

# UNIMODALITY AND COLOURED HOOK FACTORISATION

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(Received 15 April 2015; accepted 21 August 2015; first published online 11 November 2015)

## Abstract

We prove the unimodality of some coloured  $q$ -Eulerian polynomials, which involve the flag excedances, the major index and the fixed points on coloured permutation groups, via two recurrence formulas. In particular, we confirm a recent conjecture of Mongelli about the unimodality of the flag excedances over type B derangements. Furthermore, we find the coloured version of Gessel's hook factorisation, which enables us to interpret these two recurrences combinatorially. We also provide a combinatorial proof of a symmetric and unimodal expansion for the coloured derangement polynomial, which was first established by Shin and Zeng using continued fractions.

2010 *Mathematics subject classification*: primary 05A05; secondary 05A15, 05A19.

*Keywords and phrases*: coloured permutations, derangements, Eulerian polynomials, unimodality,  $\gamma$ -positivity, hook factorisation.

## 1. Introduction

Let  $l$  be a fixed positive integer. Consider the wreath product  $C_l \wr \mathfrak{S}_n$  of the cyclic group  $C_l$  of order  $l$  by the symmetric group  $\mathfrak{S}_n$  of order  $n$ . The groups  $C_l \wr \mathfrak{S}_n$  are also known as the *coloured permutation groups*. In case  $l = 1, 2$ , they are respectively the symmetric groups (or *permutation groups*)  $\mathfrak{S}_n$  and the *type B Coxeter groups*  $B_n$ . Various statistics on permutation groups have been generalised to coloured permutation groups, including the four classical ones in the literature: inversions, descents, excedances and the major index (see [1–3, 7, 10, 17] and the references therein).

An element of  $C_l \wr \mathfrak{S}_n$  is called a *coloured permutation* and can be viewed as an ordered pair  $(\pi, \epsilon)$ , with  $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$  and  $\epsilon = \epsilon_1 \cdots \epsilon_n$  a word on  $\{0, 1, \dots, l-1\}$  of length  $n$ . For convenience, we usually write  $(\pi, \epsilon)$  in one line as  $\pi_1^{\epsilon_1} \pi_2^{\epsilon_2} \cdots \pi_n^{\epsilon_n}$ . Now define the *excedance number*,  $\text{exc}(\pi, \epsilon)$ , the *major index*,  $\text{maj}(\pi, \epsilon)$ , and the *number of*

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The author is supported by the National Science Foundation of China grant number 11501244.

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fixed points,  $\text{fix}(\pi, \epsilon)$ , of a coloured permutation  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$  by

$$\begin{aligned} \text{exc}(\pi, \epsilon) &:= \#\{1 \leq j \leq n - 1 : \pi_j > j \text{ and } \epsilon_j = 0\}, \\ \text{DES}(\pi, \epsilon) &:= \{1 \leq j \leq n - 1 : \text{either } \epsilon_j < \epsilon_{j+1}, \text{ or } \epsilon_j = \epsilon_{j+1} \text{ and } \pi_j > \pi_{j+1}\}, \\ \text{maj}(\pi, \epsilon) &:= \sum_{j \in \text{DES}(\pi, \epsilon)} j, \\ \text{fix}(\pi, \epsilon) &:= \#\{1 \leq j \leq n : \pi_j = j \text{ and } \epsilon_j = 0\}. \end{aligned}$$

For example, if  $(\pi, \epsilon) = 6^0 3^2 1^2 4^0 5^0 2^1 8^0 7^0 \in C_3 \wr \mathfrak{S}_8$ , then  $\text{exc}(\pi, \epsilon) = 2$ ,  $\text{DES}(\pi, \epsilon) = \{1, 2, 5, 7\}$ ,  $\text{maj}(\pi, \epsilon) = 1 + 2 + 5 + 7 = 15$  and  $\text{fix}(\pi, \epsilon) = 2$ . Bagno and Garber [1] introduced the flag excedance of  $(\pi, \epsilon)$ , denoted  $\text{fexc}(\pi, \epsilon)$ , as

$$\text{fexc}(\pi, \epsilon) := l \cdot \text{exc}(\pi, \epsilon) + \sum_{j=1}^n \epsilon_j.$$

Our main focus is the coloured  $(q, r)$ -Eulerian polynomial  $A_n^{(l)}(t, r, q)$  defined by

$$A_n^{(l)}(t, r, q) := \sum_{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n} t^{\text{fexc}(\pi, \epsilon)} r^{\text{fix}(\pi, \epsilon)} q^{(\text{maj} - \text{exc})(\pi, \epsilon)}.$$

By convention,  $A_0^{(l)}(t, r, q) = 1$ . Clearly,  $A_n^{(1)}(t, 1, 1) = A_n(t)$ , where  $A_n(t)$  is the classical  $n$ th Eulerian polynomial (see [4]). The values of  $A_n^{(2)}(t, r, q)$  for  $1 \leq n \leq 3$  are

$$\begin{aligned} A_1^{(2)}(t, r, q) &= r + t, \\ A_2^{(2)}(t, r, q) &= r^2 + (1 + r + rq)t + (2 + q)t^2 + t^3, \\ A_3^{(2)}(t, r, q) &= r^3 + [1 + (1 + q + q^2)(r + r^2)]t + [1 + (1 + q + q^2)(2 + 2r + rq)]t^2 \\ &\quad + [1 + (1 + q + q^2)(3 + r + q)]t^3 + (3 + 2q + 2q^2)t^4 + t^5. \end{aligned}$$

Let  $(q; q)_n := \prod_{i=1}^n (1 - q^i)$  for  $n \geq 1$  and  $(q; q)_0 = 1$ . The  $q$ -exponential function  $e(z; q)$  is defined by  $e(z; q) := \sum_{n \geq 0} z^n / (q; q)_n$ . The following elegant expression for the exponential generating function of  $A_n^{(l)}(t, r, q)$  can be derived from the work of Foata and Han [7, Theorem 1.3] or Hyatt [10, Theorem 1.4]:

$$\sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} = \frac{(1 - t)e(rz; q)}{e(tz; q) - te(z; q)}. \tag{1.1}$$

Let us recall some necessary definitions. Let  $\mathbb{Q}[q]$  be the ring of all polynomials in  $q$  with rational coefficients. Define the partial order relation on  $\mathbb{Q}[q]$  by

$$f(q) \leq_q g(q) \Leftrightarrow g(q) - f(q) \text{ has nonnegative coefficients.}$$

A polynomial  $h(t) = \sum_{k=0}^n a_k(q)t^k \in \mathbb{Q}[q][t]$  is symmetric (with centre of symmetry  $n/2$ ) if  $a_k(q) = a_{n-k}(q)$  for all  $0 \leq k \leq n$  and it is unimodal if there exists a  $c$ ,  $0 \leq c \leq n$ , such that

$$a_0(q) \leq_q a_1(q) \leq_q \cdots \leq_q a_c(q) \geq_q a_{c+1}(q) \geq_q \cdots \geq_q a_n(q).$$

It is well known [16] that the Eulerian polynomial  $A_n(t)$  is symmetric and unimodal. A coloured permutation  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$  is called a derangement if  $\text{fix}(\pi, \epsilon) = 0$ . In [20],

Zhang proved that the roots of the type A derangement polynomial  $A_n^{(1)}(t, 0, 1)$  are all real, which implies its unimodality. Shareshian and Wachs [13] proved the symmetry and unimodality of  $A_n^{(1)}(t, 1, q)$  and  $A_n^{(1)}(t, 0, q)$ . Recently, Mongelli [12] noticed that  $A_5^{(2)}(t, 0, 1)$  has nonreal complex roots and conjectured that the type B derangement polynomial  $A_n^{(2)}(t, 0, 1)$  is unimodal for any  $n \geq 1$ . Motivated by this conjecture and the above results, we will investigate the unimodality of  $A_n^{(l)}(t, 1, q)$  and  $A_n^{(l)}(t, 0, q)$ .

The first result confirms the unimodality conjecture of Mongelli.

**THEOREM 1.1.** *For all  $n, l \geq 1$ , the coloured  $q$ -Eulerian polynomials  $A_n^{(l)}(t, 1, q)$  and  $A_n^{(l)}(t, 0, q)$  are symmetric and unimodal.*

Theorem 1.1 generalises parts (3) and (4) of [13, Theorem 5.3] from permutations to coloured permutations. Our approach is slightly different, being an easy application of the recurrence relations for  $A_n^{(l)}(t, r, q)$  in Theorem 1.2.

For  $n \geq 0$  and  $0 \leq k \leq n$ , define  $[n]_q := (1 - q^n)/(1 - q)$  and the  $q$ -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

**THEOREM 1.2.** *For  $n \geq 0$ ,  $A_n^{(l)}(t, r, q)$  satisfies the two recurrence relations:*

$$A_{n+1}^{(l)}(t, r, q) = r^{n+1} + \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A_k^{(l)}(t, r, q) t[l(n+1-k) - 1]_t, \tag{1.2}$$

$$A_{n+1}^{(l)}(t, r, q) = (r + t[l-1]_t q^n) A_n^{(l)}(t, r, q) + t[l]_t \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k^{(l)}(t, r, q) A_{n-k}^{(l)}(t, 1, q). \tag{1.3}$$

**REMARK 1.3.** When  $l = 1$ , these two recurrence relations reduce to the recurrences in [13, Corollary 4.3] and [11, Theorem 2], respectively.

A nice property that is stronger than the symmetry and unimodality of a polynomial is the so-called  $\gamma$ -positivity. For each permutation  $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ , a *double excedance* is an index  $i$  such that  $i < \pi_i < \pi_{\pi_i}$ . Let  $\text{cda}(\pi)$  be the number of double excedances of  $\pi$ . Shin and Zeng [14] proved the  $\gamma$ -positivity of the derangement polynomial  $A_n^{(1)}(t, 0, 1)$ :

$$A_n^{(1)}(t, 0, 1) = \sum_{k=1}^{\lfloor n/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-2k}, \tag{1.4}$$

where  $\gamma_{n,k} := \#\{\pi \in \mathfrak{S}_n : \text{fix}(\pi) = \text{cda}(\pi) = 0, \text{exc}(\pi) = k\}$ . Very recently, using the machinery of continued fractions, Shin and Zeng [15, Theorem 3] generalised their result from derangements to coloured derangements, by obtaining the following symmetric and unimodal expansion of  $A_n^{(l)}(t, 0, 1)$ .

**THEOREM 1.4 [15].** *For all  $n, l \geq 1$ ,*

$$A_n^{(l)}(t, 0, 1) = \sum_{1 \leq i+2j \leq n} \binom{n}{i} \gamma_{n-i,j} t^j (1+t)^{n-i-2j} (t[l-1]_t)^i [l]_t^{n-i}. \tag{1.5}$$

When  $l = 2$ , expansion (1.5) becomes

$$A_n^{(2)}(t, 0, 1) = \sum_{1 \leq k \leq n} \left( \sum_{i+j=k} \binom{n}{i} \gamma_{n-i,j} \right) t^k (1+t)^{2n-2k}, \tag{1.6}$$

which implies the  $\gamma$ -positivity of  $A_n^{(2)}(t, 0, 1)$ . Note that expansion (1.5) also implies the symmetry and unimodality of  $A_n^{(l)}(t, 0, 1)$ , since each summand on the right-hand side of (1.5) is symmetric and unimodal with the same centre of symmetry at  $\frac{1}{2}ln$ . We will provide a combinatorial proof of expansion (1.5).

The proofs of Theorems 1.1 and 1.2 are given in the next section. The combinatorial interpretation of the recurrences in Theorem 1.2 inspired us to find the coloured analogue of Gessel’s hook factorisation [8], which is developed in Section 3. In particular, we obtain another interpretation of  $A_n^{(l)}(t, r, q)$ . In Section 4, we give the combinatorial proof of expansion (1.5). We end with some remarks and a conjecture.

### 2. Proofs of Theorems 1.1 and 1.2

We first prove Theorem 1.2 and derive Theorem 1.1 from it. For simplicity, set  $A_n(t, q) := A_n(t, 1, q)$ .

**PROOF OF THEOREM 1.2.** We first prove recurrence relation (1.2). By (1.1),

$$\sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} = \frac{e(rz; q)}{1 - \sum_{n \geq 1} t[ln - 1]_t z^n / (q; q)_n}.$$

Multiplying both sides by  $1 - \sum_{n \geq 1} t[ln - 1]_t z^n / (q; q)_n$  and taking the coefficients of  $z^{n+1} / (q; q)_{n+1}$  gives the following recurrence, which is equivalent to (1.2):

$$A_{n+1}^{(l)}(t, r, q) - \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A_k^{(l)}(t, r, q) t[l(n+1-k) - 1]_t = r^{n+1}.$$

Next, we prove (1.3). Let  $\delta_x$  be the Eulerian differential operator defined as

$$\delta_x(f(x)) := \frac{f(x) - f(qx)}{x}$$

for any  $f(x) \in \mathbb{Q}[q][[x]]$ , the ring of formal power series in  $x$  over  $\mathbb{Q}[q]$ . It is not difficult to show that, for any variable  $y$ ,

$$\delta_z(e(yz; q)) = ye(yz; q).$$

Now, applying  $\delta_z$  to both sides of (1.1) and using the above property and the quotient

rule for the differential operator (see [11, Lemma 7]),

$$\begin{aligned} \sum_{n \geq 0} A_{n+1}^{(l)}(t, r, q) \frac{z^n}{(q; q)_n} &= \delta_z \left( \frac{(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)} \right) \\ &= \frac{r(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)} + \frac{(1-t)e(rzq; q)(te(z; q) - t^l e(t^l z; q))}{(e(t^l qz; q) - te(qz; q))(e(t^l z; q) - te(z; q))} \\ &= \frac{r(1-t)e(rz; q)}{e(t^l z; q) - te(z; q)} + \frac{(1-t)e(rzq; q)}{e(t^l qz; q) - te(qz; q)} \\ &\quad \times \left( \frac{t^l e(z; q) - t^l e(t^l z; q)}{e(t^l z; q) - te(z; q)} + \frac{te(z; q) - t^l e(z; q)}{e(t^l z; q) - te(z; q)} \right) \\ &= \left( \sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{(qz)^n}{(q; q)_n} \right) \\ &\quad \times \left( (t + \dots + t^{l-1}) \sum_{n \geq 0} A_n^{(l)}(t, q) \frac{z^n}{(q; q)_n} + t^l \sum_{n \geq 1} A_n^{(l)}(t, q) \frac{z^n}{(q; q)_n} \right) \\ &\quad + r \sum_{n \geq 0} A_n^{(l)}(t, r, q) \frac{z^n}{(q; q)_n}. \end{aligned}$$

Taking the coefficient of  $z^n/(q; q)_n$  on both sides of the above equality, we get (1.3). This completes the proof of Theorem 1.2.  $\square$

We shall apply the following fact [16, Proposition 1] to prove Theorem 1.1.

**LEMMA 2.1.** *The product of two symmetric and unimodal polynomials in  $\mathbb{Q}[q][t]$  with respective centres of symmetry  $c_1$  and  $c_2$  is symmetric and unimodal with centre of symmetry  $c_1 + c_2$ .*

**PROOF OF THEOREM 1.1.** We will show that Theorem 1.1 follows from recurrence relation (1.3) and Lemma 2.1 by induction on  $n$ . A similar discussion is also available with recurrence relation (1.3) replaced by (1.2).

For  $n = 1$ , the result is clear, as  $A_1^{(l)}(t, r, q) = r + t + t^2 + \dots + t^{l-1}$ . Suppose that Theorem 1.1 is true for all  $n \leq m$ . Setting  $r = 1$  in (1.3),

$$\begin{aligned} A_{m+1}^{(l)}(t, q) &= (1 + t[l-1]_t q^m) A_m^{(l)}(t, q) + t[l]_t \sum_{k=0}^{m-1} \begin{bmatrix} m \\ k \end{bmatrix}_q q^k A_k^{(l)}(t, q) A_{m-k}^{(l)}(t, q) \\ &= (t[l-1]_t q^m + [l+1]_t) A_m^{(l)}(t, q) + \sum_{k=1}^{m-1} \begin{bmatrix} m \\ k \end{bmatrix}_q t[l]_t q^k A_k^{(l)}(t, q) A_{m-k}^{(l)}(t, q). \end{aligned}$$

By the induction hypothesis and Lemma 2.1,

$$(t[l-1]_t q^m + [l+1]_t) A_m^{(l)}(t, q) \quad \text{and} \quad t[l]_t q^k A_k^{(l)}(t, q) A_{m-k}^{(l)}(t, q) \quad (1 \leq k < m)$$

are all symmetric and unimodal with the same centre of symmetry at  $\frac{1}{2}(l(m+1) - 1)$ . Hence,  $A_{m+1}^{(l)}(t, q)$  is symmetric and unimodal with centre of symmetry  $\frac{1}{2}(l(m+1) - 1)$ .

Similarly, setting  $r = 0$  in recurrence (1.3),

$$A_{m+1}^{(l)}(t, 0, q) = t[l - 1]_t q^n A_m^{(l)}(t, 0, q) + t[l]_t \sum_{k=0}^{m-1} \begin{bmatrix} m \\ k \end{bmatrix}_q q^k A_{m-k}^{(l)}(t, q) A_k^{(l)}(t, 0, q). \tag{2.1}$$

Now, by the induction hypothesis and Lemma 2.1, all the polynomials

$$t[l - 1]_t q^n A_m^{(l)}(t, 0, q) \quad \text{and} \quad t[l]_t \begin{bmatrix} m \\ k \end{bmatrix}_q q^k A_{m-k}^{(l)}(t, q) A_k^{(l)}(t, 0, q) \quad (0 \leq k < m)$$

are symmetric and unimodal with centre of symmetry  $\frac{1}{2}l(m + 1)$ , which implies the same for  $A_{m+1}^{(l)}(t, 0, q)$  in view of (2.1). This completes the proof by induction.  $\square$

### 3. Hook factorisation of coloured permutations

In this section, we introduce the hook factorisation of coloured permutations and give combinatorial interpretations of the two recurrences in Theorem 1.2.

Let us first recall the *hook factorisation* of permutations due to Gessel [8]. A word  $w = w_1 w_2 \cdots w_m$  over  $\mathbb{N}$  is called a *hook* if  $w_1 > w_2$  and either  $m = 2$ , or  $m \geq 3$  and  $w_2 < w_3 < \cdots < w_m$ . As shown by Gessel [8], each permutation  $\pi$  admits a unique factorisation, called its hook factorisation,  $p\tau_1\tau_2 \cdots \tau_r$ , where  $p$  is an increasing word and each factor  $\tau_1, \tau_2, \dots, \tau_r$  is a hook. The hook factorisation has applications in various combinatorial problems (see [6, 9, 11, 19]).

We can extend the hooks to coloured hooks. Let

$$\mathbb{N}^l := \{1^0, 1^1, \dots, 1^{l-1}, 2^0, 2^1, \dots, 2^{l-1}, \dots, i^0, i^1, \dots, i^{l-1}, \dots\}.$$

A letter  $i^k \in \mathbb{N}^l$  is called a  $k$ -coloured letter and  $k$  is referred to as the colour of  $i^k$ . Let  $|i^k| := i$ . A word  $w = w_1 w_2 \cdots w_m$  over  $\mathbb{N}^l$  is called a *coloured hook* if:

- $m \geq 2$  and  $|w| := |w_1||w_2| \cdots |w_m|$  is a hook in which only  $w_1$  may have positive colour; or
- $m \geq 1$  and  $|w|$  is an increasing word and only  $w_1$  may have positive colour.

As in the permutation case, each coloured permutation  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$  admits a unique factorisation, called its *coloured hook factorisation*,  $p\tau_1\tau_2 \cdots \tau_r$ , where  $p$  is a word formed by 0-coloured letters,  $|p|$  is an increasing word over  $\mathbb{N}$  and each factor  $\tau_1, \tau_2, \dots, \tau_r$  is a coloured hook. To derive the coloured hook factorisation of a coloured permutation, one can start from the right and factor out each coloured hook step by step. Clearly, coloured hook factorisation of coloured permutations is a generalisation of hook factorisation of permutations.

For example, the coloured hook factorisation of

$$2^0 4^0 5^1 8^0 3^0 7^0 10^1 1^0 9^0 6^1 \in C_2 \wr \mathfrak{S}_{10} \tag{3.1}$$

is  $2^0 4^0 |5^1 |8^0 3^0 7^0 |10^1 1^0 9^0 |6^1$ .

If  $w = w_1 w_2 \cdots w_m$  is a word over  $\mathbb{N}$ , define the *inversion number*  $\text{inv}(w)$  of  $w$  by

$$\text{inv}(w) := \#\{(i, j) : i < j, w_i > w_j\}.$$

For a coloured permutation  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$  with coloured hook factorisation  $p\tau_1\tau_2 \cdots \tau_r$ , we define

$$\text{inv}(\pi, \epsilon) := \text{inv}(\pi) \quad \text{and} \quad \text{lec}(\pi, \epsilon) := \sum_{i=1}^r \text{inv}(|\tau_i|).$$

We also define

$$\text{flec}(\pi, \epsilon) := l \cdot \text{lec}(\pi, \epsilon) + \sum_{i=1}^n \epsilon_i \quad \text{and} \quad \text{pix}(\pi, \epsilon) := \text{length of } p.$$

For example, if  $(\pi, \epsilon)$  is the coloured permutation in (3.1), then  $\text{inv}(\pi, \epsilon) = 16$ ,  $\text{lec}(\pi, \epsilon) = 4$ ,  $\text{flec}(\pi, \epsilon) = 11$  and  $\text{pix}(\pi, \epsilon) = 2$ .

The following result generalises [5, Theorem 1.4] from permutations to coloured permutations.

**THEOREM 3.1.** For  $n \geq 1$ ,

$$A_n^{(l)}(t, r, q) = \sum_{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)}. \tag{3.2}$$

**PROOF.** By the same discussions as in the proof of [6, Theorem 4], we can show that the exponential generating function for  $\sum_{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)}$  is exactly the right-hand side of (1.1). We leave the details to the reader.  $\square$

**REMARK 3.2.** This result answers a question of Han *et al.* [9] and leads to a combinatorial interpretation of a coloured symmetric  $q$ -Eulerian identity.

For the interpretations of the two recurrences in Theorem 1.2 by means of (3.2), we need the following well-known interpretation of  $q$ -binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{(\mathcal{A}, \mathcal{B})} q^{\text{inv}(\mathcal{A}, \mathcal{B})}, \tag{3.3}$$

where the sum is over all ordered partitions  $(\mathcal{A}, \mathcal{B})$  of  $[n]$  such that  $|\mathcal{A}| = k$  and

$$\text{inv}(\mathcal{A}, \mathcal{B}) := \#\{(i, j) : i \in \mathcal{A}, j \in \mathcal{B} \text{ with } i > j\}.$$

**3.1. A combinatorial interpretation of (1.2).** Note that a coloured hook of length  $k$  may contribute  $1, 2, \dots, k - 1$  to the ‘flec’ statistic of a coloured permutation. Consider the last coloured hook (possibly empty) of each coloured permutation. This gives

$$\begin{aligned} \sum_{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_{n+1}} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)} &= \sum_{k=0}^{n+1} \sum_{\substack{(\pi, \epsilon) = p\tau_1 \cdots \tau_r \\ \#\tau_r = n+1-k}} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)} \\ &= r^{n+1} + \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_q t[l(n+1-k)-1]_t \sum_{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_k} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)}, \end{aligned}$$

where we apply (3.3) to the last equality. This shows that the right-hand side of (3.2) satisfies recurrence relation (1.2).

**3.2. A combinatorial interpretation of (1.3).** Recall that, to derive the hook factorisation of a coloured permutation, one can start from the right and factor out each hook step by step. We distinguish two cases according as 1 appearing in a coloured permutation is 0-coloured or not. For this purpose, we write  $A_n^{(l)}(t, r, q)$  as

$$A_n^{(l)}(t, r, q) = B_n^{(l)}(t, r, q) + C_n^{(l)}(t, r, q), \tag{3.4}$$

where

$$B_n^{(l)}(t, r, q) = \sum_{\substack{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_{n+1} \\ 1 \text{ is 0-coloured}}} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)}$$

and

$$C_n^{(l)}(t, r, q) = \sum_{\substack{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_{n+1} \\ 1 \text{ is not 0-coloured}}} t^{\text{flec}(\pi, \epsilon)} r^{\text{pix}(\pi, \epsilon)} q^{(\text{inv} - \text{lec})(\pi, \epsilon)}.$$

*Case 1: 1 is 0-coloured in  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_{n+1}$ .* Then the coloured hook factorisation of  $(\pi, \epsilon) = \pi_1^{\epsilon_1} \cdots \pi_{j-1}^{\epsilon_{j-1}} \pi_j^{\epsilon_j} 1^0 \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}$  is  $p\tau_1 \cdots \tau_s \tau'_1 \cdots \tau'_r$ , where  $p\tau_1 \cdots \tau_s$  and  $\tau'_1 \cdots \tau'_r$  are coloured hook factorisations of  $\pi_1^{\epsilon_1} \cdots \pi_{j-1}^{\epsilon_{j-1}}$  and  $\pi_j^{\epsilon_j} 1^0 \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}$ , respectively. When  $1 \leq j \leq n$ , it is not difficult to see that

$$\begin{aligned} \text{flec}(\pi_j^{\epsilon_j} 1^0 \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) &= l + \text{flec}(\pi_j^{\epsilon_j} \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}), \\ (\text{inv} - \text{lec})(\pi_j^{\epsilon_j} 1^0 \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) &= (\text{inv} - \text{lec})(\pi_j^{\epsilon_j} \pi_{j+2}^{\epsilon_{j+2}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) \end{aligned}$$

and

$$\text{pix}(\pi, \epsilon) = \text{pix}(\pi_1^{\epsilon_1} \cdots \pi_{j-1}^{\epsilon_{j-1}}).$$

Thus, by (3.3),

$$B_n^{(l)}(t, r, q) = rA_n^{(l)}(t, r, q) + t^l \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k^{(l)}(t, r, q) A_{n-k}^{(l)}(t, 1, q). \tag{3.5}$$

*Case 2: 1 is not 0-coloured in  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_{n+1}$ .*

For  $(\pi, \epsilon) = \pi_1^{\epsilon_1} \cdots \pi_{j-1}^{\epsilon_{j-1}} 1^{\epsilon_j} \pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}$  with  $\epsilon_j \geq 1$ , the coloured hook factorisation of  $(\pi, \epsilon)$  is  $p\tau_1 \cdots \tau_s \tau'_1 \cdots \tau'_r$ , where  $p\tau_1 \cdots \tau_s$  and  $\tau'_1 \cdots \tau'_r$  are coloured hook factorisations of  $\pi_1^{\epsilon_1} \cdots \pi_j^{\epsilon_j}$  and  $1^{\epsilon_j} \pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}$ , respectively. Now

$$\begin{aligned} \text{flec}(1^{\epsilon_j} \pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) &= \epsilon_j + \text{flec}(\pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}), \\ (\text{inv} - \text{lec})(1^{\epsilon_j} \pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) &= (\text{inv} - \text{lec})(\pi_{j+1}^{\epsilon_{j+1}} \cdots \pi_{n+1}^{\epsilon_{n+1}}) \end{aligned}$$

and

$$\text{pix}(\pi, \epsilon) = \text{pix}(\pi_1^{\epsilon_1} \cdots \pi_{j-1}^{\epsilon_{j-1}}).$$

Therefore, again by (3.3),

$$C_n^{(l)}(t, r, q) = t[l-1]_t \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k^{(l)}(t, r, q) A_{n-k}^{(l)}(t, 1, q). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we get (1.3).



### 4. Combinatorial proof of expansion (1.5)

The symmetric and unimodal expansion (1.5) can be obtained by using (1.4) in the following relationship between derangement polynomials and coloured derangement polynomials.

**PROPOSITION 4.1.** *For all  $n, l \geq 1$ ,*

$$A_n^{(l)}(t, 0, 1) = \sum_{i=0}^n \binom{n}{i} (t[l-1]_t)^i [l]_t^{n-i} A_{n-i}^{(1)}(t, 0, 1). \tag{4.1}$$

In [18], Sun and Wang introduced a group action on derangements and provided a combinatorial proof of (1.4). Therefore, to provide a combinatorial proof of expansion (1.5), we just need to give a combinatorial interpretation of relationship (4.1).

**COMBINATORIAL PROOF OF (4.1).** Let  $\mathfrak{D}_n^{(l)} := \{(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n : \text{fix}(\pi, \epsilon) = 0\}$  be the set of derangements in  $C_l \wr \mathfrak{S}_n$ . We will use the following interpretation:

$$A_n^{(l)}(t, 0, 1) = \sum_{(\pi, \epsilon) \in \mathfrak{D}_n^{(l)}} t^{\text{exc}(\pi, \epsilon)}.$$

An index  $j$  is called a *coloured fixed point* of  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$  if  $\pi_j = j$  but  $\epsilon_j \neq 0$ . Let  $\mathfrak{D}_{n,i}^{(l)}$  be the set of derangements in  $\mathfrak{D}_n^{(l)}$  with  $i$  coloured fixed points. We claim that

$$\sum_{(\pi, \epsilon) \in \mathfrak{D}_{n,i}^{(l)}} t^{\text{exc}(\pi, \epsilon)} = \binom{n}{i} (t[l-1]_t)^i [l]_t^{n-i} A_{n-i}^{(1)}(t, 0, 1), \tag{4.2}$$

from which we get (4.1). So, it remains to prove this claim.

For each  $(\pi, \epsilon) \in C_l \wr \mathfrak{S}_n$ , we define the function

$$t_j(\pi, \epsilon) := \begin{cases} t^l & \text{if } \epsilon_j = 0 \text{ and } \pi_j > j, \\ t^{\epsilon_j} & \text{otherwise.} \end{cases}$$

Clearly,  $t^{\text{exc}(\pi, \epsilon)} = \prod_{j=1}^n t_j(\pi, \epsilon)$ . Let  $\mathfrak{S}_{n,i} := \{\pi \in \mathfrak{S}_n : \text{fix}(\pi) = i\}$ . For a fixed permutation  $\sigma \in \mathfrak{S}_{n,i}$  with  $\text{exc}(\sigma) = k$ ,

$$\begin{aligned} \sum_{(\sigma, \epsilon) \in \mathfrak{D}_{n,i}^{(l)}} t^{\text{exc}(\sigma, \epsilon)} &= \sum_{(\sigma, \epsilon) \in \mathfrak{D}_{n,i}^{(l)}} \prod_{j=1}^n t_j(\sigma, \epsilon) \\ &= \prod_{\sigma_j > j} (t + t^2 + \dots + t^l) \prod_{\sigma_j < j} (1 + t + \dots + t^{l-1}) \prod_{\sigma_j = j} (t + \dots + t^{l-1}) \\ &= (t[l]_t)^k [l]_t^{n-k-i} (t[l-1]_t)^i. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{(\pi, \epsilon) \in \mathfrak{D}_{n,i}^{(l)}} t^{\text{exc}(\pi, \epsilon)} &= \sum_{\sigma \in \mathfrak{S}_{n,i}} \sum_{(\sigma, \epsilon) \in \mathfrak{D}_{n,i}^{(l)}} t^{\text{exc}(\pi, \epsilon)} \\ &= \sum_{k=0}^{n-1} d_{n,k,i} (t[l]_t)^k [l]_t^{n-k-i} (t[l-1]_t)^i \\ &= \sum_{k=0}^{n-1} d_{n,k,i} t^k [l]_t^{n-i} (t[l-1]_t)^i, \end{aligned}$$

where  $d_{n,k,i} = \#\{\sigma \in \mathfrak{S}_{n,i} : \text{exc}(\sigma) = k\}$ . Our claim (4.2) then follows from the above expression and the simple fact that

$$d_{n,k,i} = \binom{n}{i} \#\{\pi \in \mathfrak{D}_{n-i}^{(1)} : \text{exc}(\pi) = k\}.$$

This completes the proof of (4.1). □

**REMARK 4.2.** Proposition 4.1 in the special case  $l = 2$  was first proved by Mongelli [12, Proposition 3.4]. It can also be proved analytically using generating functions. In fact, from (1.1),

$$\sum_{n \geq 0} A_n^{(l)}(t, 0, 1) \frac{z^n}{n!} = \frac{1-t}{e^{tz} - te^z}. \tag{4.3}$$

Setting  $l = 1$  and then substituting  $z \leftarrow [l]_t z$  yields

$$\sum_{n \geq 0} A_n^{(1)}(t, 0, 1) \frac{([l]_t z)^n}{n!} = \frac{1-t}{e^{t[l]_t z} - te^{[l]_t z}} = \frac{1-t}{(e^{tz} - te^z)e^{t[l-1]_t z}}.$$

Comparing with (4.3),

$$e^{t[l-1]_t z} \sum_{n \geq 0} A_n^{(1)}(t, 0, 1) [l]_t^n \frac{z^n}{n!} = \sum_{n \geq 0} A_n^{(l)}(t, 0, 1) \frac{z^n}{n!}.$$

Identifying the coefficient of  $z^n/n!$  on both sides gives (4.1).

### 5. Final remarks

Let  $B_d = \{t^k(1+t)^{d-2k}\}_{k=0}^{\lfloor d/2 \rfloor}$ . Using an unpublished result of Gessel, Shareshian and Wachs [13, Remark 5.5] proved the result of  $q$ - $\gamma$ -positivity:  $A_n^{(1)}(t, 1, q)$  (respectively  $A_n^{(1)}(t, 0, q)$ ) has coefficients in  $\mathbb{N}[q]$  when expanded in  $B_n$  (respectively  $B_{n-1}$ ), which implies the  $l = 1$  case of Theorem 1.1. In view of (1.6), one may wonder if there are similar  $q$ - $\gamma$ -positivity results for the type B groups. This is not the case, because

$$A_2^{(2)}(t, 1, q) = (1+t)^3 + (q-1)t(1+t)$$

and

$$A_3^{(2)}(t, 0, q) = t(1+t)^4 + (2q+2q^2-1)t^2(1+t)^2 + q^3t^3,$$

neither of which has all  $\gamma$ -coefficients in  $\mathbb{N}[q]$ .

Recall that a polynomial  $h(t) = \sum_{k=0}^n a_k t^k \in \mathbb{Q}[t]$  is said to be *log-concave* if  $a_i^2 \geq a_{i-1}a_{i+1}$  for all  $1 \leq i \leq n-1$ . If the coefficients of  $h(t)$  have *no internal zero*, that is, there do not exist integers  $0 \leq i < j < k \leq n$  such that  $a_i \neq 0, a_j = 0, a_k \neq 0$ , then the log-concavity of  $h(t)$  implies its unimodality.

**PROPOSITION 5.1.** *For all  $n, l \geq 1$ , the coloured Eulerian polynomial  $A_n^{(l)}(t, 1, 1)$  is log-concave.*

**PROOF.** It was shown by Foata and Han [7, (5.15)] that

$$A_n^{(l)}(t, 1, 1) = (1 + t + \cdots + t^{l-1})^n A_n^{(1)}(t, 1, 1).$$

The result then follows from this relationship and the known fact [16, Proposition 2] that the product of two log-concave polynomials with nonnegative coefficients and no internal zero coefficients is again log-concave.  $\square$

Actually, in [12, Conjecture 8.1], Mongelli also conjectured that the type B derangement polynomial  $A_n^{(2)}(t, 0, 1)$  is log-concave. His conjecture can be extended to the coloured derangement polynomials.

**CONJECTURE 5.2.** *For all  $n, l \geq 1$ , the coloured derangement polynomial  $A_n^{(l)}(t, 0, 1)$  is log-concave.*

It is well known that a polynomial with nonnegative coefficients and with only real roots is log-concave; thus, the  $l = 1$  case of Conjecture 5.2 follows from Zhang's result [20] that the roots of  $A_n^{(1)}(t, 0, 1)$  are all real.

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