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ON MODULES HAVING SMALL COFINITE IRREDUCIBLES

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ABSTRACT. In this paper we obtain several new characterizations of modules having small cofinite irreducibles. One of these characterizations involves a metric topology defined on the submodule lattice.

A condition was sought in [2] on a commutative Noetherian ring R such that whenever R satisfied the condition and R was cyclically pure in an R-algebra S, then R was necessarily pure in S. Such a condition was found and turned out not only to be a sufficient condition but also a necessary condition. In addition, modules having small cofinite irreducibles were introduced in [2] to help prove the existence of certain irreducible ideals contained in arbitrarily high powers of the maximal ideals of certain Noetherian local rings. In this paper we extend the results of [5] to the case of modules over local rings and obtain several new characterizations of modules having small cofinite irreducibles, one of which is topological in nature and involves a metric topology defined on the lattice of submodules. These concepts have proved useful in many research problems in commutative algebra.

In this paper all rings are assumed to be commutative with identity and all modules are unital left modules. In general we adopt the terminology of [7], [9], [10], and [12]. By a local ring we mean a commutative Noetherian ring with identity which has a unique maximal ideal. Also an irreducible submodule of an R-module M is a submodule of M which cannot be written as the intersection of two larger submodules of M.

We now recall the definition of a module having small cofinite irreducibles given in [2]. Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then M is said to have *small cofinite irreducibles* if for every positive integer n, there exists an irreducible submodule Q of M such that $Q \subseteq m^nM$ and M/Q has finite length. Our first result establishes two useful characterizations of modules having small cofinite irreducibles.

OBSERVATION 1. Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then the following are equivalent:

(1.1) M has small cofinite irreducibles

(1.2) for every positive integer n, there exists an irreducible m-primary submodule Q such that $Q \subseteq m^nM$

(1.3) for every m-primary submodule Q' of M, there exists an irreducible m-primary submodule Q such that $Q \subseteq Q'$.

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PROOF. We begin by showing that (1.1) implies (1.2). Suppose that M has small cofinite irreducibles and that n is a positive integer. Then there exists an irreducible submodule Q of M such that $Q \subseteq m^n M$ and M/Q has finite length. By [10, Proposition 7, p. 194], Q is m-primary.

To show that (1.2) implies (1.3), suppose (1.2) holds. Suppose further that Q' is an m-primary submodule of M. Then there is a positive integer n such that $m^n M \subseteq Q'$. By (1.2), there exists an irreducible m-primary submodule Q such that $Q \subseteq m^n M$. Thus, it follows that $Q \subseteq Q'$.

Finally we show that (1.3) implies (1.1). Suppose that (1.3) holds and that n is a positive integer. Since $m^n M$ is an m-primary submodule of M, then by (1.3) there exists an irreducible m-primary submodule Q of M such that $Q \subseteq m^n M$. By [10, Proposition 7, p. 194], M/Q has finite length, which completes the proof. ∎

Our first theorem characterizes modules having small cofinite irreducibles in terms of the existence of a certain type of decreasing sequence of submodules.

THEOREM 2. *Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then M has small cofinite irreducibles if and only if there exists a decreasing sequence $\{Q_n\}$ of irreducible m-primary submodules of M such that for all m-primary submodules Q' of M, there is a positive integer n such that $Q_n \subseteq Q'$.*

PROOF. Suppose M has small cofinite irreducibles. Since mM is an m-primary submodule of M, by using (1.3) pick Q_1 to be an irreducible m-primary submodule of M such that $Q_1 \subseteq mM$. For $n > 1$, recursively define Q_n as follows: choose Q_n to be an irreducible m-primary submodule of M such that $Q_n \subseteq Q_{n-1} \cap m^n M$. This is possible using (1.3) since $\operatorname{Rad}(Q_{n-1} \cap m^n M) = m$ and so $Q_{n-1} \cap m^n M$ is an m-primary submodule of M. By construction, $\{Q_n\}$ is a decreasing sequence of irreducible m-primary submodules of M. Suppose also that Q' is an m-primary submodule of M. Then there exists a positive integer n such that $m^n M \subseteq Q'$. Thus, we have that $Q_n \subseteq Q'$.

Conversely, suppose that $\{Q_n\}$ is a decreasing sequence of irreducible m-primary submodules of M such that for all m-primary submodules Q' of M, there is a positive integer n such that $Q_n \subseteq Q'$. To show that M has small cofinite irreducibles, we will show that (1.2) holds. Suppose that n is a positive integer. Since $m^n M$ is an m-primary submodule of M, then by hypothesis, there is an integer k such that $Q_k \subseteq m^n M$. Also, Q_k is an irreducible m-primary submodule of M, so (1.2) holds. This completes the proof. ∎

Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Define a metric d, called the *m-adic metric*, on the lattice of R-submodules $\mathcal{L}_R(M)$ of M as follows (where $m^0 = R$):

$$d(A,B) = 0 \quad \text{if } A + m^n M = B + m^n M \text{ for all nonnegative integers } n;$$

otherwise,

$$d(A,B) = 2^{-s(A,B)} \quad \text{where } s(A,B) = \sup\{n \mid A + m^n M = B + m^n M\}.$$

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This metric occurs naturally in various module-theoretic settings ([1], [3], and [8]) and gives rise to the m-adic completion of the lattice of R-submodules $\mathcal{L}_R(M)$ of M [4]. The next result gives a topological characterization of modules having small cofinite irreducibles in terms of the m-adic topology on $\mathcal{L}_R(M)$ given above.

THEOREM 3. *Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then M has small cofinite irreducibles if and only if $\{0\}$ is in the closure of the set of all irreducible m-primary submodules of M in the m-adic topology on $\mathcal{L}_R(M)$.*

PROOF. Suppose M has small cofinite irreducibles. Suppose further that $\epsilon > 0$. Pick a positive integer n such that $2^{-n} < \epsilon$. Using (1.2) there exists an irreducible m-primary submodule Q of M such that $Q \subseteq m^n M$. Thus, $Q + m^n M = m^n M$, so $d(Q, \{0\}) \leq 2^{-n} < \epsilon$. Hence, $\{0\}$ is in the closure of the set of all irreducible m-primary submodules of M in the m-adic topology on $\mathcal{L}_R(M)$.

Conversely, suppose $\{0\}$ is in the closure of the set of all irreducible m-primary submodules of M in the m-adic topology on $\mathcal{L}_R(M)$. We will show that (1.2) holds. Suppose that n is a positive integer. Then there exists an irreducible m-primary submodule Q of M such that $d(Q, \{0\}) < 2^{-n}$. Thus, $Q + m^n M = m^n M$, and so $Q \subseteq m^n M$. Hence, (1.2) holds and so M has small cofinite irreducibles, which completes the proof. ∎

We now introduce the concept of principal module system. This concept is analogous to the notion of principal systems in rings studied by Northcott and Rees in [11]. Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. For a proper submodule A of M, A is said to be a *principle module system* in M if for every m-primary submodule Q' of M containing A, there exists an irreducible m-primary submodule Q of M satisfying $A \subseteq Q \subseteq Q'$.

THEOREM 4. *Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then M has small cofinite irreducibles if and only if $\{0\}$ is a principal module system in M.*

PROOF. This follows from the equivalence of (1.1) and (1.3) as well as the definition of principal module system (with $A = \{0\}$). ∎

For a finitely generated module M over a local ring R with maximal ideal m, we let M^* and R^* denote the completions of M and R, respectively, in their m-adic topologies. It is well known that R^* is a local ring with maximal ideal mR^* and that M^* is a finitely generated R^*-module [9, §9.10]. For an ideal a of R, we let aR^* denote the ideal of R^* generated by a. Also, for a submodule B of M, we let R^*B denote the R^*-submodule of M^* generated by B. Besides summarizing the previous results, our next result introduces a new characterization of modules having small cofinite irreducibles which involves the canonical ring and module completions.

COROLLARY 5. *Let R be a local ring with maximal ideal m and let M be a finitely generated R-module. Then the following are equivalent:*

(5.1) M has small cofinite irreducibles

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(5.2) for every positive integer n, there exists an irreducible submodule Q of M such that $Q \subseteq m^n M$ and M/Q has finite length

(5.3) $\{0\}$ is a principal module system in M

(5.4) for every m-primary submodule Q' of M, there exists an irreducible m-primary submodule Q of M such that $Q \subseteq Q'$

(5.5) there exists a decreasing sequence $\{Q_n\}$ of irreducible m-primary submodules of M such that for all m-primary submodules Q' of M, there is a positive integer n such that $Q_n \subseteq Q'$

(5.6) $\{0\}$ is in the closure of the set of all irreducible m-primary submodules of M in the m-adic topology on $\mathcal{L}_R(M)$

(5.7) the R^*-module M^* has small cofinite irreducibles.

PROOF. The equivalence of (5.1)–(5.6) follows from Observation 1 and Theorems 2–4. The equivalence of (5.1) and (5.7) follows from Theorem 4 and the fact that $\{0\}$ is a principal system in M if and only if $\{0\}$ is a principal system in the R^*-module M^*. ∎

Let R be a local ring with maximal ideal m. Then R is said to be *Gorenstein* if its injective dimension is finite [6] and R is said to be *approximately Gorenstein* [2] if for every positive integer n there is an ideal q of R such that $q \subseteq m^n$ and R/q is Gorenstein. Furthermore, R is said to have small cofinite irreducibles if R considered as an R-module has small cofinite irreducibles. We now obtain several characterizations of rings having small cofinite irreducibles.

COROLLARY 6. *Let R be a local ring with maximal ideal m. Then the following are equivalent:*

(6.1) R has small cofinite irreducibles

(6.2) for every positive integer n, there exists an irreducible ideal q of R such that $q \subseteq m^n$ and R/q has finite length

(6.3) $\{0\}$ is a principal system in R

(6.4) for every m-primary ideal q' of R, there exists an irreducible m-primary ideal q of R such that $q \subseteq q'$

(6.5) there exists a decreasing sequence $\{q_n\}$ of irreducible m-primary ideals of R such that for all m-primary ideals q' of R, there is a positive integer n such that $q_n \subseteq q'$

(6.6) $\{0\}$ is in the closure of the set of all irreducible m-primary ideals of R in the m-adic topology on $\mathcal{L}(R)$

(6.7) R^* has small cofinite irreducibles

(6.8) R is an approximately Gorenstein ring

(6.9) R^* is an approximately Gorenstein ring.

PROOF. The equivalence of (6.1)–(6.7) follows from Corollary 5 by considering R as an R-module. By [2, (1.11) Proposition (c)], we have (6.1) and (6.8) are equivalent and that (6.7) and (6.9) are equivalent. ∎

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