Canad. Math. Bull. Vol. 39 (2), 1996 pp. 216–218

FURTHER ON THE POINTS OF INFLECTION OF BESSEL FUNCTIONS

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ABSTRACT. We offer here a substantial simplification and shortening of a proof of the monotonicity of the abscissae of the points of inflection of Bessel functions of the first kind and positive order.

1. **Introduction.** The abscissa, $j''_{\nu k}$, of the k-th positive point of inflection of the Bessel function $J_\nu(x)$ of positive order ν is, like any positive zero of $J_\nu(x)$ and $J'_\nu(x)$, an increasing function of ν for fixed $k = 1, 2, 3, \ldots$. Each point of inflection occurs (uniquely) in the second half of each arch of the graph of $J_\nu(x)$, except [3, Section 15.3(2), p. 486] for $1 < \nu < \infty$ when an additional point of inflection comes into being in the first arch alone. There it reposes in the first half of the arch. This causes a change in rank of the inflection points, with consequent awkwardness in nomenclature.

For this case, *i.e.*, $j''_{\nu 1}$, $1 < \nu < \infty$, the monotonicity is proved completely in [1, Theorem 4.1] and again, by quite different methods, in [2]. The method in [2] includes the remaining cases of $j''_{\nu k}$ as well. The method in [1] appeared to yield only partial results; the monotonicity of $j''_{\nu k}$, $k = 2, 3, \ldots$, was established there only for $0 < \nu < 3838$ [1, Theorem 10.1]. This was completed in [4] where the monotonicity of these remaining $j''_{\nu k}$ was demonstrated for $10 < \nu < \infty$, using delicate asymptotic estimates.

It turns out, however, that the approach in [1] can prove monotonicity not only for $j''_{\nu 1}$, $0 < \nu < \infty$, but also for *all* $j''_{\nu k}$, $0 < \nu < \infty$, $k = 2, 3, \ldots$; [1, Sections 5–10, pp. 938–944] and, for this purpose, all of [4] can be replaced by what follows here. We note that [4] contains other valuable information and led to the derivation of asymptotic expansions for $j''_{\nu k}$ [5].

2. **Theorem and Proof.** The monotonicity of $j''_{\nu 1}$, $1 < \nu < \infty$, was shown in [1]. We dispose here of the remaining cases:

THEOREM. *If $0 < \nu < 1$, then $j''_{\nu 1}$ is an increasing function of ν. If $0 < \nu < \infty$, then $j''_{\nu k}$ is an increasing function of ν for each $k = 2, 3, \ldots$.*

Both sentences of the theorem refer to the same situation, that is, of points of inflection occurring in the second half of each arch of the graph, so that [1, Theorem 3.2]

$$J_\nu(j''_{\nu k})J'''_\nu(j''_{\nu k}) > 0. \tag{1}$$

For the proof of the theorem we require the following lemma.

Partially supported by the Natural Sciences and Engineering Research Council of Canada.
Received by the editors January 31, 1995.
AMS subject classification: 33C10.

LEMMA. *If $\nu > 0$, then*

$$\frac{1}{2}\lambda^3 J_\nu(\lambda) J_\nu'''(\lambda) = \int_0^\lambda t J_\nu^2(t)\,dt - \lambda^2 J_\nu^2(\lambda), \tag{2}$$

where λ denotes any $j_{\nu k}''$, $k = 1, 2, 3, \ldots$.

PROOF OF LEMMA. We use [1, p. 933] the Bessel differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \tag{3}$$

and its corollary

$$x^2(x^2 - \nu^2)y''' + x(x^2 - 3\nu^2)y'' + [x^4 - (2\nu^2 + 1)x^2 + \nu^4 - \nu^2]y' = 0. \tag{4}$$

Putting $x = \lambda$ in (4) yields

$$\lambda^2(\lambda^2 - \nu^2)J_\nu'''(\lambda) = -[\lambda^4 - (2\nu^2 + 1)\lambda^2 + \nu^4 - \nu^2]J_\nu'(\lambda),$$

since $J_\nu''(\lambda) = 0$. Replacing $J_\nu'(\lambda)$ now by its value from (3), we obtain for the left member of (2)

$$\frac{1}{2}\lambda^3 J_\nu(\lambda) J_\nu'''(\lambda) = \frac{1}{2}[\lambda^4 - (2\nu^2 + 1)\lambda^2 + \nu^4 - \nu^2]J_\nu^2(\lambda). \tag{5}$$

To show that the right member of (2) also equals the right member of (5), we note [3, Section 5.11 (11), p. 135] that

$$\int_0^\lambda t J_\nu^2(t)\,dt = \frac{1}{2}[(\lambda^2 - \nu^2)J_\nu^2(\lambda) + \lambda^2\{J_\nu'(\lambda)\}^2],$$

since the lower limit of integration yields a vanishing quantity when $\nu > 0$, as here.

Again replacing $J_\nu'(\lambda)$ from (3), we obtain

$$\int_0^\lambda t J_\nu^2(t)\,dt - \lambda^2 J_\nu^2(\lambda) = \frac{1}{2}[\lambda^4 - (2\nu^2 + 1)\lambda^2 + \nu^4 - \nu^2]J_\nu^2(\lambda),$$

i.e., again the right member of (5).

This proves the lemma.

PROOF OF THEOREM. We recall the formula [1, Theorem 3.1] for $\nu > 0$,

$$\frac{d\lambda}{d\nu} = \frac{2\nu}{\lambda^2 J_\nu(\lambda) J_\nu'''(\lambda)}\left\{\int_0^\lambda \frac{J_\nu^2(t)}{t}\,dt - J_\nu^2(\lambda)\right\}. \tag{6}$$

In view of (2), we may rewrite the expression in braces as

$$\frac{1}{2}\lambda J_\nu(\lambda) J_\nu'''(\lambda) - \frac{1}{\lambda^2}\int_0^\lambda t J_\nu^2(\lambda)\,dt + J_\nu^2(\lambda) + \int_0^\lambda \frac{J_\nu^2(t)}{t}\,dt - J_\nu^2(\lambda).$$

Accordingly, for $\nu > 0$,

$$\frac{d\lambda}{d\nu} = \frac{2\nu}{\lambda^2 J_\nu(\lambda) J_\nu'''(\lambda)}\left\{\frac{1}{2}\lambda J_\nu(\lambda) J_\nu'''(\lambda) + \int_0^\lambda \left(\frac{1}{t} - \frac{t}{\lambda^2}\right) J_\nu^2(t)\,dt\right\}. \tag{7}$$

The right member is positive, in view of (1), for the λ under consideration in the theorem, *i.e.*, for $\lambda = j_{\nu k}''$, $k = 2, 3, \ldots$, when $0 < \nu < \infty$, and also for $\lambda = j_{\nu 1}''$ when $0 < \nu < 1$.

The theorem is proved.

REMARK. Although the proof above stands on its own as presented, we wish to acknowledge that we were led to its cornerstone, namely (7), by noting the fashion in which Mercer [2] formulated his expression for $dj''/d\nu$.

https://doi.org/10.4153/CMB-1996-027-7 Published online by Cambridge University Press

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https://doi.org/10.4153/CMB-1996-027-7 Published online by Cambridge University Press