J. Plasma Physics (2009), *vol.* 75, *part* 1, *pp.* 9–13. © 2008 Cambridge University Press
doi:10.1017/S0022377808007605 Printed in the United Kingdom

Letter to the Editor

On the parametric decay of waves in magnetized plasmas

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(Received 19 August 2008 and in revised form 3 September 2008, first published online 24 October 2008)

Abstract. We reconsider the theory for three-wave interactions in cold plasmas. In particular, we demonstrate that previously overlooked formulations of the general theory are highly useful when deriving concrete expressions for specific cases. We also point out that many previous results deduced directly from the basic plasma equations contain inappropriate approximations leading to unphysical results. Finally, generalizations to more elaborate plasma models containing, for example, kinetic effects are given.

1. Introduction

Three-wave interactions are fundamental in nonlinear plasma science (e.g. Sagdeev and Galeev 1964; Sjölund and Stenflo 1967; Tsytovich 1970; Davidson 1972; Weiland and Wilhelmsson 1976; Shukla 1999; Stenflo and Shukla 2007). The coupling coefficients are in general derived by means of straightforward calculations. However, it has then turned out that there are many ways to end up with erroneous results. For example, in the calculation process it is tempting to neglect some of the smallest nonlinear terms from the outset. However, the larger nonlinear terms often cancel each other and the neglected terms are therefore important in many situations. Thus, there are numerous previous papers that contain incorrect final results. Instead of restarting our calculations from the basic nonlinear plasma equations we therefore stress in the present paper an alternative method to deduce the desired coupling coefficients for specific cases. Accordingly, we start directly from the general, although somewhat formal, results for the coupling coefficients, which we then evaluate in the appropriate limits. To illustrate our approach here we are going to consider a very simple specific example of the resonant interaction of three waves in a cold magnetized plasma. However, despite its simplicity, it is rather tricky to handle the algebra correctly. That is probably the reason why the explicit results presented here cannot be found in the previous literature.

2. General results for a cold plasma

In order to demonstrate the usefulness of our approach we shall, for simplicity, consider the resonant interaction between three waves with frequencies ω_j ($j = 1, 2, 3$) and wavevectors $\mathbf{k}_j$, with $\omega_3 = \omega_1 + \omega_2$ and $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$, in a cold multi-component magnetized plasma. Using (3a,b) and (4a,b) in Stenflo and Brodin (2006) and considering wave 3 as the pump wave we then write the growth rate γ of the excited waves 1 and 2 as

$$\gamma^2 = \frac{M_1 M_2 |CE_{3z}|^2}{[\partial D(\omega_1, \mathbf{k}_1)/\partial\omega_1][\partial D(\omega_2, \mathbf{k}_2)/\partial\omega_2]} \tag{1}$$

where

$$\begin{aligned} D(\omega, \mathbf{k}) = {} & \left(1 - \frac{k^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_{\mathrm{c}}^2}\right)\left[\left((1 - \frac{k_z^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_{\mathrm{c}}^2}\right)\right. \\ & \left. \times \left(1 - \frac{k_\perp^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right) - \frac{k_\perp^2k_z^2c^4}{\omega^4}\right] \\ & - \left(\sum \frac{\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}}{\omega(\omega^2 - \omega_{\mathrm{c}}^2)}\right)^2\left(1 - \frac{k_\perp^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right), \end{aligned} \tag{2}$$

$$\begin{aligned} M_j = {} & \left(1 - \frac{k_j^2c^2}{\omega_j^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega_j^2 - \omega_{\mathrm{c}}^2}\right)\left(1 - \frac{k_{jz}^2c^2}{\omega_j^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega_j^2 - \omega_{\mathrm{c}}^2}\right) \\ & - \left(\sum \frac{\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}}{\omega_j(\omega_j^2 - \omega_{\mathrm{c}}^2)}\right)^2 \end{aligned} \tag{3}$$

and

$$\begin{aligned} C = \sum_\sigma \frac{q\omega_{\mathrm{p}}^2}{m\omega_1\omega_2\omega_3k_{1z}k_{2z}k_{3z}} & \left[\frac{\mathbf{k}_1\cdot\mathbf{K}_1}{\omega_1}\mathbf{K}_2\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_2\cdot\mathbf{K}_2}{\omega_2}\mathbf{K}_1\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_3\cdot\mathbf{K}_3^*}{\omega_3}\mathbf{K}_1\cdot\mathbf{K}_2\right. \\ & \left. - \frac{i\omega_{\mathrm{c}}}{\omega_3}\left(\frac{k_{2z}}{\omega_2} - \frac{k_{1z}}{\omega_1}\right)\mathbf{K}_3^*\cdot(\mathbf{K}_1 \times \mathbf{K}_2)\right] \end{aligned} \tag{4}$$

with

$$\begin{aligned} \mathbf{K} = {} & -\left[\mathbf{k}_\perp + i\frac{\omega_{\mathrm{c}}}{\omega}\mathbf{k} \times \widehat{\mathbf{z}} + \left(\frac{\sum i(\omega_{\mathrm{c}}/\omega)\omega_{\mathrm{p}}^2/(\omega^2 - \omega_{\mathrm{c}}^2)}{1 - (k^2c^2/\omega^2) - \sum \omega_{\mathrm{p}}^2/(\omega^2 - \omega_{\mathrm{c}}^2)}\right)\left(\mathbf{k} \times \widehat{\mathbf{z}} - i\frac{\omega_{\mathrm{c}}}{\omega}\mathbf{k}_\perp\right)\right] \\ & \times \frac{(\omega^2 - k_\perp^2c^2 - \sum \omega_{\mathrm{p}}^2)\omega^2}{(\omega^2 - \omega_{\mathrm{c}}^2)k_\perp^2c^2} + k_z\widehat{\mathbf{z}}. \end{aligned} \tag{5}$$

Here E_{3z} denotes the electric field component along the external magnetic field $\mathbf{B}_0 = B_0\widehat{\mathbf{z}}$. Furthermore, q is the particle charge, m the mass, c the velocity of light, ω_{p} the plasma frequency, ω_{c} the gyrofrequency, and $k^2 = k_\perp^2 + k_z^2$. The sum sign stands for summation over all particle species. All the three waves satisfy of course the dispersion relation $D(\omega_j, \mathbf{k}_j) = 0$.

3. A specific example

In order to demonstrate the usefulness of the general formula, we now focus on the interaction between one electromagnetic wave propagating parallel to the external magnetic field and two electrostatic waves (with $kc \gg \omega$). For the latter waves the expression (5) simplifies to (e.g. Stenflo 1973)

$$\mathbf{K} = \left[\mathbf{k}_\perp - i\frac{\omega_{\mathrm{c}}}{\omega}\widehat{\mathbf{z}} \times \mathbf{k}\right]\frac{\omega^2}{\omega^2 - \omega_{\mathrm{c}}^2} + k_z\widehat{\mathbf{z}}, \tag{6}$$

https://doi.org/10.1017/S0022377808007605 Published online by Cambridge University Press

in which case the dispersion function is $D_{\rm es}(\omega,\mathbf{k})=(k^4c^4/\omega^4)\varepsilon(\omega,\mathbf{k})$, where

$$\varepsilon(\omega,\mathbf{k})=1-\frac{k_\perp^2}{k^2}\sum\frac{\omega_{\rm p}^2}{(\omega^2-\omega_{\rm c}^2)}-\frac{k_z^2}{k^2}\sum\frac{\omega_{\rm p}^2}{\omega^2}. \tag{7}$$

When the electromagnetic wave is the pump wave, we note that the use of E_{3z} in (1) as the pump amplitude is not appropriate, as $E_{3z}\to 0$ in the limit of parallel propagation. This is, however, easily dealt with by taking the limit $k_\perp\to 0$, in which case $\mathbf{K}$ for that wave can be related to the perpendicular electric field amplitude $E_\perp$ through

$$\mathbf{K}=-(\widehat{\mathbf{x}}+i\widehat{\mathbf{y}})\frac{k_z\omega E_\perp}{(\omega+\omega_{\rm c})E_z}, \tag{8}$$

where we have chosen the right-hand polarization and $\mathbf{k}_\perp=k_x\widehat{\mathbf{x}}$, for definiteness. The corresponding dispersion function is thus $D_\parallel(\omega,\mathbf{k})=-(1-\sum\omega_{\rm p}^2/\omega^2)D_T(\omega,\mathbf{k})$, where

$$D_T(\omega,\mathbf{k})=\left(\sum\frac{\omega_{\rm p}^2\omega_{\rm c}}{\omega(\omega^2-\omega_{\rm c}^2)}\right)^2-\left(1-\frac{k^2c^2}{\omega^2}-\sum\frac{\omega_{\rm p}^2}{\omega^2-\omega_{\rm c}^2}\right)^2. \tag{9}$$

The expression for CE_{3z} then reduces to (introducing $k_\perp=|k_{1,2x}|$)

$$\begin{aligned}CE_{3z}=-\frac{q\omega_{\rm p}^2}{m\omega_1\omega_2k_{1z}k_{2z}(\omega_3+\omega_{\rm c})}&\left[\left(\frac{k_\perp^2\omega_1^2\omega_2^2}{(\omega_1^2-\omega_{\rm c}^2)(\omega_2^2-\omega_{\rm c}^2)}\right)\left(\frac{k_{2x}}{\omega_1}+\frac{k_{1x}}{\omega_2}\right)\right.\\ &\left.-\frac{\omega_{\rm c}k_{3z}}{\omega_3}\left(\frac{k_{2x}k_{1z}}{(\omega_2+\omega_{\rm c})}+\frac{k_{1x}k_{2z}}{(\omega_1+\omega_{\rm c})}\right)+\frac{k_{1z}^2k_{2x}}{\omega_1}+\frac{k_{2z}^2k_{1x}}{\omega_2}\right]E_{3\perp}.\end{aligned} \tag{10}$$

In (10) we have used the fact that the ion mass is much larger than the electron mass, and dropped the sum sign as well as the index e on q, m, $\omega_{\rm p}$ and $\omega_{\rm c}$.

Next we consider the case when all wave frequencies are much smaller than the electron cyclotron frequency, to reduce (10) to

$$CE_{3z}=\frac{q\omega_{\rm p}^2}{mk_{1z}k_{2z}\omega_{\rm c}\omega_3}\left[\frac{k_{1z}^2}{\omega_1^2}k_{2x}+\frac{k_{2z}^2}{\omega_2^2}k_{1x}\right]E_{3\perp}. \tag{11}$$

From the expression (1) we then obtain the squared growth rate

$$\gamma_{\rm em}^2=\frac{q^2\omega_{\rm p}^4k_\perp^6}{m^2k_1^4k_2^4\omega_{\rm c}^2\omega_3^2[\partial\varepsilon(\omega_1,\mathbf{k}_1)/\partial\omega_1][\partial\varepsilon(\omega_2,\mathbf{k}_2)/\partial\omega_2]}\left[\frac{k_{1z}^2}{\omega_1^2}-\frac{k_{2z}^2}{\omega_2^2}\right]^2|E_{3\perp}|^2, \tag{12}$$

where the dispersion functions for the electrostatic waves are given by (7).

The growth rate if one of the electrostatic waves is the pump wave can be found quite similarly. Our result is

$$\gamma_{\rm es}^2=\frac{q^2\omega_{\rm p}^4k_\perp^4}{m^2k_{3z}^2k_{2z}^2k_1^4c^2\omega_{\rm c}^2[\partial D_T(\omega_2,\mathbf{k}_2)/\partial\omega_2][\partial\varepsilon(\omega_1,\mathbf{k}_1)/\partial\omega_1]}\left[\frac{k_{1z}^2}{\omega_1^2}-\frac{k_{3z}^2}{\omega_3^2}\right]^2|E_{3z}|^2, \tag{13}$$

where $(\omega_2,\mathbf{k}_2)$ now represents the electromagnetic wave. If, for example, the electrostatic waves are lower hybrid waves with $k_z\ll k_\perp$ and $\omega\ll|\omega_{\rm c}|$ (Kumar and Tripathi 2008) we have $\varepsilon(\omega_1,\mathbf{k}_1)=1+(\omega_{\rm p}^2/\omega_{\rm c}^2)-(k_{1z}^2\omega_{\rm p}^2/k_1^2\omega_1^2)-(\omega_{\rm pi}^2/\omega_1^2)$ where $\omega_{\rm pi}$ is the ion plasma frequency. Furthermore, if the second decay wave is a whistler wave we can reduce (13) by inserting $D_T(\omega_2,\mathbf{k}_2)=(\omega_{\rm p}^4/\omega_2^2\omega_{\rm c}^2)-(k_2^4c^4/\omega_2^4)$ to obtain our final result. The squared growth rate (20) in the paper by Kumar and Tripathi (2008) is, however, not positive in all parameter ranges and a detailed comparison is therefore of no interest.

Finally, when thermal effects are important, kinetic theory leads to a replacement of the growth rate according to

$$\gamma^2 = \frac{\omega_1 \omega_2 |V|^2}{W_1 W_2}, \tag{14}$$

where the wave energies $W_{1,2}$ are given by $W = \varepsilon_0 \mathbf{E}^* \cdot (1/\omega)\partial(\omega^2 \boldsymbol{\varepsilon})\mathbf{E}$, $\boldsymbol{\varepsilon}$ is the usual textbook dielectric tensor, and the expression for V can be found in Stenflo (1994) or Stenflo and Brodin (2006).

4. Discussion

Here we have considered two somewhat different basic decay processes, leading to the two formulas (12) and (13). Inspecting these two growth rate expressions, we clearly see that the sign of γ_{em}^2 is determined only by $\partial\varepsilon/\partial\omega_1$ and $\partial\varepsilon/\partial\omega_2$ (the values of which in this simple case with no equilibrium drift velocities are both positive). A similar relation holds for γ_{es}^2. This shows that if the waves 1 and 2 are both positive energy waves, the squared growth rate is always positive. However, in numerous previous papers (e.g. Laham et al. 2000; Panwar and Sharma 2007; Kumar and Tripathi 2008) this is not the case. This means that some inappropriate approximations have been adopted in all such previous papers.

Let us also stress that the squared coupling coefficient is a key ingredient in the final expression for the pump-wave-enhanced fluctuation spectrum (Stenflo 2004). The calculations above are thus highly relevant when stimulated electromagnetic emissions in the ionospheric plasma are to be analyzed.

Finally, it should be mentioned that the present formulas can be extended to also cover plasmas where relativistic effects are modifying the electron mass (e.g. Stenflo 1971; Shukla et al. 1986; Brodin 1999; Lazar and Merches 2003) or to plasmas with slightly non-uniform equilibrium densities (e.g. Stenflo and Shukla 1992). The amended squared coupling coefficients will then still be positive. This holds also for the limiting case of a Hall-magnetohydrodynamic plasma (Brodin and Stenflo 1990).

5. Conclusion

Inspecting for example the expression (14) it is obvious that the squared growth rate is always positive if the frequencies as well as the energies of the decay waves are positive. It is thus advisable to start the analysis for a particular wave interaction process from (14), or more simply from the expression (1) if the plasma is cold, if the algebra for the considered specific case is expected to be complicated and/or lengthy. Otherwise one can end up with unphysical results where, for example, the squared growth rate can change sign for certain values of the background parameters ω_{p} and ω_{c}.

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