\lambda_n \to +\infty$ as $n \to +\infty$.

Proof See [4, Theorem 2.2]. ■

For any $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$ and $n \in \mathbb{Z}$, we have that $\lambda_n(\mathbf{S}_{\alpha,\beta}) < \lambda_{n+1}(\mathbf{S}_{\alpha,\beta})$, since $w > 0$ a.e. on (a, b). We also mention that the derivative formulas in [15] for continuous eigenvalue branches over $\mathcal{T}$ with respect to α and β still hold here. Similar to the case where p does not change sign, we have the following facts.

Lemma 1.29 *Fix a Sturm-Liouville equation in $\boldsymbol{\Omega}$, and let $n \in \mathbb{Z}$. As a function of (α, β), $\lambda_n(\mathbf{S}_{\alpha,\beta})$ is continuous on $[0, \pi) \times (0, \pi]$, strictly decreasing in α, and strictly increasing in β. Moreover, for each $\alpha \in [0, \pi)$,*

$$\lim_{\beta \to 0^+} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{\alpha,\pi}), \tag{1.30}$$

and for each $\beta \in (0, \pi]$,

$$\lim_{\alpha \to \pi^-} \lambda_n(\mathbf{S}_{\alpha,\beta}) = \lambda_{n-1}(\mathbf{S}_{0,\beta}). \tag{1.31}$$

Proof See the proof of the Theorem in [7]. ■

2 Inequalities Among Eigenvalues

In this section, we recall from [6] some information about the characteristic function, show the existence of eigenvalues for coupled self-adjoint BC's, establish inequalities among eigenvalues, and then derive a consequence of these inequalities.

Consider the SLE (0.1), and let $K \in \mathrm{SL}(2, \mathbb{R})$. For any $\lambda \in \mathbb{C}$ we define

$$\tau(\lambda) = \tau_K(\lambda) = k_{22}\phi_{11}(b, \lambda) - k_{21}\phi_{12}(b, \lambda) - k_{12}\phi_{21}(b, \lambda) + k_{11}\phi_{22}(b, \lambda). \tag{2.1}$$

For any $\omega \in [0, 2\pi)$, since the characteristic function for the self-adjoint BC $[e^{i\omega}K \mid -I]$ is

$$\Delta(\lambda) = 2\cos\omega - \tau(\lambda), \tag{2.2}$$

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a number λ is an eigenvalue for $[e^{i\omega}K \mid -I]$ if and only if

$$\tau(\lambda) = 2\cos\omega. \tag{2.3}$$

Associated with the coupled self-adjoint BC $[K \mid -I]$ are the separated self-adjoint BC's

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_{22} & -k_{12} \end{bmatrix} \tag{2.4}$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{11} \end{bmatrix}. \tag{2.5}$$

Note that $(k_{22}, -k_{12}) \neq (0,0) \neq (-k_{21}, k_{11})$, since $\det K = 1$. Thus, (2.4) and (2.5) are well-defined BC's. We use $\{\mu_n = \mu_n(K)\ ;\ n \in \mathbb{Z}\}$ to denote the eigenvalues for the self-adjoint BC (2.4) and $\{\nu_n = \nu_n(K)\ ;\ n \in \mathbb{Z}\}$ the eigenvalues for the self-adjoint BC (2.5), all indexed in terms of the Prüfer angle. Note also that $\mu_n(K) = \mu_n(-K)$ and $\nu_n(K) = \nu_n(-K)$ for any $n \in \mathbb{Z}$. Part of the following lemma motivates the introduction of the μ_n's and ν_n's.

Lemma 2.6

(i) *For any $\lambda \in \{\mu_n\ ;\ n \in \mathbb{Z}\} \cup \{\nu_n\ ;\ n \in \mathbb{Z}\}$, we have that $|\tau(\lambda)| \geq 2$.*
(ii) *If $\lambda_* \in \mathbb{R}$ satisfies $|\tau(\lambda_*)| < 2$, then $\tau'(\lambda_*) \neq 0$. Hence, any eigenvalue for a non-real self-adjoint boundary condition is simple.*
(iii) *If $\lambda_* \in \mathbb{R}$ satisfies $\tau(\lambda_*) = 2$, then $\tau''(\lambda_*) < 0$; if $\lambda_* \in \mathbb{R}$ satisfies $\tau(\lambda_*) = -2$, then $\tau''(\lambda_*) > 0$.*

Proof See the proofs of Lemma 4.2 Part iii), Corollary 4.1 and Theorem 4.2 in [6]. Although p is assumed positive in [6], the same proofs are valid when p changes sign. ∎

Remark 2.7 Lemma 2.6 (ii) and (iii) imply the following properties of the curve $\tau = \tau(\lambda)$:

(i) it is always strictly monotone between the two lines $\tau = \pm 2$;
(ii) when it meets the line $\tau = 2$, it is either strictly monotone, corresponding to a simple eigenvalue for $[K \mid -I]$, or is concave downward with a local maximum point on the line $\tau = 2$, corresponding to a double eigenvalue for $[K \mid -I]$;
(iii) when it meets the line $\tau = -2$, it is either strictly monotone, corresponding to a simple eigenvalue for $[-K \mid -I]$, or is concave upward with a local minimum point on the line $\tau = -2$, corresponding to a double eigenvalue for $[-K \mid -I]$.

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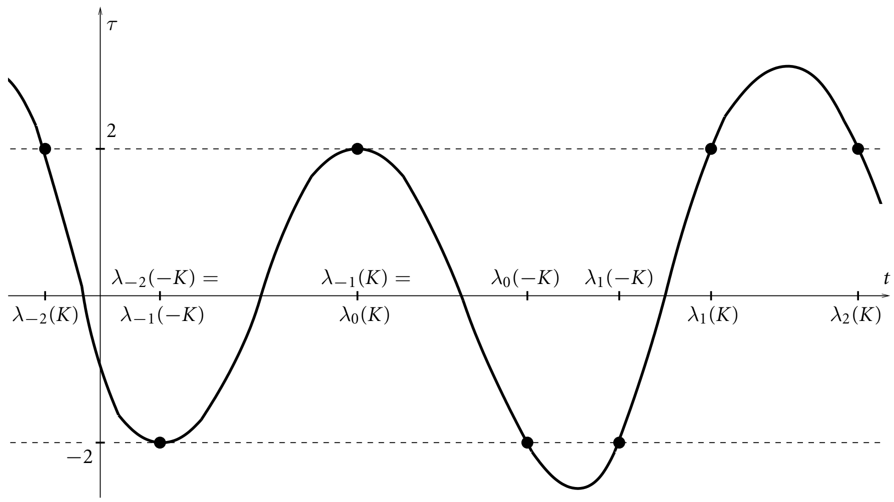

From these properties we see that the intersections, with their multiplicities counted, of the curve with the line $\tau = 2$ move continuously when $(a, b, 1/p, q, w) \in \boldsymbol{\Omega}$ and $K \in \mathrm{SL}(2, \mathbb{R})$ vary continuously, unless some of them disappear at $-\infty$ or $+\infty$. There is a similar statement for the intersections of this curve with the line $\tau = -2$.

To discuss the dependence of $\mu_n(K)$ and $\nu_n(K)$ on $K \in \mathrm{SL}(2, \mathbb{R})$, we define

$$\mathcal{L}_1 = \{K \in \mathrm{SL}(2, \mathbb{R})\ ;\ k_{11} > 0, k_{12} \leq 0\} \simeq \mathbb{R}_+ \times (-\infty, 0] \times \mathbb{R}, \tag{2.8}$$

$$\mathcal{L}_2 = \{K \in \mathrm{SL}(2, \mathbb{R})\ ;\ k_{11} \leq 0, k_{12} < 0\} \simeq (-\infty, 0] \times \mathbb{R}_- \times \mathbb{R}, \tag{2.9}$$

where $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Note that $\mathcal{L}_1 \cap \mathcal{L}_2 = \varnothing$, while the common part of the boundaries of $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\partial\mathcal{L}_1 \cap \partial\mathcal{L}_2 = \{K \in \mathrm{SL}(2, \mathbb{R})\ ;\ k_{11} = 0, k_{12} < 0\} \subset \mathcal{L}_2. \tag{2.10}$$

The following lemma is very similar to Lemma 4.3 in [6].

Lemma 2.11 *Let $n \in \mathbb{Z}$. Then, μ_n is continuous on $\mathcal{L}_1 \cup \mathcal{L}_2$, ν_n is continuous on $\mathcal{L}_1$ and on $\mathcal{L}_2$, and for any $K \in \partial\mathcal{L}_1 \cap \partial\mathcal{L}_2$,*

$$\lim_{\mathcal{L}_1 \ni L \to K} \nu_n(L) = \nu_{n-1}(K). \tag{2.12}$$

Proof From Lemma 1.29, (2.4) and (2.5) we see that $\mu_n(K)$ depends continuously on $K \in \mathrm{SL}(2, \mathbb{R})$ as long as $k_{12} \neq 0$, and $\nu_n(K)$ depends continuously on $K \in \mathrm{SL}(2, \mathbb{R})$ as long as $k_{11} \neq 0$. In particular, ν_n is continuous on $\mathcal{L}_1$.

Let $K \in \mathcal{L}_1 \cup \mathcal{L}_2$ with $k_{12} = 0$. Then, $K \in \mathcal{L}_1$, *i.e.*, $k_{11} > 0$, and hence $k_{22} > 0$ since now $k_{11}k_{22} = \det K = 1$. If $L \in \mathcal{L}_1 \cup \mathcal{L}_2$ is sufficiently close to K, then $l_{11}, l_{22} > 0$, which implies that $L \in \mathcal{L}_1$, *i.e.*, $l_{12} \leq 0$. Hence, $l_{12}/l_{22} \leq 0$. Thus, also by Lemma 1.29, $\mu_n(L) \to \mu_n(K)$ as $L \to K$ in $\mathcal{L}_1 \cup \mathcal{L}_2$. Therefore, μ_n is continuous on $\mathcal{L}_1 \cup \mathcal{L}_2$.

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Let $K \in \mathcal{L}_1 \cup \mathcal{L}_2$ with $k_{11} = 0$. Then, $K \in \mathcal{L}_2$, *i.e.*, $k_{12} < 0$, and hence $k_{21} > 0$ since now $-k_{12}k_{21} = \det K = 1$. If $L \in \mathcal{L}_1$ is sufficiently close to K, then $l_{11}, l_{21} > 0$, which implies that $l_{11}/l_{21} > 0$. Therefore, by Lemma 1.29 again, (2.12) is proved. If $L \in \mathcal{L}_2$ is sufficiently close to K, then $l_{11} \le 0$ and $l_{21} > 0$, which implies that $l_{11}/l_{21} \le 0$. Thus, $\nu_n(L) \to \nu_n(K)$ as $L \to K$ in $\mathcal{L}_2$. Therefore, ν_n is also continuous on $\mathcal{L}_2$. ■

Lemma 2.13 *Fix a Sturm-Liouville equation in $\boldsymbol{\Omega}$. Then, for any $K \in \mathcal{L}_1$,*

$$\tau_K\big(\nu_0(K)\big) \ge 2. \tag{2.14}$$

Proof Consider the SLE

$$-(py')' = \lambda y \quad \text{on } (-1,1), \tag{2.15}$$

where

$$p(t) = \begin{cases} -1 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{2.16}$$

Then, $\lambda = 0$ is an eigenvalue with an eigenfunction $y \equiv 1$ for both the periodic BC $[I \mid -I]$ and the Neumann BC $\mathbf{N} = \mathbf{S}_{\pi/2,\pi/2}$, and the Prüfer angle for 0 as an eigenvalue for $\mathbf{N}$ is $\theta \equiv \pi/2$. Thus, $\nu_0(I) = 0$ since the BC in (2.5) becomes $\mathbf{N}$ when $K = I$. Direct calculations also show that in this case, $\Phi(1,0) = I$, which implies (2.14) for the SLE (2.15) and $K = I$. Then, we obtain (2.14) for any SLE in $\boldsymbol{\Omega}$ and any $K \in \mathcal{L}_1$ from the connectedness of $\boldsymbol{\Omega}$ and $\mathcal{L}_1$, Lemma 2.6 (i) and Lemma 2.11. ■

Now, we are ready to prove the following results, the idea of whose proof is taken from [9]. Recall that by Theorem 1.14, the eigenvalues for any SLP in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ are all real.

Theorem 2.17 *Fix a Sturm-Liouville equation in $\boldsymbol{\Omega}$.*

(i) *If $K \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $\omega \in (-\pi,\pi)$, then: $\mu_n(K)$ is not an eigenvalue for $[e^{i\omega}K \mid -I]$ for any even $n \in \mathbb{Z}$; for each even $n \in \mathbb{Z}$, there are exactly two eigenvalues (counting multiplicity) for $[e^{i\omega}K \mid -I]$ in the interval $\big(\mu_n(K), \mu_{n+2}(K)\big)$, to be denoted by $\lambda_{n+1}(e^{i\omega}K)$ and $\lambda_{n+2}(e^{i\omega}K)$ in non-decreasing order.*

(ii) *If $K \in \mathrm{SL}(2,\mathbb{R}) \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$, then: $\mu_n(K)$ is not an eigenvalue for $[K \mid -I]$ for any odd $n \in \mathbb{Z}$; for each odd $n \in \mathbb{Z}$, there are exactly two eigenvalues (counting multiplicity) for $[K \mid -I]$ in the interval $\big(\mu_n(K), \mu_{n+2}(K)\big)$, to be denoted by $\lambda_{n+1}(K)$ and $\lambda_{n+2}(K)$ in non-decreasing order.*

In particular, there are infinitely many eigenvalues, unbounded from both below and above, for any coupled self-adjoint boundary condition.

(iii) *If $K \in \mathcal{L}_1 \cup \mathcal{L}_2$, then for any $\omega \in (-\pi, 0) \cup (0, \pi)$, we have that*

$$
\begin{aligned}
\cdots \leq \lambda_{-2}(K) &< \lambda_{-2}(e^{i\omega}K) < \lambda_{-2}(-K) \leq \{\mu_{-2}(K), \nu_{-1-\delta}(K)\} \\
&\leq \lambda_{-1}(-K) < \lambda_{-1}(e^{i\omega}K) < \lambda_{-1}(K) \leq \{\mu_{-1}(K), \nu_{0-\delta}(K)\} \\
&\leq \lambda_{0}(K) < \lambda_{0}(e^{i\omega}K) < \lambda_{0}(-K) \leq \{\mu_{0}(K), \nu_{1-\delta}(K)\} \\
&\leq \lambda_{1}(-K) < \lambda_{1}(e^{i\omega}K) < \lambda_{1}(K) \leq \{\mu_{1}(K), \nu_{2-\delta}(K)\} \leq \cdots,
\end{aligned} \tag{2.18}
$$

where $\delta = 0$ if $K \in \mathcal{L}_1$ and $\delta = 1$ if $K \in \mathcal{L}_2$.

(iv) *If $K \in \mathrm{SL}(2, \mathbb{R}) \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$, then (iii) applies to $-K$.*

Proof Our claims for the non-real self-adjoint BC's are consequences of those for the real self-adjoint BC's and Lemma 2.6 (ii). So, we only need to prove the claims for the real self-adjoint BC's.

For $h \in (0, 1]$, consider

$$K_h = \begin{pmatrix} 1/h & -1/h \\ 0 & h \end{pmatrix} \in \mathcal{L}_1. \tag{2.19}$$

Let $\mathbf{B} = \mathbf{S}_{\pi/4, \pi/2}$, *i.e.*,

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.20}$$

then we have that as $h \to 0^+$,

$$[K_h \,|\, -I] = \begin{bmatrix} 1 & -1 & -h & 0 \\ 0 & h & 0 & -1 \end{bmatrix} \longrightarrow \mathbf{B}, \tag{2.21}$$

$$[-K_h \,|\, -I] = \begin{bmatrix} 1 & -1 & h & 0 \\ 0 & -h & 0 & -1 \end{bmatrix} \longrightarrow \mathbf{B}. \tag{2.22}$$

Fix an even integer $n \geq 2$. Note that each $\lambda_i(\mathbf{B})$ is simple by Theorem 1.19. By Remarks 1.21 and 1.22, there is a connected neighborhood $\mathcal{N}$ of $\mathbf{B}$ in $\mathcal{B}^{\mathbb{R}}$ such that the simple continuous eigenvalue branch Λ_i through $\lambda_i(\mathbf{B})$ is defined on $\mathcal{N}$ if $-n-1 \leq i \leq n+2$, and for each $\mathbf{X} \in \mathcal{N}$, the only eigenvalues for $\mathbf{X}$ in the interval $(\frac{\lambda_{-n-2}(\mathbf{B})+\lambda_{-n-1}(\mathbf{B})}{2}, \frac{\lambda_{n+2}(\mathbf{B})+\lambda_{n+3}(\mathbf{B})}{2})$ are $\Lambda_{-n-1}(\mathbf{X}), \Lambda_{-n}(\mathbf{X}), \dots,$ and $\Lambda_{n+2}(\mathbf{X})$. From (2.21) and (2.22) we then deduce that for each $i \in \mathbb{Z}$ satisfying $-n-1 \leq i \leq n+2$,

$$\lim_{h \to 0^+} \Lambda_i(K_h) = \lambda_i(\mathbf{B}) = \lim_{h \to 0^+} \Lambda_i(-K_h). \tag{2.23}$$

Note that $\{\mu_i(K_h)\ ;\ i \in \mathbb{Z}\}$ are the eigenvalues for the BC

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & h & 1/h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -h^2 & -1 \end{bmatrix}, \tag{2.24}$$

which converges to

$$(2.25)\qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \mathbf{S}_{0,\pi/2}$$

as $h \to 0^+$, while $\{\nu_i(K_h)\ ;\ i \in \mathbb{Z}\}$ are the eigenvalues for the BC

$$(2.26)\qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/h \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \mathbf{S}_{\pi/2,\pi/2}.$$

Set $\mathbf{A} = \mathbf{S}_{0,\pi/2}$ and $\mathbf{C} = \mathbf{S}_{\pi/2,\pi/2}$. From (2.24), (2.25) and Lemma 1.29 we then obtain that for each $i \in \mathbb{Z}$,

$$(2.27)\qquad \lim_{h\to 0^+} \mu_i(K_h) = \lambda_i(\mathbf{A}).$$

Lemma 1.29 also implies that for any $h \in (0, 1]$,

$$(2.28)\qquad \begin{aligned} \cdots < \lambda_{-1}(\mathbf{C}) = \nu_{-1}(K_h) < \lambda_{-1}(\mathbf{B}) < \lambda_{-1}(\mathbf{A}) \\ < \lambda_0(\mathbf{C}) = \nu_0(K_h) < \lambda_0(\mathbf{B}) < \lambda_0(\mathbf{A}) \\ < \lambda_1(\mathbf{C}) = \nu_1(K_h) < \lambda_1(\mathbf{B}) < \lambda_1(\mathbf{A}) < \cdots . \end{aligned}$$

Then (2.23), (2.27) and (2.28) together yield that when $h > 0$ is sufficiently small, none of $\nu_{-n}(K_h)$ and $\mu_{n+1}(K_h)$ is an eigenvalue for any of K_h and $-K_h$, the only eigenvalues for K_h in the interval $\big(\nu_{-n}(K_h), \mu_{n+1}(K_h)\big)$ are $\Lambda_{-n}(K_h)$, $\Lambda_{-n+1}(K_h), \ldots,$ and $\Lambda_{n+1}(K_h)$, similarly for $-K_h$, and

$$(2.29)\qquad \begin{aligned} \nu_{-n}(K_h) < \big\{\Lambda_{-n}(K_h), \Lambda_{-n}(-K_h)\big\} < \mu_{-n}(K_h) \\ < \cdots \\ < \nu_{n+1}(K_h) < \big\{\Lambda_{n+1}(K_h), \Lambda_{n+1}(-K_h)\big\} < \mu_{n+1}(K_h). \end{aligned}$$

For such an h: since $\Lambda_0(-K_h)$ is simple and is the only eigenvalue for $-K_h$ in $\big(\nu_0(K_h), \mu_0(K_h)\big)$, from (2.29) and Lemma 2.6 (i) we see that $\tau_{K_h}\big(\mu_0(K_h)\big) \le -2$; similarly, $\tau_{K_h}\big(\nu_1(K_h)\big) \le -2$; since there are no eigenvalues for K_h in $\big(\mu_{-1}(K_h), \nu_0(K_h)\big]$, we must have $\tau_{K_h}\big(\mu_{-1}(K_h)\big) \ge 2, \ldots$; in total, for $i = -n, -n+2, \ldots, n$,

$$(2.30)\qquad \tau_{K_h}\big(\nu_i(K_h)\big) \ge 2, \quad \tau_{K_h}\big(\nu_{i+1}(K_h)\big) \le -2,$$

$$(2.31)\qquad \tau_{K_h}\big(\mu_i(K_h)\big) \le -2, \quad \tau_{K_h}\big(\mu_{i+1}(K_h)\big) \ge 2.$$

Note that each $\mu_i(K_h)$ and every $\nu_i(K_h)$ continuously depend on $h \in (0, 1]$. Thus, for *any* $h \in (0, 1]$, (2.30) and (2.31) are true, and hence (2.29) implies that

$$(2.32)\qquad \begin{aligned} \nu_{-n}(K_h) < \big\{\mu_{-n}(K_h), \nu_{-n+1}(K_h)\big\} \\ < \cdots \\ < \big\{\mu_n(K_h), \nu_{n+1}(K_h)\big\} < \mu_{n+1}(K_h). \end{aligned}$$

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For $h \in (0, 1]$, since these hold for an arbitrary even $n \in \mathbb{N}$, we have (2.30) and (2.31) for every even $i \in \mathbb{N}$, and

$$\begin{aligned}(2.33) \qquad \cdots < \{\mu_{-2}(K_h), \nu_{-1}(K_h)\} < \{\mu_{-1}(K_h), \nu_0(K_h)\} \\ < \{\mu_0(K_h), \nu_1(K_h)\} < \{\mu_1(K_h), \nu_2(K_h)\} < \cdots .\end{aligned}$$

Hence, for $h \in (0, 1]$, $\mu_i(K_h)$ is not an eigenvalue for K_h ($-K_h$, resp.) if i is even (odd, resp.). Let $m \in \mathbb{Z}$ be even. By Remarks 1.21 and 1.22 together with (2.29), Λ_{m+1} and Λ_{m+2} have continuous extensions to the whole $\{K_h ; 0 < h \leq 1\}$ such that for each $h \in (0, 1]$, $\Lambda_{m+1}(K_h)$ and $\Lambda_{m+2}(K_h)$ are the only eigenvalues for K_h in $\big(\mu_m(K_h), \mu_{m+2}(K_h)\big)$, while Λ_{m+2} and Λ_{m+3} have continuous extensions to the entire $\{-K_h ; 0 < h \leq 1\}$ such that for each $h \in (0, 1]$, $\Lambda_{m+2}(-K_h)$ and $\Lambda_{m+3}(-K_h)$ are the only eigenvalues for $-K_h$ in $\big(\mu_{m+1}(K_h), \mu_{m+3}(K_h)\big)$. These together with (2.30), (2.31) and (2.33) yield our claims for K_h with $h \in (0, 1]$. Our claims for the general $K \in \mathcal{L}_1$ and for any $K \in \mathcal{L}_2$ follow from those for K_h together with Remark 2.7, Lemma 2.6 (i) and Lemma 2.11. Finally, if $K \in \mathrm{SL}(2, \mathbb{R})$ is not in $\mathcal{L}_1 \cup \mathcal{L}_2$, then $-K$ is in $\mathcal{L}_1 \cup \mathcal{L}_2$. This finishes the proof. ■

Remark 2.34 By Remark 2.7, Lemma 2.6 (i), Lemma 2.11 and the connectedness of $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$, we only need to show the inequalities for one particular SLE in $\boldsymbol{\Omega}$, such as (2.15), and one particular K in $\mathrm{SL}(2, \mathbb{R})$, such as I. On the other hand, such a direct proof may involve some detailed estimates, which are avoided in our proof above.

Remark 2.35 The unboundedness from below of the eigenvalues of any problem in $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ is proved in [16] using operator theory.

Remark 2.36 Fix an SLE in $\boldsymbol{\Omega}$, and let $n \in \mathbb{Z}$. The eigenvalue $\lambda_n(\mathbf{S}_{\alpha,\beta})$ for a separated self-adjoint BC $\mathbf{S}_{\alpha,\beta}$ can be computed from its Prüfer angle definition given in Theorem 1.27. Our inequalities in Theorem 2.17 can be used to construct an algorithm for computing the eigenvalue $\lambda_n(e^{i\omega}K)$ for an *arbitrary coupled* self-adjoint BC $[e^{i\omega}K \mid -I]$: one finds $\mu_{n-1}(K)$ and $\mu_n(K)$ first, then $\lambda_n(e^{i\omega}K)$ is the only zero (possibly double) of $\tau_K(\lambda) = 2\cos\omega$ in the interval $[\mu_{n-1}(K), \mu_n(K)]$ and can be located by using a root finder. Note that when n is even, $\lambda_n(K) \neq \lambda_{n+1}(K)$ if $K \in \mathcal{L}_1 \cup \mathcal{L}_2$, and $\lambda_n(K) \neq \lambda_{n-1}(K)$ if $K \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathcal{L}_1 \cup \mathcal{L}_2$. Such an algorithm has been implemented in the code `SLEIGN2` for the case where p is positive, see [2].

In the following consequence of our inequalities, for any real function f, we use f_- and f_+ to denote the positive and negative parts of f, respectively, such that $f = f_+ - f_-$.

Theorem 2.37 *The eigenvalues* $\{\lambda_n ; n \in \mathbb{Z}\}$ *of any Sturm-Liouville problem* $(a, b, 1/p, q, w ; \mathbf{A}) \in \boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ *satisfy*

$$(2.38) \qquad \lambda_n \sim \pm \frac{n^2\pi^2}{\left[\int_a^b \frac{w(t)}{p_\pm(t)}\, dt\right]^2} \quad \text{as } n \to \pm\infty.$$

Proof When the self-adjoint BC in the problem is a separated one, the result has been proved in [3]; when the BC is a coupled one, the result follows from the separated case and Theorem 2.17 (i), (ii). ■

3 Discontinuity and Range of λ_n

For each $n \in \mathbb{Z}$, we now have a function λ_n on $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ given by Theorem 2.17. In this section, we characterize the discontinuities of λ_n on $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$ and determine the range of λ_n on $\mathcal{B}^{\mathbb{C}}$.

To study the discontinuities of λ_n on $\mathcal{B}^{\mathbb{C}}$, we will need the notation

$$\mathcal{F}^{\mathbb{C}}_{-} = \{[e^{i\omega}K \mid -I]\,;\, K \in \mathrm{SL}(2,\mathbb{R}),\ k_{11}k_{12} \le 0,\ \omega \in [0,\pi)\}, \tag{3.1}$$

$$\mathcal{G}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} a_1 & 1 & 0 & -\bar{z} \\ z & 0 & -1 & b_2 \end{bmatrix}\,;\, b_2 \le 0,\ a_1 \in \mathbb{R},\ z \in \mathbb{C}\right\}, \tag{3.2}$$

$$\mathcal{H}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} 1 & a_2 & -\bar{z} & 0 \\ 0 & z & b_1 & -1 \end{bmatrix}\,;\, a_2 \le 0,\ b_1 \in \mathbb{R},\ z \in \mathbb{C}\right\}, \tag{3.3}$$

$$\mathcal{F}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{6} \setminus \mathcal{F}^{\mathbb{C}}_{-}, \quad \mathcal{G}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{4} \setminus \mathcal{G}^{\mathbb{C}}_{-}, \quad \mathcal{H}^{\mathbb{C}}_{+} = \mathcal{O}^{\mathbb{C}}_{3} \setminus \mathcal{H}^{\mathbb{C}}_{-}, \tag{3.4}$$

$$\mathcal{I}^{\mathbb{C}}_{-} = \left\{\begin{bmatrix} 1 & a_2 & 0 & \bar{z} \\ 0 & z & -1 & b_2 \end{bmatrix}\,;\, a_2, b_2 \le 0,\ z \in \mathbb{C},\ a_2b_2 \ge z\bar{z}\right\}, \tag{3.5}$$

$$\mathcal{I}^{\mathbb{C}}_{+} = \left\{\begin{bmatrix} 1 & a_2 & 0 & \bar{z} \\ 0 & z & -1 & b_2 \end{bmatrix}\,;\, a_2, b_2 > 0,\ z \in \mathbb{C},\ a_2b_2 > z\bar{z}\right\}, \tag{3.6}$$

$$\mathcal{I}^{\mathbb{C}}_{0} = \mathcal{O}^{\mathbb{C}}_{2} \setminus (\mathcal{I}^{\mathbb{C}}_{-} \cup \mathcal{I}^{\mathbb{C}}_{+}), \tag{3.7}$$

$$\begin{aligned} \mathcal{J}^{\mathbb{C}} = {} & \{[e^{i\omega}K \mid -I]\,;\, K \in \mathrm{SL}(2,\mathbb{R}),\ k_{12} = 0,\ \omega \in [0,\pi)\} \\ & \cup \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{B}^{\mathbb{R}}\,;\, a_2b_2 = 0\right\}. \end{aligned} \tag{3.8}$$

Note that the separated BC's in $\mathcal{J}^{\mathbb{C}}$ other than the Dirichlet BC are in $\mathcal{G}^{\mathbb{C}}_{-} \cup \mathcal{H}^{\mathbb{C}}_{-}$. To discuss the discontinuities of λ_n on $\mathcal{B}^{\mathbb{R}}$, we will use $\mathcal{F}^{\mathbb{R}}_{\pm}$, $\mathcal{G}^{\mathbb{R}}_{\pm}$, $\mathcal{H}^{\mathbb{R}}_{\pm}$, $\mathcal{I}^{\mathbb{R}}_{\pm}$, $\mathcal{I}^{\mathbb{R}}_{0}$ and $\mathcal{J}^{\mathbb{R}}$, which can be defined using (3.1)–(3.8) with $\omega = 0$ and $\mathbb{C}$ replaced by $\mathbb{R}$. Note that the coupled BC's in $\mathcal{J}^{\mathbb{R}}$ are all in $\mathcal{F}^{\mathbb{R}}_{-}$, and

$$\mathcal{J}^{\mathbb{R}} \cap \mathcal{T} = (\mathcal{J}^{\mathbb{R}} \cap \mathcal{G}^{\mathbb{R}}_{-}) \cup (\mathcal{J}^{\mathbb{R}} \cap \mathcal{H}^{\mathbb{R}}_{-}) \cup \{\mathbf{D}\}. \tag{3.9}$$

Theorem 3.10

(i) *Fix a self-adjoint boundary condition. Then, for any $n \in \mathbb{Z}$, the function λ_n is continuous on $\boldsymbol{\Omega}$.*

(ii) *For any $n \in \mathbb{Z}$, the function λ_n is continuous on $\boldsymbol{\Omega} \times (\mathcal{B}^{\mathbb{R}} \setminus \mathcal{J}^{\mathbb{R}})$ and at each problem with a coupled boundary condition in $\mathcal{J}^{\mathbb{R}}$ where $\lambda_n = \lambda_{n-1}$, and discontinuous at any other point of $\boldsymbol{\Omega} \times \mathcal{J}^{\mathbb{R}}$. More precisely, if $n \in \mathbb{Z}$, then we have the following: for each problem $(\boldsymbol{\omega}, \mathbf{A}) \in \boldsymbol{\Omega} \times \mathcal{J}^{\mathbb{R}}$ with a coupled boundary condition $\mathbf{A}$, the*

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restriction of λ_n to $\boldsymbol{\Omega} \times \mathcal{F}^{\mathbb{R}}_{-}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\times\mathcal{F}^{\mathbb{R}}_{+}\ni(\boldsymbol{\sigma},\mathbf{B})\to(\boldsymbol{\omega},\mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \mathbf{A}); \tag{3.11}$$

for each problem $(\boldsymbol{\omega}, \mathbf{A}) \in \boldsymbol{\Omega} \times \mathcal{J}^{\mathbb{R}}$ with $\mathbf{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{G}^{\mathbb{R}}_{-}$, the restriction of λ_n to $\boldsymbol{\Omega} \times \mathcal{G}^{\mathbb{R}}_{-}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\times\mathcal{G}^{\mathbb{R}}_{+}\ni(\boldsymbol{\sigma},\mathbf{B})\to(\boldsymbol{\omega},\mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \mathbf{A}); \tag{3.12}$$

for each problem $(\boldsymbol{\omega}, \mathbf{A}) \in \boldsymbol{\Omega} \times \mathcal{J}^{\mathbb{R}}$ with $\mathbf{A} \in \mathcal{J}^{\mathbb{R}} \cap \mathcal{H}^{\mathbb{R}}_{-}$, the restriction of λ_n to $\boldsymbol{\Omega} \times \mathcal{H}^{\mathbb{R}}_{-}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\times\mathcal{H}^{\mathbb{R}}_{+}\ni(\boldsymbol{\sigma},\mathbf{B})\to(\boldsymbol{\omega},\mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \mathbf{A}); \tag{3.13}$$

while for each problem $(\boldsymbol{\omega}, \mathbf{D}) \in \boldsymbol{\Omega} \times \mathcal{J}^{\mathbb{R}}$ with the Dirichlet boundary condition $\mathbf{D}$, the restriction of λ_n to $\boldsymbol{\Omega} \times \mathcal{I}^{\mathbb{R}}_{-}$ is continuous at $(\boldsymbol{\omega}, \mathbf{D})$, and

$$\lim_{\boldsymbol{\Omega}\times\mathcal{I}^{\mathbb{R}}_{0}\ni(\boldsymbol{\sigma},\mathbf{B})\to(\boldsymbol{\omega},\mathbf{D})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n-1}(\boldsymbol{\omega}, \mathbf{D}), \tag{3.14}$$

$$\lim_{\boldsymbol{\Omega}\times\mathcal{I}^{\mathbb{R}}_{+}\ni(\boldsymbol{\sigma},\mathbf{B})\to(\boldsymbol{\omega},\mathbf{D})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n-2}(\boldsymbol{\omega}, \mathbf{D}). \tag{3.15}$$

(iii) *The conclusions of (ii) still hold when all the super indices $\mathbb{R}$ are replaced by $\mathbb{C}$.*

Proof (i) From the Prüfer angle characterization of the eigenvalues for separated self-adjoint BC's we see that λ_n depends continuously on $\boldsymbol{\omega} \in \boldsymbol{\Omega}$. For coupled self-adjoint BC's, the continuity of λ_n in $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ is a consequence of Remark 2.7 and Theorem 2.17.

(ii) and (iii) Fix $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ and let $n \in \mathbb{Z}$. From the proof of Lemma 2.11 we see the following: μ_n is continuous on $\mathcal{F}^{\mathbb{R}}_{-}$, which yields that so is λ_n; and for any $K \in \mathrm{SL}(2, \mathbb{R})$ with $k_{12} = 0$,

$$\lim_{\mathcal{F}^{\mathbb{R}}_{+}\ni[L|-I]\to[K|-I]} \mu_n(L) = \mu_{n-1}(K), \tag{3.16}$$

which implies that

$$\lim_{\mathcal{F}^{\mathbb{R}}_{+}\ni[L|-I]\to[K|-I]} \lambda_n(\boldsymbol{\omega}, [L \mid -I]) = \lambda_{n-1}(\boldsymbol{\omega}, [K \mid -I]). \tag{3.17}$$

Then, by arguments similar to those in the proof of Theorem 3.76 in [11], the continuity of λ_n on $\mathcal{F}^{\mathbb{R}}_{-}$ gives the continuity of λ_n on $\boldsymbol{\Omega} \times \mathcal{F}^{\mathbb{R}}_{-}$, and (3.17) yields (3.11). The other claims can be shown similarly (see also the proof of Theorem 3.39 in [11]). ∎

As a direct consequence of Theorem 3.18, we have the following *global* existence of continuous eigenvalue branches, which can be directly established using the reality and unboundedness from both below and above of the eigenvalues. In this result, by a *coordinate subset* we mean an open subset that can be used in a coordinate system.

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Corollary 3.18 *Fix a Sturm-Liouville equation in $\boldsymbol{\Omega}$, and let $\mathcal{O}$ be a coordinate subset of $\mathcal{T}$ or $\mathcal{B}^{\mathbb{R}}$ or $\mathcal{B}^{\mathbb{C}}$. Then, there are continuous eigenvalue branches $\dots, \Lambda_{-1}, \Lambda_0, \Lambda_1, \dots$ on $\mathcal{O}$ such that for any $\mathbf{A} \in \mathcal{O}$,*

(i) $\dots \le \Lambda_{-1}(\mathbf{A}) \le \Lambda_0(\mathbf{A}) \le \Lambda_1(\mathbf{A}) \le \dots,$
(ii) $\dots, \Lambda_{-1}(\mathbf{A}), \Lambda_0(\mathbf{A}), \Lambda_1(\mathbf{A}), \dots$ *are the eigenvalues for* $\mathbf{A}$.

Proof This follows from the fact that as $\mathbf{A}$ varies in $\mathcal{O}$, none of the eigenvalues for $\mathbf{A}$ disappears, even though the indices of the eigenvalues may change at some BC's. ∎

Remark 3.19 There are smooth manifold structures on $\mathcal{B}^{\mathbb{R}}$ and $\mathcal{B}^{\mathbb{C}}$ [13], and one can study the differentiability of λ_n at its continuous points. Since the discontinuities of λ_n are completely characterized by Theorem 3.10, the derivative formulas (in [15]) for continuous eigenvalue branches imply derivative formulas for λ_n and hence monotonicity results. We omit the details here, and refer interested readers to Section 5 in [11] and Section 4 in [14].

Fix an SLE in $\boldsymbol{\Omega}$ and consider λ_n as a function on $\mathcal{T}$ or $\mathcal{B}^{\mathbb{R}}$ or $\mathcal{B}^{\mathbb{C}}$. Let $\lambda_n^{\mathbf{D}}$ be the value of λ_n at the Dirichlet BC $\mathbf{D}$. We have the following results.

Theorem 3.20

(i) *For each $n \in \mathbb{Z}$, the range of λ_n on the space $\mathcal{T}$ of separated self-adjoint boundary conditions is $(\lambda_{n-2}^{\mathbf{D}}, \lambda_n^{\mathbf{D}}]$.*
(ii) *For each $n \in \mathbb{Z}$, the range of λ_n on the space $\mathcal{B}^{\mathbb{R}}$ of real self-adjoint boundary conditions is the same as that on $\mathcal{T}$, and the range of λ_n on the space $\mathcal{B}^{\mathbb{C}}$ of complex self-adjoint boundary conditions is also the same as that on $\mathcal{T}$.*

Proof The proof of Theorem 4.1 in [11] can be used here with only minor and obvious modifications, so we omit the details. ∎

4 Number of Zeros of Eigenfunctions

In this section, we study the number of zeros of eigenfunctions for an arbitrarily fixed SLE $(a, b, 1/p, q, w) \in \boldsymbol{\Omega}$ and illustrate our results with examples.

In the case where p is positive, for each $n \in \mathbb{N}$, an eigenfunction for the n-th eigenvalue for a separated self-adjoint BC has exactly $n-1$ zeros in the interior of the SLE's interval. In the case considered in this paper, an eigenfunction may have infinitely many zeros (see, for example, the introduction of [1] and Theorem 3.2 in [4]). In the definition of the Prüfer angle, we used the following ranges of α and β: $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$. These choices imply that if either $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ or both $\alpha, \beta \in (0,\pi)$, then any eigenfunction y_n for λ_n for a separated self-adjoint BC has at least $|n|$ zeros in (a, b). For the remaining situations (all with negative $n \in \mathbb{Z}$), the minimum number of zeros of y_n is as follows: $|n+1|$ if either $\alpha = 0$ and $\beta \in (0,\pi)$, or $\alpha \in (0,\pi)$ and $\beta = \pi$; 0 if $\alpha = 0$, $\beta = \pi$ and $n = -1$; and $|n+2|$ if $\alpha = 0$, $\beta = \pi$ and $n \le -2$. These minimum numbers of zeros (even up to $|n|$ when $n < 0$) are achieved, and examples can be easily constructed. The following example shows that the number of zeros of y_n, when finite, can be any integer larger than the corresponding minimum given above. In this example, for $r \in \mathbb{R}$, $\lceil r \rceil$ stands for the smallest integer larger than or equal to r; and for any two constants c and d, the function $f_{c,d}$ is defined by

$$f_{c,d}(t) = \begin{cases} c & \text{if } 0 < t < 1, \\ d & \text{if } 1 < t < 2. \end{cases} \tag{4.1}$$

Example 4.2 Let $w \in \mathrm{L}\big((0,2), \mathbb{R}_+\big)$, $\alpha \in [0,\pi)$, $\beta \in (0,\pi]$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. If we set

$$\hat{c} = \left(\left\lceil \frac{\min\{\alpha, \beta + n\pi\}}{\pi} \right\rceil - \frac{k+1}{2} \right) \pi, \quad c = \alpha - \hat{c}, \quad d = \beta + n\pi - \hat{c}, \tag{4.3}$$

then $c, d > 0$, the eigenvalue λ_n of the SLP consisting of

$$-(f_{-1/c,1/d}\, y')' + f_{c,-d}\, y = \lambda w y \quad \text{on } (0,2) \tag{4.4}$$

and $\mathbf{S}_{\alpha,\beta}$ equals 0, and its eigenfunctions have in $(0,2)$ exactly $|n| + k$ zeros if $\alpha \neq 0$ and exactly $|n+1| + k$ zeros if $\alpha = 0$.

To see these, we just need to notice that from the Prüfer transformation of the SLE (4.4) with $\lambda = 0$ we obtain

$$\theta' = f_{-c,d}, \tag{4.5}$$

and hence

$$\theta(t) = \begin{cases} \alpha - ct & \text{if } 0 \le t \le 1, \\ \hat{c} + d(t-1) & \text{if } 1 < t \le 2. \end{cases} \tag{4.6}$$

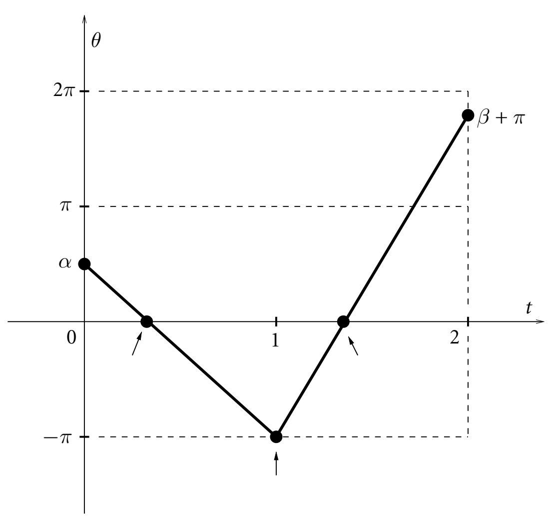

Figure 1

Figure 1 indicates the situation with $n = 1$ and $k = 3$. In the figure, the 3 points corresponding to the "extra" zeros of the eigenfunctions are pointed to by arrows.

Recall that p is said to *change sign finitely often* if there are an integer k and numbers $a_0 = a < a_1 < a_2 < \cdots < a_{k+1} = b$ such that either $p < 0$ a.e. on each (a_i, a_{i+1}) with an even i and $p > 0$ a.e. on each (a_i, a_{i+1}) with an odd i, or $p > 0$ a.e. on each (a_i, a_{i+1}) with an even i and $p < 0$ a.e. on each (a_i, a_{i+1}) with an odd i; in this case, $a_1, a_2, \ldots$ and a_k are called the *turning points* of p, and we sometimes also say that p *changes sign k times*.

When p changes sign only finitely often, an asymptotic for the number $N(\lambda)$ of zeros of a solution of the SLE is obtained in [1]:

$$N(\lambda) \sim \frac{\sqrt{\pm\lambda}}{\pi} \int_a^b \sqrt{\frac{w(t)}{p_{\pm}(t)}}\, \mathrm{d}t \quad \text{as } \lambda \to \pm\infty. \tag{4.7}$$

We are interested not only in the number of zeros of an eigenfunction, but also in the relation between the index of an eigenvalue and the zeros of an eigenfunction for this eigenvalue. In comparison with the case where p is positive, from the DE

$$\theta' = \frac{1}{p}\cos^2\theta + (\lambda w - q)\sin^2\theta \quad \text{on } (a, b) \tag{4.8}$$

for the Prüfer angle θ of an eigenfunction we see the following new features in the case where p changes sign:

(i) θ can cross an integer multiple of π downward; on the open subintervals of (a, b) on which $p < 0$ a.e., θ is strictly decreasing when passing through an integer multiple of π;

(ii) at some points in (a, b), θ can have integer multiples of π as local extrema, corresponding precisely to the zeros of the eigenfunction that are local extreme points of the eigenfunction; if p changes sign only finitely often, then each zero that is a local extreme point of the eigenfunction must be a turning point of p;

(iii) θ can take an integer multiple of π infinitely many times, corresponding precisely to accumulating zeros of the eigenfunction.

Now, we are ready to determine the index of an eigenvalue for a separated self-adjoint BC from a certain weighted count of the zeros of its eigenfunctions.

Theorem 4.9 *Assume that $n \in \mathbb{Z}$, a real eigenfunction y_n for the eigenvalue λ_n for a separated self-adjoint boundary condition has a finite set $\mathcal{Z}$ of zeros on (a, b) which are not local extreme points of y_n, and each zero in $\mathcal{Z} \cup \{a, b\}$ has a neighborhood in $[a, b]$ on which p does not change sign. Let n_+ be the number of zeros of y_n in (a, b) having neighborhoods on which $p > 0$ a.e., and n_- the number of zeros of y_n in $[a, b]$ having neighborhoods in $[a, b]$ on which $p < 0$ a.e. Then $n = n_+ - n_-$.*

Proof Let θ_n be the Prüfer angle of y_n, and $\mathcal{Z}_-$ the set of zeros of y_n in $[a, b]$ having neighborhoods in $[a, b]$ on which $p < 0$ a.e. Since y_n only has finitely many zeros that are not local extreme points, θ_n crosses integer multiples of π only finitely many times. Our assumptions also imply that near each of these crossings, θ_n is either strictly decreasing or strictly increasing, depending on the a.e. sign of p on a neighborhood in $[a, b]$ of the corresponding zero. If neither a nor b is in $\mathcal{Z}_-$, then

$$(n_+ - n_-)\pi < \theta_n(b) \le (n_+ - n_- + 1)\pi \tag{4.10}$$

as in the case where p is positive. If $a \in \mathcal{Z}_-$ and $b \notin \mathcal{Z}_-$, then $\alpha = 0$, $\theta_n(\epsilon) \in (-\pi, 0)$ for any $\epsilon > 0$ sufficiently small, and hence (4.10) still holds. We can prove (4.10) for the other subcases similarly. Therefore, $n = n_+ - n_-$. ■

Corollary 4.11 *Assume that p changes sign only finitely often, $n \in \mathbb{Z}$, and y_n is a real eigenfunction for the eigenvalue λ_n for a separated self-adjoint boundary condition. Let n_+ be the number of zeros of y_n in (a, b) having neighborhoods on which $p > 0$ a.e., and n_- the number of zeros of y_n in $[a, b]$ having neighborhoods in $[a, b]$ on which $p < 0$ a.e. Then $n = n_+ - n_-$.*

Remark 4.12 In numerical approximations of an eigenfunction, a zero that is a local extreme point is not stable: it may disappear or become two nearby zeros. If p changes sign only finitely often, the identity $n = n_+ - n_-$ is not affected by this instability, since such a point must be a turning point of p (and hence n_+ and n_- are changed by the same integer during approximations) in this case.

We can extend Theorem 4.9 to cover the situation where eigenfunctions have infinitely many zeros that are not local extreme points. This situation needs more care. The main idea here is that only finitely many of these zeros are essential for our count. To indicate this idea, we only prove the following result. In this result, for a finite set $\mathcal{S}$, we will use $\#\mathcal{S}$ to denote the number of elements in $\mathcal{S}$.

Theorem 4.13 *Let $n \in \mathbb{Z}$. Assume that a real eigenfunction y_n for the eigenvalue λ_n for a separated self-adjoint boundary condition has an infinite set $\mathcal{Z}$ of zeros on (a, b) which are not local extreme points of y_n, a is the only accumulation point of $\mathcal{Z}$, and each zero in $\mathcal{Z} \cup \{b\}$ has a neighborhood in $(a, b]$ on which p does not change sign. Let $\mathcal{Z}_-$ be the set of zeros of y_n in $(a, b]$ having neighborhoods in $(a, b]$ on which $p < 0$ a.e., and $\mathcal{Z}_+$ the set of zeros of y_n in (a, b) having neighborhoods on which $p > 0$ a.e. Then,*

(i) *for some zero $\hat{a} \in (a, b)$, between any two consecutive points in $(a, \hat{a}] \cap \mathcal{Z}_-$ there is a unique point in $\mathcal{Z}_+$, and between any two consecutive points in $(a, \hat{a}] \cap \mathcal{Z}_+$ there is a unique point in $\mathcal{Z}_-$;*
(ii) *for each zero $\hat{a} \in (a, b)$ satisfying the requirements in (i), we have that*

$$n = \#\big((\hat{a}, b) \cap \mathcal{Z}_+\big) - \#\big([\hat{a}, b] \cap \mathcal{Z}_-\big). \tag{4.14}$$

Proof Now we have $\alpha = 0$. Let θ_n be the Prüfer angle of y_n. Then, there is an $\hat{a} \in (a, b)$ such that $-\pi < \theta_n(t) < \pi$ for $t \in (a, \hat{a}]$. So, $\theta_n = 0$ on $(a, \hat{a}] \cap \mathcal{Z}$. Let t_1 and t_2 be two points in $(a, \hat{a}] \cap \mathcal{Z}_-$ satisfying $t_1 < t_2$, then for some $\delta > 0$,

$$\theta_n < 0 \text{ on } (t_1, t_1 + \delta), \quad \theta_n > 0 \text{ on } (t_2 - \delta, t_2). \tag{4.15}$$

Thus, there is a zero of θ_n that is not a local extreme point of θ_n (*i.e.*, a point of $\mathcal{Z}$) in (t_1, t_2), and at least one such zero must be in $\mathcal{Z}_+$ by (4.15) and our assumptions. Similarly, we show that between each pair of points in $(a, \hat{a}] \cap \mathcal{Z}_+$, there is a point in $\mathcal{Z}_-$. Then, between each pair of consecutive points in $(a, \hat{a}] \cap \mathcal{Z}_-$, there is a unique point in $\mathcal{Z}_+$; between each pair of consecutive points in $(a, \hat{a}] \cap \mathcal{Z}_+$, there is a unique point in $\mathcal{Z}_-$. Since a is an accumulation point of $\mathcal{Z}$, we can pick a zero of y_n as $\hat{a}$. This proves (i).

For any zero $\hat{a}$ of y_n in (a, b) satisfying the requirements in (i), we notice that the SLP with its interval (a, b) replaced by $(\hat{a}, b)$ also has y_n as an eigenfunction for the eigenvalue λ_n (with the index n), since $\theta_n(\hat{a}) = 0$. Therefore, (ii) follows from Theorem 4.9. ∎

Remark 4.16 If there are numbers $b_1 = b > b_2 > b_3 > \cdots > a$ such that for each $i \in \mathbb{N}$, p does not change sign on (b_{i+1}, b_i), then p satisfies the sign requirement at the zeros of eigenfunctions in Theorem 4.13. This is the case if p is continuous on (a, b) and its zeros do not accumulate in (a, b).

The following example illustrates Theorem 4.13 and is taken from Theorem 3.2 in [4].

Example 4.17 Let $b > 0$,

$$\frac{1}{p(t)} = 2t \cos\frac{1}{t} + \sin\frac{1}{t}, \quad q \in \mathrm{L}\big((0, b), \mathbb{R}\big), \quad w \in \mathrm{L}\big((0, b), \mathbb{R}_+\big). \tag{4.18}$$

Then, by Theorem 3.2 in [4], any eigenfunction for the SLP consisting of $(0, b, 1/p, q, w)$ and $\mathbf{S}_{\alpha,\beta} \in \mathcal{T}$ has infinitely many zeros (accumulating at 0, of course). Since p is continuous on $(0, b)$ and its zeros do not accumulate in $(0, b)$, we can apply Theorem 4.13.

As a particular case of this example, we consider the eigenfunction

$$y_*(t) = t^2 \cos\frac{1}{t} \tag{4.19}$$

for the eigenvalue $\lambda_* = 0$ when $q \equiv 0$. Its set of zeros is

$$\left(\left\{ t_k := \frac{1}{k\pi + \pi/2}\ ; k \in \mathbb{N}_0 \right\} \cap (0, b]\right) \cup \{0\}. \tag{4.20}$$

Since

$$p(t_k) = (-1)^k \neq 0 \quad \text{for } k \in \mathbb{N}_0, \tag{4.21}$$

we can take $\hat{a}$ to be the largest zero in $(0, b)$. Thus, $\lambda_{-1} = 0$ if $b \in [t_k, t_{k-1}]$ for some odd $k \in \mathbb{N}$, and $\lambda_0 = 0$ if $b \in (t_k, t_{k-1})$ for some even $k \in \mathbb{N}$ or $b \in (\frac{2}{\pi}, +\infty)$. (These claims can also be obtained by directly looking at the Prüfer angle of y_*, which is also easy in this particular case, since now $py' \equiv 1$.)

By Theorem 3.4 in [4], under some additional assumptions, the quasi-derivative py_n' of an eigenfunction y_n for λ_n for a separated self-adjoint BC has exactly $|n|$ zeros in (a,b) when $|n|$ is sufficiently large. The main idea in the proof of this fact in [4] can be used to prove the following partial refinements of (4.7) under additional assumptions.

Theorem 4.22 *Assume that p changes sign k times, and $\{\lambda_n\ ;\ n \in \mathbb{Z}\}$ are the eigenvalues for a separated self-adjoint boundary condition.*

(i) *If q/w has an essential upper bound λ^*, then $\lambda_{-2} \le \lambda^*$, and for any $n \in \mathbb{Z}$ such that $\lambda_n > \lambda^*$, each eigenfunction for λ_n has at most $|n| + k + 1$ zeros in (a,b).*
(ii) *If q/w has an essential lower bound λ_*, then $\lambda_2 \ge \lambda^*$, and for any $n \in \mathbb{Z}$ such that $\lambda_n < \lambda_*$, each eigenfunction for λ_n has at most $|n| + k + 1$ zeros in (a,b).*

In particular, if $q \equiv 0$, then $\lambda_{-2} \le 0$, $\lambda_2 \ge 0$, and for any $n \in \mathbb{Z}$ such that $\lambda_n \ne 0$, each eigenfunction for λ_n has at most $|n| + k + 1$ zeros in (a,b).

Proof (i) When $\lambda > \lambda^*$, (4.8) implies that θ is strictly increasing when passing through an odd integer multiple of $\pi/2$ (*i.e.*, one of $\dots, -\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \dots$). Thus, $\lambda_{-2} \le \lambda^*$. Now, let $n \in \mathbb{Z}$ such that $\lambda_n > \lambda^*$. On each open subinterval of (a,b) on which $p < 0$ a.e., θ can cross (downward) at most one integer multiple of π. After each downward crossing of an integer multiple of π, there is exactly one upperward crossing of the same multiple of π before θ can pass through any other integer multiple of π, and this upperward crossing must occur in an open subinterval of (a,b) on which $p > 0$ a.e. Therefore, θ crosses integer multiples of π at most $|n| + k + 1$ times, *i.e.*, y_n has at most $|n| + k + 1$ zeros.

(ii) This can be shown similarly. ∎

All the above results can be generalized to cover coupled self-adjoint BC's. Here, we only give the following results, which are similar to some results in the case where p is positive (see, for example, [11, Theorem 4.8]).

Theorem 4.23 *Let $n \in \mathbb{Z}$.*

(i) *Assume that a real eigenfunction y_n for the eigenvalue λ_n for a real coupled self-adjoint boundary condition has a finite set $\mathcal{Z}$ of zeros on (a,b) which are not local extreme points of y_n, and each zero in $\mathcal{Z} \cup \{a,b\}$ has a neighborhood in $[a,b]$ on which p does not change sign. Let n_+ be the number of zeros of y_n in (a,b) having neighborhoods on which $p > 0$ a.e., and n_- the number of zeros of y_n in $[a,b]$ having neighborhoods in $[a,b]$ on which $p < 0$ a.e. Then, either $n - 1 = n_+ - n_-$ or $n = n_+ - n_-$ if $\lambda_n \le \lambda^{\mathbf{D}}_{n-1}$, and either $n = n_+ - n_-$ or $n + 1 = n_+ - n_-$ if $\lambda_n > \lambda^{\mathbf{D}}_{n-1}$.*
(ii) *Assume that y_n is an eigenfunction for the eigenvalue λ_n for a non-real coupled self-adjoint boundary condition. Then, under the same assumptions on* $\operatorname{Re} y_n$ *(or* $\operatorname{Im} y_n$*) as in (i), we have the same conclusions for* $\operatorname{Re} y_n$ *(or* $\operatorname{Im} y_n$*) as in (i). Moreover, y_n is never zero on $[a,b]$.*

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Proof (i) By Theorem 3.20, $\lambda^{\mathbf{D}}_{n-2} < \lambda_n \leq \lambda^{\mathbf{D}}_n$. Let θ_n be the Prüfer angle of y_n, $\alpha = \theta_n(a)$, and $\theta_n(b) = \beta + m\pi$, where $\beta \in (0, \pi]$ and $m \in \mathbb{Z}$. Then, $\alpha \in [0, \pi)$, $\lambda_m(\mathbf{S}_{\alpha,\beta}) = \lambda_n$, and y_n is an eigenfunction for $\lambda_m(\mathbf{S}_{\alpha,\beta})$. Thus, by Theorem 4.9, we have that $m = n_+ - n_-$. On the other hand, by Theorem 3.20 again, either $m = n - 1$ or $m = n$ if $\lambda_n \leq \lambda^{\mathbf{D}}_{n-1}$, and either $m = n$ or $m = n + 1$ if $\lambda_n > \lambda^{\mathbf{D}}_{n-1}$. Therefore, our claims are proved.

(ii) Note that $\operatorname{Re} y_n$ and $\operatorname{Im} y_n$ are nontrivial solutions to the SLE in the problem with $\lambda = \lambda_n$. Thus, our conclusions about the index can be shown as in (i). Since λ_n does not have a real eigenfunction, $\operatorname{Re} y_n$ and $\operatorname{Im} y_n$ are linearly independent on (a, b), and hence, $\operatorname{Re} y_n$ and $\operatorname{Im} y_n$ do not have a common zero on $[a, b]$. Therefore, y_n does not have a zero on $[a, b]$. ■

Remark 4.24 In the situation of Theorem 4.23, we can also show the following: if $\lambda_n \leq \lambda^{\mathbf{D}}_{n-1}$ and $n = n_+ - n_-$, or ($\lambda_n > \lambda^{\mathbf{D}}_{n-1}$ and) $n + 1 = n_+ - n_-$, then neither a nor b is a zero of the function in question, *i.e.*, y_n for (i) and $\operatorname{Re} y_n$ or $\operatorname{Im} y_n$ for (ii). The proof uses Lemma 1.29.

5 Jumps of the Direct Indices

In this section, we index the eigenvalues directly: the negative ones are numbered as $\dots, \lambda_{-3}, \lambda_{-2}, \lambda_{-1}$, while the non-negative ones are indexed as $\lambda_0, \lambda_1, \lambda_2, \dots$, all in non-decreasing order. Note that this indexing scheme is independent of the Prüfer angle, *i.e.*, independent of the indexing scheme used in Sections 1–4. We characterize the discontinuities of each λ_n under this indexing scheme on the space of SLP's considered.

Since the direct indices jump only at SLP's having 0 as an eigenvalue, the set of these problems plays a key role here. For a given SLE $(a, b, 1/p, q, w) \in \boldsymbol{\Omega}$, the set $\mathcal{S}^{\mathbb{C}}_0$ of self-adjoint BC's having 0 as an eigenvalue has been characterized in [13] and is given by $\mathcal{S}^{\mathbb{C}}_{\bullet}\Phi(b, 0)$, where

$$\mathcal{S}^{\mathbb{C}} = \{[A \,|\, B] \in \mathcal{B}^{\mathbb{C}} \,;\, \det(A + B) = 0\}. \tag{5.1}$$

The sets $\mathcal{S}^{\mathbb{R}}$ and $\mathcal{S}^{\mathbb{R}}_0$ are defined similarly.

Lemma 5.2 *We have that* $\mathcal{S}^{\mathbb{R}} \subset \mathcal{O}^{\mathbb{R}}_2 \cup \mathcal{O}^{\mathbb{R}}_3 \cup \mathcal{O}^{\mathbb{R}}_5$ *and* $\mathcal{S}^{\mathbb{C}} \subset \mathcal{O}^{\mathbb{C}}_2 \cup \mathcal{O}^{\mathbb{C}}_3 \cup \mathcal{O}^{\mathbb{C}}_5$.

Proof If $K \in \mathrm{SL}(2, \mathbb{R})$, then either $k_{11} \neq 0$ or $k_{12} \neq 0$, and hence $[K \,|\, -I] \in \mathcal{O}^{\mathbb{R}}_3 \cup \mathcal{O}^{\mathbb{R}}_5$. Thus, $\mathcal{O}^{\mathbb{R}}_1 = \mathcal{O}^{\mathbb{R}}_6 \subset \mathcal{O}^{\mathbb{R}}_3 \cup \mathcal{O}^{\mathbb{R}}_5$, and $\mathcal{B}^{\mathbb{R}} = \bigcup_{i=2}^{5} \mathcal{O}^{\mathbb{R}}_i$. Direct calculations yield that

$$\mathcal{O}^{\mathbb{R}}_4 \setminus (\mathcal{O}^{\mathbb{R}}_2 \cup \mathcal{O}^{\mathbb{R}}_3) = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & b_2 \end{bmatrix} ; b_2 \in \mathbb{R} \right\}, \tag{5.3}$$

which does not overlap with $\mathcal{S}^{\mathbb{R}}$. Therefore, $\mathcal{S}^{\mathbb{R}} \subset \mathcal{O}^{\mathbb{R}}_2 \cup \mathcal{O}^{\mathbb{R}}_3 \cup \mathcal{O}^{\mathbb{R}}_5$. Similarly, we show the other claim. ■

For $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $z \in \mathbb{C}$, we set

$$\mathbf{K}(a_2, b_2, z) = \begin{bmatrix} 1 & a_2 & 0 & \bar{z} \\ 0 & z & -1 & b_2 \end{bmatrix}, \tag{5.4}$$

$$\mathbf{M}(a_2, b_1, z) = \begin{bmatrix} 1 & a_2 & -\bar{z} & 0 \\ 0 & z & b_1 & -1 \end{bmatrix}, \tag{5.5}$$

$$\mathbf{P}(a_1, b_1, z) = \begin{bmatrix} a_1 & 1 & \bar{z} & 0 \\ z & 0 & b_1 & -1 \end{bmatrix}, \tag{5.6}$$

which are the general elements of $\mathcal{O}_2^{\mathbb{C}}$, $\mathcal{O}_3^{\mathbb{C}}$ and $\mathcal{O}_5^{\mathbb{C}}$, respectively. By definition and direct calculations,

$$\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}} = \{\mathbf{K}(a_2, b_2, r)\ ;\ a_2, b_2, r \in \mathbb{R},\ a_2 + b_2 + 2r = 0\}, \tag{5.7}$$

$$\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_3^{\mathbb{R}} = \{\mathbf{M}(a_2, b_1, r)\ ;\ a_2, b_1, r \in \mathbb{R},\ a_2 b_1 + (r-1)^2 = 0\}, \tag{5.8}$$

$$\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_5^{\mathbb{R}} = \{\mathbf{P}(a_1, b_1, r)\ ;\ a_1, b_1, r \in \mathbb{R},\ a_1 + b_1 + 2r = 0\}. \tag{5.9}$$

When dealing with the discontinuities of λ_n on $\mathcal{B}^{\mathbb{C}}$, we will need the notation

$$\mathcal{K}_-^{\mathbb{C}} = \{\mathbf{K}(a_2, b_2, z)\ ;\ a_2, b_2 \in \mathbb{R},\ z \in \mathbb{C},\ a_2 + b_2 + 2\operatorname{Re} z < 0\}, \tag{5.10}$$

$$\mathcal{M}_-^{\mathbb{C}} = \left\{\mathbf{M}(a_2, b_1, z)\ ;\ \begin{matrix} a_2 < 0,\ b_1 > 0,\ z \in \mathbb{C} \\ |z|^2 - 2\operatorname{Re} z + 1 < -a_2 b_1 \end{matrix}\right\}, \tag{5.11}$$

$$\mathcal{M}_+^{\mathbb{C}} = \left\{\mathbf{M}(a_2, b_1, z)\ ;\ \begin{matrix} a_2 \geq 0,\ b_1 \leq 0,\ z \in \mathbb{C} \\ |z|^2 - 2\operatorname{Re} z + 1 \leq -a_2 b_1 \end{matrix}\right\}, \tag{5.12}$$

$$\mathcal{P}_-^{\mathbb{C}} = \{\mathbf{P}(a_1, b_1, z)\ ;\ a_1, b_1 \in \mathbb{R},\ z \in \mathbb{C},\ a_1 + b_1 + 2\operatorname{Re} z \leq 0\}, \tag{5.13}$$

$$\mathcal{K}_+^{\mathbb{C}} = \mathcal{O}_2^{\mathbb{C}} \setminus \mathcal{K}_-^{\mathbb{C}}, \quad \mathcal{M}_0^{\mathbb{C}} = \mathcal{O}_3^{\mathbb{C}} \setminus (\mathcal{M}_-^{\mathbb{C}} \cup \mathcal{M}_+^{\mathbb{C}}), \quad \mathcal{P}_+^{\mathbb{C}} = \mathcal{O}_5^{\mathbb{C}} \setminus \mathcal{P}_-^{\mathbb{C}}. \tag{5.14}$$

When discussing the discontinuities of λ_n on $\mathcal{B}^{\mathbb{R}}$, we will need $\mathcal{K}_\pm^{\mathbb{R}}$, $\mathcal{M}_\pm^{\mathbb{R}}$, $\mathcal{M}_0^{\mathbb{R}}$ and $\mathcal{P}_\pm^{\mathbb{R}}$, which can be defined using (5.10)–(5.14) with $\mathbb{C}$ replaced by $\mathbb{R}$. By Lemma 5.2 and (5.7)–(5.9),

$$\mathcal{S}^{\mathbb{R}} \subset \mathcal{K}_+^{\mathbb{R}} \cup \mathcal{M}_0^{\mathbb{R}} \cup \mathcal{M}_+^{\mathbb{R}} \cup \mathcal{P}_-^{\mathbb{R}}. \tag{5.15}$$

By Lemma 5.2 again and three identities similar to (5.7)–(5.9), (5.15) is still true when all the super indices $\mathbb{R}$ in it are replaced by $\mathbb{C}$. If $\mathcal{X}$ is any of the sets in (5.10)–(5.14) or any of $\mathcal{K}_\pm^{\mathbb{R}}$, $\mathcal{M}_\pm^{\mathbb{R}}$, $\mathcal{M}_0^{\mathbb{R}}$, $\mathcal{P}_\pm^{\mathbb{R}}$, $\mathcal{S}^{\mathbb{R}}$ and $\mathcal{S}^{\mathbb{C}}$, then we set

$$\boldsymbol{\Omega}\mathcal{X} = \{(\boldsymbol{\omega}, \mathbf{A})\ ;\ \boldsymbol{\omega} = (a, b, 1/p, q, w) \in \boldsymbol{\Omega},\ \mathbf{A} \in \mathcal{X}_\bullet \Phi_{\boldsymbol{\omega}}(b, 0)\}. \tag{5.16}$$

The discontinuities of the direct indices are completely characterized in the following theorem, the main ideas of whose proof are from [8].

Theorem 5.17 *Index directly the eigenvalues of the Sturm-Liouville problems in* $\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{C}}$.

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(i) *Let $n \in \mathbb{Z}$. Then, the function λ_n is continuous on $(\boldsymbol{\Omega} \times \mathcal{B}^{\mathbb{R}}) \setminus \boldsymbol{\Omega}\mathcal{S}^{\mathbb{R}}$ and discontinuous at each point of $\boldsymbol{\Omega}\mathcal{S}^{\mathbb{R}}$. More precisely about the discontinuities, let $(\boldsymbol{\omega} = (a, b, 1/p, q, w), \mathbf{A}) \in \boldsymbol{\Omega}\mathcal{S}^{\mathbb{R}}$, then we have the following: for $\mathbf{A} \in \mathcal{K}^{\mathbb{R}}_{+\bullet}\Phi_{\boldsymbol{\omega}}(b, 0)$, the restriction of λ_n to $\boldsymbol{\Omega}\mathcal{K}^{\mathbb{R}}_{+}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and*

$$\lim_{\boldsymbol{\Omega}\mathcal{K}^{\mathbb{R}}_{-} \ni (\boldsymbol{\sigma}, \mathbf{B}) \to (\boldsymbol{\omega}, \mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n+1}(\boldsymbol{\omega}, \mathbf{A}); \tag{5.18}$$

for $\mathbf{A} \in \mathcal{M}^{\mathbb{R}}_{0\bullet}\Phi_{\boldsymbol{\omega}}(b, 0)$, the restriction of λ_n to $\boldsymbol{\Omega}\mathcal{M}^{\mathbb{R}}_{0}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\mathcal{M}^{\mathbb{R}}_{-} \ni (\boldsymbol{\sigma}, \mathbf{B}) \to (\boldsymbol{\omega}, \mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n+1}(\boldsymbol{\omega}, \mathbf{A}); \tag{5.19}$$

for $\mathbf{A} \in \mathcal{M}^{\mathbb{R}}_{+\bullet}\Phi_{\boldsymbol{\omega}}(b, 0)$, the restriction of λ_n to $\boldsymbol{\Omega}\mathcal{M}^{\mathbb{R}}_{+}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\mathcal{M}^{\mathbb{R}}_{0} \ni (\boldsymbol{\sigma}, \mathbf{B}) \to (\boldsymbol{\omega}, \mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n+1}(\boldsymbol{\omega}, \mathbf{A}), \tag{5.20}$$

$$\lim_{\boldsymbol{\Omega}\mathcal{M}^{\mathbb{R}}_{-} \ni (\boldsymbol{\sigma}, \mathbf{B}) \to (\boldsymbol{\omega}, \mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n+2}(\boldsymbol{\omega}, \mathbf{A}) \quad \textit{when } \mathbf{A} = [\Phi_{\boldsymbol{\omega}}(b, 0) \mid -I]; \tag{5.21}$$

for $\mathbf{A} \in \mathcal{P}^{\mathbb{R}}_{-\bullet}\Phi_{\boldsymbol{\omega}}(b, 0)$, the restriction of λ_n to $\boldsymbol{\Omega}\mathcal{P}^{\mathbb{R}}_{-}$ is continuous at $(\boldsymbol{\omega}, \mathbf{A})$, and

$$\lim_{\boldsymbol{\Omega}\mathcal{P}^{\mathbb{R}}_{+} \ni (\boldsymbol{\sigma}, \mathbf{B}) \to (\boldsymbol{\omega}, \mathbf{A})} \lambda_n(\boldsymbol{\sigma}, \mathbf{B}) = \lambda_{n+1}(\boldsymbol{\omega}, \mathbf{A}). \tag{5.22}$$

(ii) *The conclusions of (i) still hold when all the super indices $\mathbb{R}$ are replaced by $\mathbb{C}$.*

Proof (i) Fix an SLE $(a, b, 1/p, q, w) \in \boldsymbol{\Omega}$ and set $C = \Phi(b, 0)$. Among the separated self-adjoint BC's of the form

$$\mathbf{K}(0, b_2, 0)_\bullet C = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & -1 & b_2 \end{bmatrix}, \tag{5.23}$$

the only one in $\mathcal{S}^{\mathbb{R}}_0$ is the one with $b_2 = 0$. Define $\alpha_0 \in [0, \pi)$ in terms of

$$\mathbf{K}(0, 0, 0)_\bullet C = \mathbf{S}_{\alpha_0, \pi}. \tag{5.24}$$

By the derivative formula in [15] for the continuous eigenvalue branches over

$$\{\mathbf{S}_{\alpha_0, \beta}\,;\, \pi/2 < \beta < 3\pi/2\}, \tag{5.25}$$

$\lambda_0\big(\mathbf{K}(0, b_2, 0)_\bullet C\big)$ is continuous and strictly increasing in b_2 on $[0, +\infty)$, and

$$\lim_{b_2 \to 0^-} \lambda_{-1}\big(\mathbf{K}(0, b_2, 0)_\bullet C\big) = \lambda_0\big(\mathbf{K}(0, 0, 0)_\bullet C\big). \tag{5.26}$$

Now, we consider the open subset $\mathcal{O}^{\mathbb{R}}_{2\bullet}C$. The plane $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}^{\mathbb{R}}_2)_\bullet C$ divides it into two (open) halves. The half in $\mathcal{K}^{\mathbb{R}}_{+\bullet}C$ will be denoted by $\mathcal{U}^{\mathbb{R}}{}_\bullet C$. By Corollary 3.18, there is a continuous eigenvalue branch Λ on the whole $\mathcal{O}^{\mathbb{R}}_{2\bullet}C$ through

$\lambda_0\big(\mathbf{K}(0,1,0)_\bullet C\big) > 0$. Then, Λ does not change sign on each half. By what we have proven in the previous paragraph,

$$\Lambda\big(\mathbf{K}(0,b_2,0)_\bullet C\big) = \lambda_{-1}\big(\mathbf{K}(0,b_2,0)_\bullet C\big) < 0 \quad \text{if } b_2 < 0. \tag{5.27}$$

Thus,

$$\Lambda > 0 \text{ on } \mathcal{U}^{\mathbb{R}}{}_\bullet C, \quad \Lambda = 0 \text{ on } (\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}})_\bullet C, \quad \Lambda < 0 \text{ on } \mathcal{K}_-^{\mathbb{R}}{}_\bullet C. \tag{5.28}$$

Since $[C \mid -I] \notin \mathcal{O}_2^{\mathbb{R}}{}_\bullet C$, 0 is always a simple eigenvalue on $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}})_\bullet C$. So, we must have that

$$\lambda_0 = \Lambda \text{ on } \mathcal{K}_+^{\mathbb{R}}{}_\bullet C, \quad \lambda_{-1} = \Lambda \text{ on } \mathcal{K}_-^{\mathbb{R}}{}_\bullet C. \tag{5.29}$$

Hence, if $\mathbf{A} \in (\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_2^{\mathbb{R}})_\bullet C \subset \mathcal{K}_+^{\mathbb{R}}{}_\bullet C$, then

$$\lim_{\mathcal{K}_-^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_{n-1}(\mathbf{B}) = \lambda_n(\mathbf{A}) = \lim_{\mathcal{K}_+^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}). \tag{5.30}$$

Next, we look at the open subset $\mathcal{O}_3^{\mathbb{R}}{}_\bullet C$. The cone $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_3^{\mathbb{R}})_\bullet C$ cuts it into three parts, *i.e.*, the open back half cone $\mathcal{M}_-^{\mathbb{R}}{}_\bullet C$, the closed front half cone $\mathcal{M}_+^{\mathbb{R}}{}_\bullet C$ and the remaining part $\mathcal{M}_0^{\mathbb{R}}{}_\bullet C$. For $r \in \mathbb{R}$,

$$\mathbf{M}(1,-1,r) = \mathbf{K}(1-r^2,-1,r), \tag{5.31}$$

which is in $\mathcal{K}_+^{\mathbb{R}} \cap \mathcal{M}_+^{\mathbb{R}}$ if $r \in [0,2]$ and in $\mathcal{K}_-^{\mathbb{R}} \cap \mathcal{M}_0^{\mathbb{R}}$ if $r < 0$. Thus, the proven case implies that for any $\mathbf{A}$ on $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{M}_+^{\mathbb{R}})_\bullet C$,

$$\lim_{\mathcal{M}_0^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_{n-1}(\mathbf{B}) = \lambda_n(\mathbf{A}) = \lim_{\mathcal{M}_+^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}). \tag{5.32}$$

For $r \in \mathbb{R}$,

$$\mathbf{M}(-1,1,r) = \mathbf{K}(r^2-1,1,-r), \tag{5.33}$$

which is in $\mathcal{K}_+^{\mathbb{R}} \cap \mathcal{M}_0^{\mathbb{R}}$ if $r \le 0$ and in $\mathcal{K}_-^{\mathbb{R}} \cap \mathcal{M}_-^{\mathbb{R}}$ if $r \in (0,2)$. Thus, for any $\mathbf{A}$ in $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{M}_0^{\mathbb{R}})_\bullet C$,

$$\lim_{\mathcal{M}_-^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_{n-1}(\mathbf{B}) = \lambda_n(\mathbf{A}) = \lim_{\mathcal{M}_0^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}). \tag{5.34}$$

From (5.32) and (5.34) we deduce that

$$\lim_{\mathcal{M}_-^{\mathbb{R}}{}_\bullet C \ni \mathbf{B} \to \mathbf{A}} \lambda_{n-2}(\mathbf{B}) = \lambda_n(\mathbf{A}) \quad \text{when } \mathbf{A} = [C \mid -I]. \tag{5.35}$$

Finally, we handle the open subset $\mathcal{O}_5^{\mathbb{R}}{}_\bullet C$. The plane $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{O}_5^{\mathbb{R}})_\bullet C$ splits it into two parts, *i.e.*, $\mathcal{P}_-^{\mathbb{R}}{}_\bullet C$ and $\mathcal{P}_+^{\mathbb{R}}{}_\bullet C$. For $r \in \mathbb{R}$,

$$\mathbf{P}(1,-1,r) = \mathbf{K}\left(\frac{1}{1+r^2}, \frac{-1}{1+r^2}, \frac{-r}{1+r^2}\right), \tag{5.36}$$

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which is in $\mathcal{K}_+^{\mathbb{R}} \cap \mathcal{P}_-^{\mathbb{R}}$ if $r \leq 0$ and in $\mathcal{K}_-^{\mathbb{R}} \cap \mathcal{P}_+^{\mathbb{R}}$ if $r > 0$. Thus, for any $\mathbf{A}$ on $(\mathcal{S}^{\mathbb{R}} \cap \mathcal{P}_-^{\mathbb{R}})_{\bullet} C$,

$$\lim_{\mathcal{P}_-^{\mathbb{R}}{}_{\bullet} C \ni \mathbf{B} \to \mathbf{A}} \lambda_n(\mathbf{B}) = \lambda_n(\mathbf{A}) = \lim_{\mathcal{P}_+^{\mathbb{R}}{}_{\bullet} C \ni \mathbf{B} \to \mathbf{A}} \lambda_{n-1}(\mathbf{B}). \tag{5.37}$$

From (5.30), (5.32), (5.34), (5.35), (5.37) and Theorem 1.20 we obtain the claims in our theorem (see also the proof of Theorem 3.76 in [11]).

(ii) These claims can be shown similarly. ■

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