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MOLCHANOV'S DISCRETE SPECTRA CRITERION FOR A WEIGHTED OPERATOR

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1. **Introduction.** We consider the second-order operator

$$\ell(y)=\frac{1}{w}\{-(py')'+qy\} \tag{1}$$

where the coefficients are real continuous functions on an interval $\mathcal{I}$ with w and p positive. The operator is assumed singular at only one endpoint which we take to be either 0 (finite singularity) or ∞ (infinite singularity). Let $\mathcal{L}_w^2(\mathcal{I})$ be the Hilbert space of all complex-valued, measurable functions f satisfying $\int_{\mathcal{I}} w\,|f|^2<\infty$. The operator ℓ determines a minimal closed symmetric operator L_0 in $\mathcal{L}_w^2(\mathcal{I})$ with domain dense in $\mathcal{L}_w^2(\mathcal{I})$. Properties of ℓ and L_0 are established in the texts [1, 4] for $w=1$; similar considerations apply for general w. Our concern here is with property **BD** which we define by: *if L is a self-adjoint extension of L_0, then its spectrum is bounded below and discrete.*

A. M. Molchanov gave in 1952 (cf. [2, p. 90]) a necessary and sufficient condition for property **BD** for the differential expression ($w=1$),

$$\ell(y)=(-1)^n y^{(2n)}+q(x)y, \qquad A\le x<\infty,$$

when the function $q(x)$ is bounded below. This criterion is that:

$$\operatorname*{limit}_{x\to\infty}\int_x^{x+\varepsilon} q(\xi)\,d\xi=\infty \quad \text{for all} \quad \varepsilon>0.$$

In this note we show (for $n=1$) how Molchanov's criterion must be modified to allow for the functions p and w. Additional discussion of Molchanov's theorem is given in [2, p. 199].

As in [2], we employ oscillation theory. For $[a,b]\subset\mathcal{I}$, define $\mathcal{A}(a,b)$ to be the set of all real functions y on $[a,b]$ such that y is absolutely continuous, $y'\in\mathcal{L}^2(a,b)$, and $y(a)=y(b)=0$. The operator ℓ is said to be *oscillatory on* $[a,b]$ provided there is a nontrivial solution y of $\ell(y)=0$ and numbers c and d, $a\le c<d\le b$, such that $y(c)=y(d)=0$. The operator ℓ is said to be *oscillatory at* ∞ (0) provided that for each $N>0$ ($\delta>0$) there is an interval $[a,b]\subset(N,\infty)$ ($[a,b]\subset(0,\delta)$) such that ℓ is oscillatory on $[a,b]$. Oscillation and property **BD** are connected by [2, Section 10]:

THEOREM A. *Property **BD** holds for $\mathcal{I}=[A,\infty)$ ($\mathcal{I}=(0,A]$) if and only if for all real numbers $\lambda>0$, $\ell(y)-\lambda y$ is nonoscillatory at ∞(0).*

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A classical result on oscillation is the following.

THEOREM B. *The operator ℓ is oscillatory on $[a, b]$ if and only if there is a* $y \in \mathcal{A}(a, b)$, $y \not\equiv 0$, *such that*

$$\int_a^b [p(y')^2 + qy^2]\, dx \le 0.$$

2. **The infinite singularity case.** We will make appropriate modifications of the proof given in [2].

LEMMA 1. *Suppose f is a positive, continuously differentiable function on* $\mathcal{I} = [A, \infty)$ *and* $|f'(x)| \le Mf(x)^{3/2}$ *on* $\mathcal{I}$. *If* $0 < \varepsilon < \frac{1}{2}$ *and* $|x - s| \le \varepsilon f(s)^{-1/2}/\theta M$ *where* $\theta = (3/2)^{3/2}$, *then*

$$\left|\frac{f(x)}{f(s)} - 1\right| < \varepsilon. \tag{2}$$

Proof. For s fixed, let $g(x) = f(x)/f(s)$. If (2) does not hold for $|x - s| \le \varepsilon f(s)^{-1/2}/\theta M$, then there is an x^*, $|x^* - s| \le \varepsilon f(s)^{-1/2}/\theta M$ such that $|g(x^*) - 1| = \varepsilon$ and $|g(x) - 1| < \varepsilon$ for all x between s and x^*. By the mean value theorem, for some $\tilde{x}$ between s and x^*,

$$\begin{aligned}\varepsilon &= |g(x^*) - 1| = |g'(\tilde{x})|\,|x^* - s| \\ &= \frac{|f'(\tilde{x})|}{f(s)}|x^* - s| \le \frac{Mf(\tilde{x})^{3/2}}{f(s)} \cdot \frac{\varepsilon}{\theta M f(s)^{1/2}} \\ &= \frac{\varepsilon}{\theta} g(\tilde{x})^{3/2} < \frac{\varepsilon(1+\varepsilon)^{3/2}}{\theta} < \varepsilon;\end{aligned}$$

this contradiction establishes the lemma.

Note that the hypotheses of the lemma imply that

$$f(x)^{-1/2} - f(A)^{-1/2} \le M(x - A)/2,$$

and consequently $\int_A^\infty f^{1/2} = \infty$.

THEOREM 1. *Suppose ℓ is given by* (1) *where p, q and w are real continuous functions on* $\mathcal{I} = [A, \infty)$ *with p and w positive and continuously differentiable. If*

(i) *q/w is bounded below on* $\mathcal{I}$,

(ii) *There is a constant M such that on* $\mathcal{I}$,

$$\left[\frac{p(x)}{w(x)}\right]^{1/2}\left[\frac{|w'(x)|}{w(x)} + \frac{|p'(x)|}{p(x)}\right] \le M,$$

then ℓ has property **BD** *if and only if for all sufficiently small positive ε,*

$$\lim_{x\to\infty} \left[\frac{w(x)}{p(x)}\right]^{1/2} \int_x^{x_\varepsilon} q(\xi)/w(\xi)\, d\xi = \infty, \tag{3}$$

where $x_\varepsilon = x + \varepsilon(p(x)/w(x))^{1/2}$.

Proof. Since property **BD** is invariant under multiplication of w by a positive constant and under addition of a multiple of w to q, we assume without loss of generality that the lower bound for q/w is one.

(a) Sufficiency. Let $\lambda > 0$ be given and choose ε such that

$$\varepsilon < 1/2\theta M,\ 2\varepsilon\lambda < 1 - 2\varepsilon(M+1),\ (\theta = (3/2)^{3/2}),$$

and choose N such that for $x \geq N$,

$$\left(\frac{w(x)}{p(x)}\right)^{1/2} \int_x^{x_\varepsilon} [q/w] \geq 1. \tag{4}$$

Suppose now $[a, b] \subset (N, \infty)$ and $y \in \mathcal{A}(a, b)$, $y \not\equiv 0$. Partition $[a, b]$ into subintervals $\Omega_1, \ldots, \Omega_K$ such that the length $\mu(\Omega_k)$ of Ω_k satisfies (extend $[a, b]$ if necessary to make (5) an equality for $k = K$)

$$\mu(\Omega_k) = \varepsilon[p(s_k)/w(s_k)]^{1/2} \tag{5}$$

where s_k is the left-hand endpoint of Ω_k. Define $x_k \in \Omega_k$ by

$$w(x_k)y(x_k)^2 = \int_{\Omega_k} w(q/w)y^2 \Big/ \int_{\Omega_k} (q/w). \tag{6}$$

From

$$(wy^2)(x) = (wy^2)(x_k) + \int_{x_k}^{x} 2(w^{1/2}y)(w^{1/2}y)'$$
$$= (wy^2)(x_k) + \int_{x_k}^{x} [2wyy' + w'y^2],$$

and

$$2wyy' = 2(w^{1/4}p^{1/4}y')(w^{3/4}p^{-1/4}y)$$
$$\leq w^{1/2}p^{1/2}(y')^2 + w^{3/2}p^{-1/2}y^2,$$

we have that

$$\int_{\Omega_k} wy^2 \leq [(wy^2)(x_k) + \int_{\Omega_k} \{w^{1/2}p^{1/2}(y')^2 + w^{3/2}p^{-1/2}y^2 + |w'|\, y^2\}]\mu(\Omega_k). \tag{7}$$

Application of (4), (5) and (6) to (7) yields that

$$\int_{\Omega_k} wy^2 \leq \varepsilon \int_{\Omega_k} qy^2 \tag{8}$$
$$+ \varepsilon[p(s_k)/w(s_k)]^{1/2} \int_{\Omega_k} \{w^{1/2}p^{1/2}(y')^2 + w^{3/2}p^{-1/2}y^2 + |w'|\, y^2\}.$$

For $f = (w/p)$, condition (ii) gives that

$$|f'f^{-3/2}| = |(w'/w - p'/p)(p/w)^{1/2}| \leq M;$$

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thus Lemma 1 gives for $x \in \Omega_k$,

$$\tfrac{1}{2} \le 1-\varepsilon' \le \frac{(w/p)(x)}{(w/p)(s_k)} \le 1+\varepsilon' \le 2 \qquad (\varepsilon' = \varepsilon\theta M) \tag{9}$$

Applying (9) and (ii) to (8), we have

$$\int_{\Omega_k} wy^2 \le \varepsilon \int_{\Omega_k} qy^2 + \varepsilon \int_{\Omega_k} \sqrt{2}\{[p(y')^2+wy^2]+Mwy^2\}.$$

$$\le 2\varepsilon \int_{\Omega_k} [p(y')^2+qy^2] + 2\varepsilon(M+1)\int_{\Omega_k} wy^2.$$

Summing this inequality over k yields for our choice of ε,

$$\lambda \int_a^b wy^2 \le \frac{2\varepsilon\lambda}{1-2\varepsilon(M+1)} \int_a^b [p(y')^2+qy^2] < \int_a^b [p(y')^2+qy^2].$$

By Theorem B, $\ell(y)-\lambda y$ is nonoscillatory at ∞ and by Theorem A, ℓ has property **BD**.

(b). Necessity. Let $0<\varepsilon<\frac{1}{2}M\theta$ and for $k=1,2,\ldots,$ define $\Omega_k = [s_k, s_k+\varepsilon(p(s_k)/w(s_k))^{1/2}]$ where $\{s_k\}$ is a sequence satisfying $s_k \to \infty$ as $k \to \infty$. Define y_k on Ω_k by $y_k = w^{-1/2}u_k$ where $(1/m = \varepsilon(p(s_k)/w(s_k))^{1/2}/4)$

$$u_k(x) = \begin{cases} m(x-s_k), & s_k \le x \le s_k+1/m, \\ 1, & s_k+1/m \le x \le s_k+3/m, \\ m(4/m+s_k-x), & s_k+3/m \le x \le s_k+4/m. \end{cases}$$

Assuming **BD** holds, we have by Theorems A and B that for each $\lambda>0$ there corresponds an N_λ such that if $[a,b] \subset (N_\lambda, \infty)$ and $y \in \mathcal{A}(a,b)$, $y \not\equiv 0$, then

$$\lambda \int_a^b wy^2 < \int_a^b [p(y')^2+qy^2]. \tag{10}$$

For $\Omega_k \subset (N_\lambda, \infty)$,

$$\int_{\Omega_k} wy_k^2 = \int_{\Omega_k} u_k^2 \ge (\varepsilon/2)(p(s_k)/w(s_k))^{1/2}, \tag{11}$$

$$\int_{\Omega_k} qy_k^2 = \int_{\Omega_k} (q/w)u_k^2 \le \int_{\Omega_k} (q/w), \tag{12}$$

and

$$\int_{\Omega_k} p(y_k')^2 = \int_{\Omega_k} p\left[\frac{-u_k w'}{2w^{3/2}} + \frac{u_k'}{w^{1/2}}\right]^2 \tag{13}$$

$$\le 2\int_{\Omega_k} p\left[\frac{(w')^2}{4w^3} + \frac{(u_k')^2}{w}\right]$$

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$$\leq 2\int_{\Omega_k} [(p/w)(w'/w)^2+(p/w)(w(s_k)/p(s_k)(16/\varepsilon^2)]$$

$$\leq 2\int_{\Omega_k} [M^2+32/\varepsilon^2]$$

$$=2[M^2+32/\varepsilon^2]\mu(\Omega_k)$$

where the last inequality is a consequence of (ii) and (9). Substitution of (11), (12), and (13) into (10) yields

$$(\lambda\varepsilon/2)<2(M^2+32/\varepsilon^2)\varepsilon+(w(s_k)/p(s_k))^{1/2}\int_{\Omega_k} q/w.$$

Since λ is arbitrary, we conclude that ($s_{k,\varepsilon}=s_k+\varepsilon(p(s_k)/w(s_k))^{1/2}$)

$$\left[\frac{w(s_k)}{p(s_k)}\right]^{1/2}\int_{\Omega_k} q/w=\left[\frac{w(s_k)}{p(s_k)}\right]^{1/2}\int_{s_k}^{s_{k,\varepsilon}} [q/w]\to\infty$$

as $k\to\infty$; thus (3) holds since the s_k are arbitrary.

As an example, consider

$$(14)\qquad \ell_2(x)=x^{-\alpha}[(-x^{\Delta}y')'+qy],\qquad 1\leq x<\infty.$$

Then (i)-(ii) of Theorem 1 are equivalent to $x^{-\alpha}q(x)$ bounded below and $\Delta-\alpha\leq 2$. Under these conditions, ℓ_2 has property **BD** if and only if for all $\varepsilon>0$ (sufficiently small)

$$\operatorname*{limit}_{x\to\infty} x^{(\alpha-\Delta)/2}\int_x^{x_\varepsilon} \xi^{-\alpha}q(\xi)\,d\xi=\infty.$$

where $x_\varepsilon=x+\varepsilon x^{(\Delta-\alpha)/2}$.

Molchanov's result gives that a certain average of q/w must tend to ∞ as a necessary and sufficient condition for property **BD**. Theorem 1 shows how the length of the averaging interval depends on p and w. Intervals of constant length occur in the special case p/w is a constant.

The conditions of Theorem 1 imply that $\int_A^\infty (w/p)^{1/2}=\infty$. In [3], criteria for property **BD** are given for $2n$th order operators which satisfy $\int_{\mathscr{I}} (w/p)^{1/2n}<\infty$. For the operator ℓ_2 in (14), one of these criteria yields property **BD** if $\Delta-\alpha>2$, $\Delta\neq 1$, and q satisfies as $x\to\infty$,

$$\left|\int_x^\infty q(\xi)\,d\xi\right|\leq x^{\Delta-1}\left[\frac{\Delta-1}{4}+\frac{o(1)}{\ln^2 x}\right],\qquad \Delta>1$$

$$\left|\int_x^\infty \xi^{2-2\Delta}q(\xi)\,d\xi\right|\leq x^{1-\Delta}\left[\frac{\Delta-1}{4}+\frac{o(1)}{\ln^2 x}\right],\qquad \Delta<1.$$

As noted in [3], the constant $(\Delta-1)/4$ is sharp by comparison with Euler equation. For equations of this type, property **BD** may hold even if $q/w\to-\infty$ as $x\to\infty$.

3. **The finite singularity case.** We consider here an operator k defined by $(\cdot = d/dt)$

$$k(Y) = -\frac{1}{W}(-(P\dot{Y})^{\cdot} + QY), \qquad 0 < t \leq A; \tag{15}$$

where, W, P, and Q are continuous real functions with W and P positive and continuously differentiable. If we set

$$y(x) = Y(t), \qquad x = 1/t,$$

then calculations show the equation $k(Y) = \lambda Y$ is equivalent to $\ell(y) = \lambda y$ where

$$\ell(y) = \frac{1}{w}(-(py')' + qy), \tag{16}$$

$p(x) = x^2 P(1/x)$, $w(x) = W(1/x)/x^2$, and $q(x) = Q(1/x)/x^2$. By Theorems A and B, k has property **BD** at 0 if and only if ℓ has property **BD** at ∞.

THEOREM 2. *Let k be as above and assume*
(i) *Q/W is bounded below on* $(0, A]$
(ii) *There is a constant M such that for* $0 < t \leq A$,

$$\left[\frac{P(t)}{W(t)}\right]^{1/2}\left[\frac{1}{t} + \frac{|\dot{W}(t)|}{W(t)} + \frac{|\dot{P}(t)|}{P(t)}\right] \leq M;$$

then k has property **BD** *at* 0 *if and only if for all sufficiently small* $\varepsilon > 0$,

$$\operatorname*{limit}_{t \to 0} \left[\frac{W(t)}{P(t)}\right]^{1/2} \int_{t_\varepsilon}^{t} Q(\tau)/W(\tau)\, d\tau = \infty \tag{17}$$

where $t_\varepsilon = t - \varepsilon (P(t)/W(t))^{1/2}$.

Proof. Again we assume $Q/W \geq 1$. Conditions (i) and (ii) of Theorem 2 imply that conditions (i) and (ii) of Theorem 1 hold for (16); moreover $p(x)/x^2 w(x) = 0(1)$ as $x \to \infty$. With this latter condition an inspection of the proof of Theorem 1 shows that it may be repeated with intervals Ω_k of length

$$\mu(\Omega_k) = \frac{x}{1 - \dfrac{\varepsilon}{x}\left[\dfrac{p(x)}{w(x)}\right]^{1/2}} - x = \frac{\varepsilon\left[\dfrac{p(x)}{w(x)}\right]^{1/2}}{1 - \dfrac{\varepsilon}{x}\left[\dfrac{p(x)}{w(x)}\right]^{1/2}}.$$

By choosing the intervals Ω_k in this fashion equation (3) is replaced by

$$\operatorname*{limit}_{x \to \infty} \left[\frac{w(x)}{p(x)}\right]^{1/2} \int_{x}^{x_\varepsilon} (q/w) = \infty \tag{18}$$

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where $x_\varepsilon = x(1-\varepsilon x^{-1}(p(x)/w(x))^{1/2})^{-1}$. However, the transformation above yields

$$\left[\frac{w(x)}{p(x)}\right]^{1/2}\int_x^{x_\varepsilon}(q/w)=t^2\left[\frac{W(t)}{P(t)}\right]\int_{t_\varepsilon}^{t}\left[\frac{Q(\tau)}{W(\tau)}\right]\frac{d\tau}{\tau^2}$$

where $t_\varepsilon = t-\varepsilon(P(t)/W(t))^{1/2}$. For $t_\varepsilon<\tau\le t$, by (ii),

$$1-\varepsilon M\le\frac{\tau}{t}\le 1;$$

hence (17) is equivalent to (18).

ADDED IN PROOF. For $w=1$ in (1), averaging criteria for property **BD** may also be found on pp. 107–112 of the recent text by E. Müller-Pfeiffer, *Spektraleigenschaften singulärer gewöhnlicher Differentialoperatoren.* Leipzig: Teubner, 1977.

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