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CONVERGENCE IN RELAXATION SPECTRUM RECOVERY

R. J. LOY, F. R. DE HOOG and R. S. ANDERSSEN✉

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Abstract

Because of its practical and theoretical importance in rheology, numerous algorithms have been proposed and utilised to solve the convolution equation $g(x) = (\mathrm{sech} \star h)(x)$ $(x \in \mathbb{R})$ for h, given g. There are several approaches involving the use of series expansions of derivatives of g, which are then truncated to a small number of terms for practical application. Such truncations can only be expected to be valid if the infinite series converge. In this note, we examine two specific truncations and provide a rigorous analysis to obtain sufficient conditions on g (and equivalently on h) for the convergence of the series concerned.

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1. Introduction

In continuous relaxation spectrum recovery using oscillatory shear data, one is required to solve for the relaxation spectrum h from the convolution equation

$$g(x) = (\mathrm{sech} \star h)(x) \quad (x \in \mathbb{R}). \tag{1.1}$$

Here g and h are bounded nonnegative measurable functions on $\mathbb{R}$, although satisfying (1.1) means that g is necessarily C^∞. The details of obtaining this equation from considerations of storage and loss moduli are given in [1]. Some approaches to the problem of estimating h from g are discussed in [5, Ch. 4A] with different terminology (see also [9], [10, 4(f)]). Having derived an infinite series solution for h in terms of derivatives of g, such as [1, (3.13)], which is reproduced in (3.2) in Section 3 below, approximations such as

$$h(x) \sim \frac{1}{\pi} g(x), \quad h(x) \sim \frac{1}{\pi}\Big(g(x) - \frac{\pi^2}{8}\frac{d^2 g}{dx^2}(x)\Big)$$

are generated as truncations of the series. The utility and implementation of such approximations have been examined in [2]. However, to verify their validity, a proof of the convergence of the series is required. The provision of a direct proof of this convergence is the goal of the current paper.

The paper has been organised in the following manner. Section 2 discusses the related power series approach of Gureyev *et al.* [7] as motivation for the subsequent deliberations where weaker conditions which guarantee convergence are derived. The mollifier approach of [1, Section 3] is briefly outlined in Section 3 in order to set the scene for the subsequent analysis. Sections 4 and 5 derive properties of Gaussians and the sech function required to establish, in Section 6, conditions under which convergence will hold. In particular, it is established that pointwise and L^1 convergence hold for any g (and, equivalently, h) in a dense subset of L^1. This, in turn, justifies the truncation procedures used to approximate, as outlined above. Some remarks about the possible extension to the more general situation where $h \in L^1$ are given in Section 7.

2. The power series approach of Gureyev *et al.*

Following Gureyev *et al.* [7], we consider a 'local' approach of assuming a power series expansion for h and obtaining a series expansion for g in terms of derivatives of h, which can then be 'inverted' to yield an expansion for h in terms of derivatives of g. Under a suitable hypothesis, the formal solution can be confirmed to solve (1.1).

To be clear about the nature of the hypotheses needed, we examine the steps needed to make the confirmation. One tries for a solution of the form

$$h(x) = \sum_{n=0}^{\infty} b_n \frac{d^n g}{dx^n}(x). \tag{2.1}$$

Formal substitution in (1.1) and equating coefficients of $d^n h/dx^n(x)$ results in b_n satisfying

$$b_0 = \frac{1}{a_0}, \quad b_n = -b_0 \sum_{k=0}^{n-1} b_k a_{n-k},$$

where

$$a_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} y^n \operatorname{sech}(y)\, dy$$

are multiples of the moments of sech. Now $a_{2k-1} = 0$, $k = 1, 2, \ldots$, because sech is even, and hence it follows that $b_{2k-1} = 0$, $k = 1, 2, \ldots$, so odd derivatives of g are absent from the solution. Since $a_0 = \pi$ and, for $k = 1, 2, \ldots$,

$$a_{2k} = \frac{1}{(2k)!} \int_{-\infty}^{\infty} x^{2k} \operatorname{sech}(x)\, dx = 2 \cdot (2k)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^{2k+1}} \le 2,$$

one shows that there are geometric bounds $|b_n| \le (2/\pi^2)(1 + 2/\pi)^{n-1}, n = 1, 2, \ldots$. Indeed, this is clear for $n = 1$. Suppose it true for $1 \le n \le j$,

$$|b_{j+1}| \le \frac{1}{\pi} \sum_{k=0}^{j} |b_k a_{j+1-k}| \le \frac{2}{\pi^2} + \sum_{k=1}^{j} \frac{4}{\pi^3} \left(1 + \frac{2}{\pi}\right)^{k-1}.$$

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Summing the geometric series gives

$$|b_{j+1}| \le \frac{2}{\pi^2}\Big(1+\frac{2}{\pi}\Big)^j,$$

which is the result for $j+1$, as required. The bounds follow by mathematical induction.

Thus, (2.1) converges absolutely and uniformly on any interval for which there are constants $C > 0$ and $0 < \delta < (1+2/\pi)^{-1}$ for which

$$\left|\frac{d^n g(x)}{dx^n}\right| \le C\delta^n \quad n \in \mathbb{N}. \tag{2.2}$$

In fact, for (2.1), only even values of n are needed, but, for later steps, odd powers enter the calculation. In particular, the same convergence holds for

$$h^{(k)}(x) = \sum_{n=0}^{\infty} b_{2n}\frac{d^{k+2n}g}{dx^{k+2n}}(x) \quad k = 1, 2, \ldots. \tag{2.3}$$

Note that, for each $k = 1, 2, \ldots$, assuming that (2.2) holds at $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}|b_n|\Big(\sum_{k=0}^{\infty}\left|\frac{d^{k+n}g}{dx^{k+n}}(x)\right|\frac{|y|^k}{k!}\Big)\operatorname{sech}(y)\,dy &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{d^{k+n}g}{dx^{k+n}}(x)\right||a_k b_n| \\
&= \sum_{r=0}^{\infty}\Big(\sum_{n+k=r}|a_k b_n|\Big)\left|\frac{d^r g}{dx^r}(x)\right| \\
&\le 2\sum_{r=0}^{\infty}\Big(\sum_{n=0}^{r}|b_n|\Big)\left|\frac{d^r g}{dx^r}(x)\right| \\
&\le C'\sum_{r=0}^{\infty}\delta^r(1+2/\pi)^r < \infty, \qquad (2.4)
\end{aligned}$$

for some constant $C' > 0$. This finiteness is needed below.

Thus, substituting (2.1) in (1.1),

$$\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} b_{2n}\frac{d^{2n}g}{dx^{2n}}(x-y)\operatorname{sech}(y)\,dy \tag{2.5}$$

$$= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty} b_{2n}\frac{d^{2n}}{dx^{2n}}\Big(\sum_{k=0}^{\infty}\frac{d^k g}{dx^k}(x)\frac{(-y)^k}{k!}\Big)\operatorname{sech}(y)\,dy \tag{2.6}$$

$$= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty} b_{2n}\Big(\sum_{k=0}^{\infty}\frac{d^{k+2n}g}{dx^{k+2n}}(x)\frac{(-y)^k}{k!}\Big)\operatorname{sech}(y)\,dy \tag{2.7}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{d^{2k+2n}g}{dx^{2k+2n}}(x)a_{2k}b_{2n} \tag{2.8}$$

$$= \sum_{r=0}^{\infty}\Big(\sum_{n+k=r}a_{2k}b_{2n}\Big)\frac{d^{2r}g}{dx^{2r}}(x) \tag{2.9}$$

$$= g(x). \tag{2.10}$$

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The step (2.5) → (2.6) requires the Taylor expansion of g about x to converge to g everywhere on $\mathbb{R}$; (2.6) → (2.7) is the usual term-by-term differentiation with the resulting series absolutely convergent by (2.3); (2.7) → (2.8) is the usual term-by-term integration; and the rearrangement (2.8) → (2.9) is valid because of finiteness at (2.4).

The regularity required for this argument to be valid for any point $x \in \mathbb{R}$, is that:

the Taylor series of g about x converges to g everywhere on $\mathbb{R}$, and so g extends from $\mathbb{R}$ to an entire function on $\mathbb{C}$; and (2.11)

the bounds (2.2) hold. (2.12)

But then, given $\rho > 0$ and $x \in \mathbb{R}$, set $z = x + \rho\mathrm{i}$ and estimate

$$|g(z)| = \left|\sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n g}{dx^n}(x)(z-x)^n\right| \le C\sum_{n=0}^{\infty}\frac{1}{n!}(\delta\rho)^n.$$

It follows that

$$g \text{ is bounded on the strip } |\Im(z)| \le \rho \text{ for every } \rho > 0. \tag{2.13}$$

Remark 2.1. Since we assume that g is bounded on $\mathbb{R}$, all these conditions are satisfied, for example, if g is of exponential type less than $(1 + 2/\pi)^{-1}$ (see [3, Ch. 6]). Our approach below is valid under more general conditions (see (6.3)).

In order to justify the basic assumption of [7] that g has an expansion in terms of derivatives of h, one argues that

$$g(x) = \int_{-\infty}^{\infty} h(x-y)\,\mathrm{sech}(y)\,dy \tag{2.14}$$

$$= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(-y)^n}{n!}\frac{d^n h}{dx^n}(x)\,\mathrm{sech}(y)\,dy \tag{2.15}$$

$$= \sum_{n=0}^{\infty}\frac{d^n h}{dx^n}(x)\frac{(-1)^n}{n!}\int_{-\infty}^{\infty} y^n\,\mathrm{sech}(y)\,dy \tag{2.16}$$

$$= \sum_{n=0}^{\infty} a_{2n}\frac{d^{2n}h}{dx^{2n}}(x), \tag{2.17}$$

where the a_n are as before. For (2.14) → (2.15), we need the Taylor series of h about x to converge to h on all of $\mathbb{R}$, so that h is the restriction to $\mathbb{R}$ of an entire function. For (2.15) → (2.16), it would suffice that

$$\int_0^{\infty}\sum_{n=0}^{\infty}\frac{y^n}{n!}\left|\frac{d^n h}{dx^n}(x)\right|\mathrm{sech}(y)\,dy < \infty. \tag{2.18}$$

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This is a very strong restriction, but holds, for example, if there exist $0 < r < 1$ and $K > 0$ such that

$$\left|\frac{d^n h}{dx^n}(x)\right| \le K r^n,$$

for then

$$\int_0^\infty \sum_{n=0}^\infty \frac{y^n}{n!}\left|\frac{d^n h}{dx^n}(x)\right| \operatorname{sech}(y)\,dy \le \int_0^\infty e^{ry}\operatorname{sech}(y)\,dy < \infty.$$

This is the case for the Gaussians $\exp(-\beta x^2)$ for $\beta < 1$, but note that (2.18) fails for $h(x) = \exp(-x^2)$.

3. The mollifier approach of Anderssen *et al.*

The argument in [1, Section 3], going from their (3.6) and (3.7) to their (3.12) and (3.13), respectively, lacks any detail to justify the interchanging of the limit operations. Here, we consider the matter more carefully, focusing on obtaining their (3.13)

$$h(t) = \frac{1}{\pi}\sum_{r=0}^\infty (-1)^r \left(\frac{\pi}{2}\right)^{2r} \frac{1}{(2r)!}\left[\frac{d^{2n}g}{dt^{2n}}(t)\right], \tag{3.1}$$

from their (3.7)

$$h(t) \tag{3.2}$$
$$= \lim_{\varepsilon\to 0}\frac{1}{2\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty g(x)\sum_{n=0}^\infty \frac{1}{(2n)!}\left(\frac{\pi}{2}p\right)^{2n}\exp\left(-\frac{1}{2}\varepsilon^2p^2\right)\exp(ip(t-x))\,dp\,dx,$$

where $t > 0$ is constant as far as the integration is concerned. This integral comes from taking the Fourier transform of (1.1), rearranging as a formula for $\widehat{h}$, introducing the factor $\exp(-\varepsilon^2 p^2/2)$, applying standard inversion, and then changing the order of integration.

Since

$$\sum_{n=0}^\infty \frac{1}{(2n)!}\left(\frac{\pi}{2}p\right)^{2n}\exp\left(-\frac{1}{2}\varepsilon^2p^2\right) \le \exp\left(\frac{\pi}{2}p - \frac{1}{2}\varepsilon^2p^2\right)$$

is integrable for any $\varepsilon > 0$, the inner integration and summation in (3.1) can be interchanged by dominated convergence, giving

$$\frac{1}{2\pi^2}\int_{-\infty}^\infty g(x)\,dx\sum_{n=0}^\infty\frac{1}{(2n)!}\int_{-\infty}^\infty\left(\frac{\pi}{2}p\right)^{2n}\exp\left(-\frac{1}{2}\varepsilon^2p^2\right)\exp(ip(t-x))\,dp. \tag{3.3}$$

4. Some observations about Gaussians

Set

$$F_\varepsilon(\xi) = \int_{-\infty}^\infty \exp(-\tfrac{1}{2}\varepsilon^2p^2)\exp(-ip\xi)\,dp, \tag{4.1}$$

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which is the Fourier transform of the Gaussian $\exp(-\frac{1}{2}\varepsilon^2 p^2)$. Of course,

$$F_\varepsilon(\xi) = \frac{\sqrt{2\pi}}{\varepsilon} \exp\Big(-\frac{\xi^2}{2\varepsilon^2}\Big). \tag{4.2}$$

Also note that $F_\varepsilon/(2\pi)$ is an approximate identity in $L^1(\mathbb{R})$ [8, Section 21.36], with all elements having norm one. In particular, $F_\varepsilon \geq 0$ and $\int_{-\delta}^{\delta} F_\varepsilon(\xi)\, d\xi \to 2\pi$ as $\varepsilon \to 0$, for any $\delta > 0$. Thus, for any bounded measurable f which is continuous at t,

$$\int_{-\infty}^{\infty} F_\varepsilon(x-t) f(x)\, dx \to 2\pi f(t) \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

Further, for any $f \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} F_\varepsilon(x-t) f(x)\, \mathrm{d}x - 2\pi f(t) \Big|\, dt \to 0.$$

Differentiating (4.1), for each $n \in \mathbb{N}$,

$$\frac{d^{2n} F_\varepsilon}{d\xi^{2n}}(\xi) = \int_{-\infty}^{\infty} (-ip)^{2n} \exp\Big(-\frac{1}{2}\varepsilon^2 p^2\Big) \exp(-ip\xi)\, dp, \tag{4.4}$$

so that, by the Riemann–Lebesgue theorem, F_ε, together with all its derivatives, vanish at $\pm\infty$. Note the similarity between the inner integrand of (3.3) and that of (4.4).

We shall require the finiteness of integrals of the form

$$\int_{-\infty}^{\infty} g(x) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \Big(\frac{\pi}{2}\Big)^{2n} \Big| \frac{d^{2n} F_\varepsilon}{dx^{2n}}(x-t) \Big|\, dx, \tag{4.5}$$

where g is a given nonnegative bounded measurable function. For this, we recall some properties of Hermite functions. In particular,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad n = 1, 2, \ldots,$$

so that

$$\int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2}\, dx = 2^n \cdot n! \cdot \sqrt{\pi}. \tag{4.6}$$

From (4.2),

$$\begin{aligned}
\frac{d^{2n} F_\varepsilon}{dx^{2n}}(x-t) &= \frac{\sqrt{2\pi}}{\varepsilon} \frac{d^{2n}}{dx^{2n}} \Big(\exp -\frac{(x-t)^2}{2\varepsilon^2}\Big) \\
&= \frac{\sqrt{2\pi}}{\varepsilon} (2\varepsilon^2)^{-2n} \frac{d^{2n}}{d\eta^{2n}} \Big(\exp -\eta^2\Big)\Big|_{\eta=(x-t)(\sqrt{2}\varepsilon)} \\
&= \frac{\sqrt{2\pi}}{\varepsilon} (2\varepsilon^2)^{-2n} H_{2n}\Big(\frac{x-t}{\sqrt{2}\varepsilon}\Big) \exp(-(x-t)^2/(2\varepsilon^2)),
\end{aligned}$$

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and hence

$$H_{2n}\Big(\frac{x-t}{\sqrt{2}\varepsilon}\Big) = e^{(x-t)^2/(2\varepsilon^2)}(2\varepsilon^2)^{2n}\frac{\varepsilon}{\sqrt{2\pi}}\frac{d^{2n}F_\varepsilon}{dx^{2n}}(x-t).$$

Now, from (4.6),

$$\int_{-\infty}^{\infty} H_{2n}\Big(\frac{x-t}{\sqrt{2}\varepsilon}\Big)^2 e^{-(x-t)^2/(2\varepsilon^2)}\,dx = 2^{2n}\cdot(2n)!\cdot\sqrt{2\pi}\varepsilon,$$

so that

$$\int_{-\infty}^{\infty} e^{(x-t)^2/(2\varepsilon^2)}\Big|\frac{d^{2n}F_\varepsilon}{dx^{2n}}(x-t)\Big|^2\,dx = 2^{2n}\cdot(2n)!\cdot\sqrt{2\pi}\varepsilon\cdot(2\varepsilon^2)^{-4n}\cdot\frac{2\pi}{\varepsilon^2}.$$

Thus,

$$\Big\|e^{(x-t)^2/(4\varepsilon^2)}\frac{d^{2n}F_\varepsilon}{dx^{2n}}(x-t)\Big\|_2 = 2^{-n+3/4}\cdot\sqrt{(2n)!}\cdot\varepsilon^{-4n-1/2}\cdot\pi^{3/4}.$$

Consequently,

$$\frac{2^{-n+3/4}\cdot\sqrt{(2n)!}\cdot\varepsilon^{-4n-1/2}\cdot\pi^{3/4}\cdot(\pi/2)^{2n}}{(2n)!} = A_\varepsilon^n\cdot B_\varepsilon\cdot((2n)!)^{-1/2}$$

for constants $A_\varepsilon, B_\varepsilon$, which is summable over n for each $\varepsilon > 0$. The integral (4.5) can be rewritten as

$$\int_{-\infty}^{\infty}\Big(g(x)e^{-(x-t)^2/(4\varepsilon^2)}\Big)\Big(\sum_{n=0}^{\infty}\frac{1}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n}\Big|\frac{d^{2n}F_\varepsilon}{dx^{2n}}(x-t)\Big|e^{(x-t)^2/(4\varepsilon^2)}\Big)\,dx,$$

where each $(\cdots)$ has finite L^2-norm. Thus, the integral (4.5) is finite, as required.

5. Properties of the function sech

It is known [4, Proposition 3] that, for $t \in \mathbb{R}$,

$$\operatorname{sech}^{(m)}(t) = \operatorname{sech}(t)S_m(\tanh(t)),$$

where the S_m are the polynomials given by

$$S_m(z) = \sum_{j=0}^{m}\Big[(-1)^j j!\sum_{k=j}^{m}\binom{m}{k}\left\{{k \atop j}\right\}2^{k-j}\Big](z+1)^j$$

and the $\left\{{k \atop j}\right\}$ denote the Stirling numbers of the second kind. Since $|\tanh(t)| \le 1$, this establishes continuity, boundedness and integrability of each of the derivatives $\operatorname{sech}^{(m)}$.

The meromorphic function $z \mapsto \operatorname{sech}(z)$ has (simple) poles at the points of the set $\{(k+1/2)i\pi, k \in \mathbb{Z}\}$. The closest such points to the real line are $S = \{\pm i\pi/2\}$. Thus, for real $t \neq 0$,

$$\operatorname{sech}(t+z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\operatorname{sech}^{(n)}(t)$$

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has radius of convergence $d(t, S) > \pi/2$, and so, in particular,

$$\operatorname{sech}(t + i\pi/2) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i\pi}{2}\Big)^n \operatorname{sech}^{(n)}(t),$$

where the series is absolutely convergent. Since $\operatorname{sech}(t + i\pi/2)$ is purely imaginary for any real t, it follows that, for $t \in \mathbb{R} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} \operatorname{sech}^{(2n)}(t) = 0. \tag{5.1}$$

Since

$$\operatorname{sech}^{(2n)}(0) = (-1)^n 2(2n)!\Big(\frac{2}{\pi}\Big)^{2n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \sim (-1)^n 2(2n)!\Big(\frac{2}{\pi}\Big)^{2n+1}$$

[6, Section 9.652], at $t = 0$, the terms of the series (5.1) converge to $4/\pi$, so the series itself diverges to $+\infty$.

In fact, much more is true. For $u \in (0, \infty)$, set $M_u = \max\{|\operatorname{sech}(z)| : |z - u| \le \pi/2\}$. Note that $u \mapsto M_u$ is continuous and $M_u \to 0$ as $u \to \infty$, $M_u \to \infty$ as $u \downarrow 0$. Take $\varepsilon > 0$, $0 < s < t$ and $\pi/2 < d < \min\{d(s, S), d(t, S)\}$. Set $M = \max\{M_u : s \le u \le t\}$. Finally, take natural numbers $p \le q$. Then, making use of Cauchy's inequalities,

$$\begin{aligned}\sum_{n=p}^{q} \frac{1}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n} |\operatorname{sech}^{(2n)}(t) - \operatorname{sech}^{(2n)}(s)| &\le \sum_{n=2p}^{2q} \frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n |\operatorname{sech}^{(n)}(t) - \operatorname{sech}^{(n)}(s)| \\ &\le \sum_{n\ge 2p} \frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n (|\operatorname{sech}^{(n)}(t)| + |\operatorname{sech}^{(n)}(s)|) \\ &\le 2M \sum_{n\ge 2p} \Big(\frac{\pi}{2d}\Big)^n < \varepsilon,\end{aligned}$$

provided p is sufficiently large. Thus, (5.1) is uniformly (absolutely) convergent on compact subsets of $\mathbb{R} \setminus \{0\}$. As remarked earlier, it is divergent to $+\infty$ at $t = 0$.

For norm convergence in $L^1(\mathbb{R})$, similarly, on the interval $\{t : |t - u| \le \pi/2\}$ for some $u > 0$,

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n} |\operatorname{sech}^{(2n)}(t)| \le 2M_u \sum_{n=0}^{\infty} \Big(\frac{\pi}{2d}\Big)^n.$$

Further, on the disc with the interval as diameter, setting $z = x + iy$,

$$|\cosh^2(z)| = \cosh^2(x) - 1 + \cos^2(y) \ge \cosh^2(x) - 1 \ge \cosh^2(u - \pi/2) - 1.$$

It follows that, for $u \ge \pi$ (to ensure the second inequality),

$$M_u \le \frac{1}{\sqrt{\cosh^2(u - \pi/2) - 1}} \le \sqrt{2}\exp(-u + \pi/2).$$

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So, finally, for $u \geq \pi$ and $|t - u| \leq \pi/2$, so that $d > \pi/\sqrt{2}$,

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n} |\mathrm{sech}^{(2n)}(t)| \leq 2\sqrt{2}\exp\Big(-u+\frac{\pi}{2}\Big)\sum_{n=0}^{\infty}\Big(\frac{\pi}{2d}\Big)^{n} \leq 9\exp\Big(-u+\frac{\pi}{2}\Big).$$

It follows, from this and the earlier uniform estimate, that the series in (5.1) is L^1-norm convergent on $\mathbb{R} \setminus (-\delta, \delta)$ for each $\delta > 0$.

In summary,

$$e_k(t) = \sum_{n=0}^{k} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} \mathrm{sech}^{(2n)}(t) \overset{k\to\infty}{\to} \begin{cases} 0 & t \neq 0, \\ +\infty & t = 0, \end{cases} \tag{5.2}$$

uniformly and in L^1 on each $\mathbb{R} \setminus \{-\delta, \delta\}$.

Since $M_u = O(u^{-1})$ near 0, as well as $d(u, S) \to \pi/2$ as $u \to 0$, this argument does not extend to give convergence in $L^1(\mathbb{R})$. In view of the above, the boundedness of (e_k) in $L^1(\mathbb{R})$ would follow from boundedness in $L^1[0, \delta]$ for some $\delta > 0$. Further discussion is given in Section 7.

6. Convergence arguments

Using (4.1), the summation in (3.3) can be written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n} \frac{d^{2n}F_\varepsilon}{dt^{2n}}(x-t),$$

so that (3.3) becomes

$$\frac{1}{2\pi^2}\int_{-\infty}^{\infty} g(x) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n} \frac{d^{2n}F_\varepsilon}{dt^{2n}}(x-t)\,dx. \tag{6.1}$$

Furthermore, g and its derivatives vanish at $\pm\infty$, so that integration by parts yields

$$\int_{-\infty}^{\infty} g(x)\frac{d^{2n}F_\varepsilon}{dt^{2n}}(x-t)\,dx = \int_{-\infty}^{\infty} F_\varepsilon(x-t)\frac{d^{2n}g(x)}{dx^{2n}}\,dx \quad n = 0, 1, \ldots.$$

To justify the interchange of integration and summation in (6.1), we simply invoke (4.5), to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n}\Big(\int_{-\infty}^{\infty} F_\varepsilon(x-t)\frac{d^{2n}g(x)}{dx^{2n}}\,dx\Big), \tag{6.2}$$

which is an absolutely convergent series.

For the case of interest $g(t) = (\mathrm{sech} \star h)(t)$,

$$\frac{d^{2n}g(t)}{dt^{2n}} = (\mathrm{sech}^{(2n)} \star h)(t).$$

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So, as noted in (4.3), for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} F_\varepsilon(x-t)\frac{d^{2n}g(x)}{dx^{2n}}\,dx \to 2\pi\frac{d^{2n}g(t)}{dt^{2n}}$$

pointwise and also in $L^1(\mathbb{R})$. Rather than termwise convergence, on the *assumption* that the sum

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n}\frac{d^{2n}g(x)}{dx^{2n}} \tag{6.3}$$

is continuous at $x = t$ and bounded, (4.3) with $f(x)$ given by (6.3) gives

$$h(t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n}\frac{d^{2n}g(t)}{dt^{2n}}, \tag{6.4}$$

with convergence being pointwise and in $L^1(\mathbb{R})$. This is the required equation (3.13) of [1].

In particular, we note that (6.4) certainly holds when g satisfies the growth condition (2.12)

$$\left|\frac{d^n g(t)}{dt^n}\right| \le C\delta^n \quad n \in \mathbb{N}, \tag{6.5}$$

for some $C > 0$ and some $0 < \delta < (1 + 2/\pi)^{-1}$. In fact, any power growth, that is, any fixed finite $\delta > 0$ in (6.5), is sufficient.

In fact, we can do much better that this. Suppose that g is the restriction to $\mathbb{R}$ of a function analytic and bounded (by M) on a strip $|\Im z| \le \rho$ for some $\rho > \pi/2$. This is much weaker that the entire function assumption of (2.11) and (2.13). (Relevant examples are $\operatorname{sech}(x/a)$ for $a > 1$, and Gaussians $\exp(-b(x-c)^2)$ for $b > 0, c \in \mathbb{R}$.) From Cauchy's inequalities,

$$\left|\frac{d^n g(t)}{dt^n}\right| \le \frac{n!M}{\rho^n} \quad n \in \mathbb{N},$$

so that

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n}\left|\frac{d^{2n}g(t)}{dt^{2n}}\right| \le M\sum_{n=0}^{\infty}\Big(\frac{\pi}{2\rho}\Big)^{2n} < \infty.$$

Thus (6.3) is uniformly convergent and equality holds at (6.4). Indeed, (6.4) extends to the strip $|\Im z| \le \rho$, giving h as analytic and bounded on $|\Im z| \le \rho$.

On the other hand, supposing that h is analytic and bounded on $|\Im z| \le \rho$, for $t \in \mathbb{R}$,

$$g(t) = (\operatorname{sech} \star h)(t) = \int_{-\infty}^{\infty}\operatorname{sech}(s)h(t-s)\,ds,$$

so this also extends to $|\Im z| \le \rho$, giving

$$g(z) = \int_{-\infty}^{\infty}\operatorname{sech}(s)h(z-s)\,ds.$$

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It follows that the assumption of being analytic and bounded on $|\Im z| \le \rho$ for h gives the same condition on g, so the two are equivalent. They (both) imply that (6.4) holds with convergence at a geometric rate (depending on $\rho > \pi/2$).

Looking at the question more generally, we can rephrase the problem, so as to understand the convergence of (6.3), as

$$\sum_{n=0}^{k} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n}\Big(\operatorname{sech}^{(2n)} \star h\Big) \overset{?}{\to} h, \tag{6.6}$$

keeping in mind (5.2). We have shown convergence, both pointwise and in $L^1(\mathbb{R})$, in (6.6) for $g(x) = (\operatorname{sech} \star h)(x)$ lying in a dense subspace of $L^1(\mathbb{R})$, namely, the span of the Gaussians. Indeed, if g is a function for which convergence in $L^1(\mathbb{R})$ holds in (6.6), convolving both sides with k shows that we have the same for $g \star k$ for any $k \in L^1(R)$. In particular, we have convergence in $L^1(\mathbb{R})$ for any suitably mollified function.

7. Technical remarks

The above argument requires a severe restriction on g, or, equivalently, on h. As shown in Section 5, for each $\delta > 0$, on the set $\mathbb{R} \setminus (-\delta, \delta)$,

$$\sum_{n=0}^{k} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} \operatorname{sech}^{(2n)} \to 0 \tag{7.1}$$

uniformly and in L^1-norm. It is, however, divergent to $+\infty$ at $t = 0$. In particular, it cannot be $L^1(\mathbb{R})$-convergent. Otherwise, suppose that (7.1) did converge in $L^1(\mathbb{R})$. Then, since a subsequence converges almost everywhere [8, Theorem 13.11] to the L^1 limit, that limit is necessarily zero, by (5.2). Then (6.2) could be written as

$$\int_{-\infty}^{\infty} F_\varepsilon(x-t)\Big(\Big(\sum_{n=0}^{\infty} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} \operatorname{sech}^{(2n)}\Big) \star h\Big)(x)\,dx,$$

which is 0 since the inner $(\cdots) = 0$. But we know that this last expression converges to h, for suitable restricted $h \ne 0$, which is a contradiction.

In spite of (7.1) not converging in $L^1(\mathbb{R})$, there is the possibility that (6.6) converges in $L^1(\mathbb{R})$ for each $h \in L^1(\mathbb{R})$. If such was the case, then, by the Banach–Steinhaus theorem [8, Corollary 14.24], there is a constant K such that

$$\sup_k \Big\| \sum_{n=0}^{k} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} (\operatorname{sech}^{(2n)} \star h)\Big\|_1 \le K\|h\|_1.$$

It would then follow from this and the denseness result above that the infinite sum would be h itself for all $h \in L^1(\mathbb{R})$. Then letting h range over an approximate identity of norm one, it would follow that

$$\sup_k \Big\| \sum_{n=0}^{k} \frac{1}{(2n)!}\Big(\frac{i\pi}{2}\Big)^{2n} \operatorname{sech}^{(2n)}\Big\|_1 \le K \tag{7.2}$$

so that (e_k) would be a bounded approximate identity in $L^1(\mathbb{R})$.

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Conversely, if we knew that (7.2) was the case (by (5.2), even in $L^1[0, \delta]$ for some $\delta > 0$ would be enough), then, because (6.6) holds on a dense subset, we could conclude that it holds for all $h \in L^1(\mathbb{R})$.

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R. J. LOY, Mathematical Sciences Institute,
Australian National University, John Dedman Building 27,
Union Lane, Canberra, ACT 2601, Australia
e-mail: Rick.Loy@anu.edu.au

F. R. DE HOOG, CSIRO Data61, GPO Box 664,
Canberra, ACT 2601, Australia
e-mail: Frank.deHoog@csiro.au

R. S. ANDERSSEN, CSIRO Data61, GPO Box 664,
Canberra, ACT 2601, Australia
e-mail: Bob.Anderssen@csiro.au

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