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On ω-Strongly Measurable Cardinals

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Abstract
We prove several consistency results concerning the notion of ω-strongly measurable cardinal in HOD. In particular, we show that is it consistent, relative to a large cardinal hypothesis weaker than $o(\kappa) = \kappa$, that every successor of a regular cardinal is ω-strongly measurable in HOD.

Contents

1. Introduction

A prominent line of research in set theory is the study of the set theoretic universe V (i.e., model of the axioms of set theory, ZFC) by considering canonical inner model $M \subseteq V$ with additional strong features, which approximates V. The concept builds on the suggestion that if M is sufficiently 'close' to V, then some of the properties of M may lift to V and allow us to derive new consequences about models of set theory.

The prospects of this approach are demonstrated in the theory of Gödel's constructible universe $L \subseteq V$ and Jensen's Covering Theorem ([8]), which asserts that under the anti-large cardinal assumption

of the nonexistence of $0^{\#}$, the covering property holds for $L \subseteq V$.[1] The combination of covering for $L \subseteq V$, together with the rigid structure of L, has been shown to have many implications both on cardinal arithmetic in V,[2] as well as on the existence of incompactness phenomena in V such as an Abelian group $G \in V$ of size $\aleph_{\omega+1}$ that is not free, although every subgroup $H \lhd G$ of smaller cardinality is free.

Jensen's Covering Lemma describes one side of a sharp dichotomy: if $0^{\#}$ does not exist, then L covers subsets of V successfully. By Silver, if $0^{\#}$ does exist, then L fails to approximate even the most basic features of V. For example, in the presence of $0^{\#}$, L never computes successor cardinals correctly.

Jensen's Covering Lemma can be utilized to obtain a lower bound for the consistency strength of set theoretical statements, which do not necessarily mention large cardinals. However, it is quite restrictive. To be able to obtain various lower bounds, one has to construct canonical models that can accommodate stronger large cardinal axioms. The construction of such inner models is the subject of a prominent program in set theory known as the Inner Model program. For a large cardinal property Φ,[3] one would like to to construct a canonical L-like inner model K that is maximal with respect to inner models that do not satisfy the large cardinal property Φ (see Schimmerling-Steel [32] for the precise statement). This maximality property couples with a covering lemma: assuming there is no inner model of V with the property Φ, K approximates the universe V by satisfying a certain covering property. For example, for Φ being the existence of $0^{\#}$, $K = L$.

For stronger Φ, the existence of such an inner model K would allow us to extend Jensen's sharp dichotomy. Namely, either the large cardinal property Φ holds, or V is close to K and therefore inherits various combinatorial properties such as the existence of certain incompactness phenomena.

Starting in the 1970s, inner models for increasing large cardinal properties Φ have been constructed. Starting from the seminal studies of Kunen, Silver and Solovay on a model $L[U]$ with a measurable cardinal ([18],[35]), extended by Dodd-Jensen ([10]) and Mitchell ([23]), to large cardinal properties Φ involving coherent sequences of normal measures and many measurable cardinals. Then, following major developments and the introduction of iteration trees in Martin and Steel ([20]), Mitchell and Steel ([24]), and Steel ([36]), the theory was extended to the level of Woodin cardinals.

The program took a significant turn after Woodin showed that there cannot be a single maximal inner model, in an absolute sense, past a Woodin cardinal. This has sparked new lines of study, involving forms of the K^c construction (see Jensen, Schimmerling, Schindler, and Steel [17] and Andereta, Neeman and Steel [2]).

Another seminal development was the introduction of The Core Model Induction method, first introduced by Woodin and extensively developed by Steel, Schindler, Sargsyan, Trang and many others ([34]). The method establishes new consistency results for stronger large cardinal properties by incorporation ideas from descriptive set theory with various local construction methods. The relevant large cardinal properties are often described in terms of expansions of the Axiom of Determinacy (AD) in inner models M of ZF and can be further translated to inner models of ZFC with large cardinal properties.

First results on fine structural inner models for finite levels of supercompact cardinals were obtained by Neeman and Steel, [27] and by Woodin. It is still unknown whether similar constructions could lead to an inner model with a (full) supercompact cardinal, and some recent results of Woodin suggest that major obstructions appear past the level of finite supercompactness [41]. There are many excellent resources for the introduction of the inner model theory, the inner model program and its development. We refer the reader to [16, 22, 26, 29, 33, 37, 43].

The inner model of Hereditarily Ordinal Definable sets (HOD) plays a significant role in many of the recent advancements in the Inner Model program.

Definition 1.1. Let M be a model of set theory. A set $x \in M$ is hereditarily ordinal definable in M if both x and every set in the transitive closure of x is definable in V using some formula with ordinal parameters. The class of all hereditarily ordinal definable sets in a model M is denoted by HOD^M.

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[1] i.e., every set of ordinals $x \in V$ is contained in a set $y \in L$ such that $|y| \le |x| + \aleph_1^V$.

[2] e.g., it implies that the Singular Cardinal Hypothesis (SCH) holds in V

[3] e.g., the existence of a cardinal κ with a large cardinal property such as a measurable cardinal, a strong cardinal, a Woodin cardinal or a supercompact cardinal.

We write HOD for $\text{HOD}^V \subseteq V$.

The class HOD was first introduced by Gödel, and has been extensively studied (for example, see [25]). The study of HOD in inner models of strong forms of AD and the associated strategic-extender models plays a critical role in Descriptive Inner Model Theory. See [30, 31, 38].

In [39], Woodin presents a new approach of addressing the inner model problem for all large cardinals. Woodin analyses the possible properties and limitations of some of the current methods, and introduces the seminal notions of a suitable extender model N for a supercompact cardinal δ (in V), which in addition to several properties similar to well-known inner models, requires that N captures witnessing δ-supercompact measures in V.[4] In a following work (see [41]), Woodin presents the 'V = Ultimate-L' axiom, to assert (roughly) that Σ_2-definable properties of the universe are satisfied in canonical strategic-extender models of the form $\text{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$, for some Universally Baire set $A \subseteq \mathbb{R}$.

Combining the above notions, Woodin has formulated the 'Ultimate-L' conjecture, asserting that there exists a suitable extender model $N \subseteq \text{HOD}$ which satisfies the axiom 'V = Ultimate-L'. Following the search for some N, the theory established in [39] studies the possibility of $\text{HOD} \subseteq V$ being a suitable extender model, and possible implications. For this, Woodin introduces a new assumption known as the *HOD-conjecture* (Conjecture 1.3 below) and shows that, remarkably, if the HOD-conjecture is true, then a sufficiently strong large cardinal assumption (e.g., an extendible cardinal) guarantees a version of the covering lemma for $\text{HOD} \subseteq V$. However, if the HOD-conjecture fails in the presence of sufficiently large cardinals, then HOD is very far from V, just like the smaller inner models. See Theorem 1.5 for an exact formulation.

We remark that, in general, the inner model HOD of an arbitrary model of ZFC can be easily modified by forcing. Nevertheless, it contains every canonical inner model and thus, the HOD-conjecture might be a consequence of the covering theorem for some extremely large canonical inner model [40]. An appealing aspect of the HOD-conjecture is that it is a combinatorial statement about the HOD and V, and it does not rely on inner model theory. Even without any further development in the inner model program, Woodin established that the HOD-conjecture poses many significant limitations on the consistency of large cardinals in the choice-less context. Moreover, large cardinals beyond choice, if consistent, form a hierarchy of failures of the HOD-conjecture [3].

The HOD-conjecture centers around the notion of ω-strongly measurable cardinals in HOD. Not much was know about this notion, and previously, Woodin has raised the question ([41]) of whether more than three ω-strongly measurables in HOD can exist. In this work, we study the notion of ω-strongly measurable cardinals in HOD, we prove several consistency results concerning this notion and we establish the consistency of a model where all successors of regular cardinal are ω-strongly measurable in HOD.

Definition 1.2. Let κ be an uncountable regular cardinal, and let S be a stationary subset of κ. We say that κ is strongly measurable in HOD with respect to S if there exists some $\eta < \kappa$ such that $(2^\eta)^{\text{HOD}} < \kappa$ and there is no partition $\langle S_\alpha \mid \alpha < \eta \rangle \in \text{HOD}$ of S into sets, all stationary sets in V. We say that κ is ω-strongly measurable in HOD if it is strongly measurable in HOD with respect to the set $S = \kappa \cap \text{Cof}(\omega)$, and that it is strongly measurable in HOD if it is strongly measurable in HOD with respect to $S = \kappa$.

In general, one might replace HOD with any other inner model of V, M, obtaining a meaningful notion of strong measurability in M. Since, in this paper, we will be interested solely in strong measurability in HOD, we will occasionally omit the emphasis 'in HOD' and say simply that κ is strongly measurable.

It is shown in [39] that if κ is an ω-strongly measurable in HOD, then there are stationary sets $S \subseteq \kappa \cap \text{Cof}(\omega)$ for which the restriction of the filter $\text{CUB}_\kappa \restriction S$ to HOD forms a measure on κ in HOD. However, Woodin shows ([39]) that the existence of a class of regular cardinals which are not ω-strongly measurable in HOD, together with the existence of a HOD-supercompact, implies that HOD satisfies many appealing approximation properties with respect to V. The results promote Woodin's HOD-conjecture.

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[4]Namely, for all $\lambda > \delta$, there is a supercompact measure U for $P_\delta(\lambda)$ such that $N \cap P_\delta(\lambda) \in U$ and $U \cap N \in N$.

Conjecture 1.3 (HOD conjecture, [39, Definition 191]). *There is a proper class of regular uncountable cardinals κ which are not ω-strongly measurable in* HOD.

In light of the HOD-conjecture, it is natural to attempt forming models with as many as possible ω-strongly measurable cardinals in HOD. Woodin has established the consistency (relative to large cardinals) of models with up to three ω-strongly measurable cardinals (see [41, Remark 3.43]). The main purpose of this work is to prove that many strongly measurable cardinals can be obtained from a relatively mild large cardinal assumption of hypermeasurability.

Theorem 1.4. *It is consistent relative to the existence of an inaccessible cardinal θ for which $\{o(\kappa) \mid \kappa < \theta\}$ is unbounded in θ, that every successor of a regular cardinal is strongly measurable in* HOD.

Cummings, Friedman and Golshani ([7]) have established the consistency of a model where $(\alpha^+)^{\mathrm{HOD}} < \alpha^+$ for every infinite cardinal α. In [15, Theorem 2.2], Gitik and Merimovich prove that it is consistent relative to large cardinals that every regular uncountable cardinal is measurable in HOD. A similar result is obtained using a different technique in [5, Theorem 1.4]. Perhaps more related to our work is [5, Theorem 1.3], in which a club of cardinals which are measurable in HOD is obtained from a large cardinal axiom weaker than $o(\kappa) = \kappa$. In those models, there are no ω-strongly measurable successor cardinals.

We note that these results do not apply to models where there is an extendible cardinal. The existence of an extendible cardinal in V derives a sharp dichotomy between HOD being either very close or very far from V, as shown by Woodin's HOD-Dichotomy Theorem ([39]).

Theorem 1.5 (The HOD-Dichotomy, Woodin, [42]). *Let δ be an extendible cardinal. Then one of the following holds:*

1. *Every cardinal η above δ, which is singular in V, is singular in* HOD *and $(\eta^+)^{\mathrm{HOD}} = \eta^+$.*
2. *Every regular cardinal above δ is ω-strongly measurable in* HOD.

In the last part of this work, we prove a consistency result regarding strong measurability at successors of singular cardinals. Woodin ([39]) establishes the consistency of a successor of a singular cardinal λ, which is a ω-strongly measurable cardinal in HOD, from the large cardinal assumption I_0. Here, we prove a weaker consistency result from a weaker large cardinal assumption.

Theorem 1.6. *Suppose that $\kappa < \lambda$ are cardinals, κ is λ-supercompact and λ is measurable. Then, there is a generic extension in which κ is a singular cardinal of cofinality ω, and $\lambda = \kappa^+$ is strongly measurable in* HOD *with respect to S, for some stationary subset $S \subseteq \lambda \cap \mathrm{Cof}(\omega)$.*

A brief summary of this paper. In **section 2**, we review some basic facts about strong measurability which will be central in the proof of the main theorem. In the following sections, we gradually develop the forcing methods used to prove our main results (Theorems 1.4 and 1.6). In **section 3**, we show how to obtain a model where ω_1 is strongly measurable in HOD starting with a single measurable cardinal. The case of $\kappa = \omega_1$ is different from the general case as it does not require incorporating posets for changing cofinalities. It can also be seen as a warmup for the general case. In **section 4**, we further develop the ideas from the previous section and combine them with a suitable iteration for changing cofinalities. As a result, we establish the consistency of a strongly measurable cardinal which is a successor of an arbitrary regular cardinal λ, from the large cardinal assumption of $o(\kappa) = \lambda + 1$. In **section 6**, we introduce a method to construct a Prikry-type poset which is equivalent to the forcing from the previous section and has a direct extension order that is λ-closed. This is utilized in **section 5** to form iterations of the single cardinal forcing, thus obtaining models with many strongly measurable cardinals. In **section 7**, we prove our theorem concerning successors of singular cardinals. The results of this section do not depend on the other sections past our preliminaries.

In the appendix, we cite and prove some useful results related to homogeneous forcings and their iterations (including Prikry type forcings), and homogeneous iterations for changing cofinalities.

Our notations are mostly standard. We follow the Jerusalem forcing convention, in which for two conditions p, p' in a poset $\mathbb{P}$, the fact that p' is stronger (more informative) than p is denoted by $p' \geq p$.

2. Variations of strong measurability

We start with several observations concerning a natural generalization of the notion of ω-strong measurability.

Definition 2.1. Let $S \subseteq \kappa$, $S \in$ HOD stationary, and let η be a cardinal in HOD. κ is $(S, < \eta)$-strongly measurable if there is no partition in HOD of S into η many disjoint stationary sets. κ is (S, η)-strongly measurable if it is $(S, < (\eta^+)^{\mathrm{HOD}})$-strongly measurable.

Definition 2.2. A cardinal κ is S-strongly measurable if $S \in$ HOD, and κ is $(S, < \eta)$-strongly measurable for some η such that $(2^\eta)^{\mathrm{HOD}} < \kappa$. We say that κ is strongly measurable if κ is κ-strongly measurable.

Therefore, a cardinal κ is ω-strongly measurable if it is (S_ω^κ, η)-strongly measurable for η such that $(2^\eta)^{\mathrm{HOD}} < \kappa$. Note that if $S \subseteq T$ are stationary subsets of κ in HOD and κ is T-strongly measurable, then it is S-strongly measurable. In particular, every strongly measurable cardinal is ω-strongly measurable.

Theorem 2.3 (Woodin). *Let δ be an extendible cardinal. Then the following are equivalent:*

1. *There is a regular cardinal $\kappa \geq \delta$ which is not ω-strongly measurable.*
2. *There is a regular cardinal $\kappa \geq \delta$ which is not (S, δ)-strongly measurable for some $S \in$ HOD which consists of singular ordinals of cofinality $< \delta$.*
3. *The* HOD-*conjecture.*
4. *There is no regular ω-strongly measurable cardinal above δ.*

For the proof, see [39, Theorems 197, 212, 213]. Without the assumption that the ordinals of S have fixed V-cofinality, the equivalence might fail.

The next result provides a necessary and sufficient condition for a cardinal κ to be ω-strongly measurable in HOD. This observation will guide us in devising the main forcing construction, which will be used to prove theorem 1.4.

Lemma 2.4. *A cardinal κ is strongly measurable with respect to $S \in$ HOD if and only if κ is an inaccessible cardinal in* HOD*, and the restriction of the club filter on S to* HOD *is the intersection of η normal measures from* HOD*, $\langle U_{\kappa,i} \mid i < \eta \rangle \in$ HOD*, for some $\eta < \kappa$.*

Proof. For the backwards implication, since κ is inaccessible in HOD and $\eta < \kappa$, $(2^\eta)^{\mathrm{HOD}} < \kappa$.

Let $\langle T_\alpha \mid \alpha < (\eta^+)^{\mathrm{HOD}} \rangle \in$ HOD be a decomposition of S into stationary sets. By the assumption, for each α there is a measure $U_{\kappa,i}$ in HOD such that $T_\alpha \in U_{\kappa,i}$. Since the sets T_α are pairwise disjoint, it is impossible for $\alpha \neq \beta$ to belong to the same $U_{\kappa,i}$. Thus, we obtain an injective function from $(\eta^+)^{\mathrm{HOD}}$ to η in HOD – a contradiction.

Let us assume now that κ is strongly measurable with respect to $S \in$ HOD. In particular, κ is inaccessible in HOD. Let $\mathcal{S} \in$ HOD be a maximal collection of pairwise disjoint stationary subsets of S, in HOD, such that for all $T \in \mathcal{S}$, the club filter restricted to T is an ultrafilter in HOD. Let us denote this ultrafilter by U_T. Since this collection is a partition of S into stationary sets, $|\mathcal{S}| < \kappa$.

If $\bigcap \{U_T \mid T \in \mathcal{S}\}$ is not the club filter restricted to S in HOD, then it contains a set $S \setminus S'$, where $S' \subseteq S$ stationary, $S' \in$ HOD. In particular, $S' \notin U_T$ for all $T \in \mathcal{S}$, so $S' \cap T$ is nonstationary for all $T \in \mathcal{S}$, and $S'' = S' \setminus \bigcup_{T \in \mathcal{S}} (S' \cap T)$ is a stationary subset of S, disjoint from all members of $\mathcal{S}$. Since κ is strongly measurable with respect to S, it is also strongly measurable with respect to S''. Thus, there is some $T' \subseteq S''$ stationary such that the club filter restricted to T' is an ultrafilter. But this contradicts the maximality of $\mathcal{S}$. $\qquad \square$

Corollary 2.5. *Let κ be $(S, <\kappa)$-strongly measurable. Then S is contained in the regular cardinals of* HOD*, up to a nonstationary error.*

3. ω_1 is strongly measurable from one measurable cardinal

In this section, we would like to present a forcing that forces ω_1 to be strongly measurable. By Lemma 2.4, this means that in HOD, the club filter of ω_1 is an intersection of countably many normal measures.

In the case of ω_1, we can take a single measure. So, we would like to collapse a measurable cardinal κ with a normal measure to be ω_1 and then using a Mathias-type forcing, add a club that diagonalizes the normal measure. In order to show that this works, we need to show two things. First, we must show that the iteration is cone homogeneous. This is done in Lemma 3.3. Second, we need to show that it does not collapse ω_1. This amount to showing that the second step of the iteration is σ-distributive, which in turn requires us to be able to add a U-generic point to the generic club. See Lemma 3.1 for the precise formulation.

Let us present the forcing. Suppose that κ is a measurable cardinal in a model V, and U is a normal measure on κ. Force with Levy collapse poset $\mathrm{Coll}(\omega, <\kappa)$ over V. Let H be a V-generic filter.

Working in the generic extension $V[H]$, let $\mathbb{C}_U$ be the poset consisting of pairs $x = \langle c, A \rangle$, where $c \subseteq \kappa$ is a bounded closed subset of κ and $A \in U$. The condition $x' = \langle c', A' \rangle$ extends x if c' is an end extension of c, $A' \subseteq A$, and $c' \setminus c \subseteq A$.

It is clear that if $x = \langle c, A \rangle$ and $x' = \langle c', A' \rangle$ are two conditions with the same bounded closed set $c = c'$, then x, x' are compatible. Since $\kappa^{<\kappa} = \kappa$ in $V[H]$, then $\mathbb{C}_U$ satisfies κ^+-c.c. (which is $\aleph_2^{V[H]}$-c.c.). The forcing $\mathbb{C}_U$ adds a diagonalizing club to U. It has also been studied in [28] in the context of well-behaved posets which can introduce square sequences and was found useful in other contexts.

The following lemma is the key ingredient in the proof of the distributivity of $\mathbb{C}_U$.

Lemma 3.1. *Work in $V[H]$ and fix some regular cardinal $\theta > \kappa^+$. There exists a stationary set of structures $M \prec H_\theta$ of size $|M| < \kappa$, with the property that $\sup(M \cap \kappa) \in A$ for every $A \in U \cap M$.*

Proof. Fix any $f\colon H_\theta^{<\omega} \to H_\theta$ in $V[H]$. We would like to show that there exists some $M \subseteq H_\theta$ which is closed under f and satisfies the conditions in the statement of the lemma.

Fix in V a name $\underset{\sim}{f}$ for f and let $f'\colon \mathrm{Coll}(\omega, <\kappa) \times (H_\theta^V)^{<\omega} \to H_\theta^V$ be a function that sends $(p, x) \in \mathrm{Coll}(\omega, <\kappa) \times (H_\theta)^{<\omega}$ to y if $p \Vdash \underset{\sim}{f}(x) = \check{y}$. Note that x is a finite sequence of names.

By the definition of f', if $M' \prec H_\theta^V$ is closed under f', $M' \cap \kappa \in \kappa$, and $\kappa, U \in M'$, then $M' = M \cap H_\theta^V$ for some $M \subseteq H_\theta$ which is closed under f. Indeed, we may take $M = M'[H \cap M]$. By the chain condition of $\mathrm{Coll}(\omega, <\kappa)$, every name for a ground model object that belongs to M' can be refined to a nice name which is contained in M'.

Since $U \subseteq H_\theta^V$, it is therefore sufficient to prove that there exists some $M' \subseteq H_\theta^V$ which is closed under f' and satisfies $|M'| < \kappa$ and $\sup(M' \cap \kappa) \in A$ for all $A \in M' \cap U$.

Working in V, take an elementary substructure $N \prec H_\theta^V$ satisfying $f'[N] \subseteq N$, $N^{<\kappa} \subseteq N$, $|N| = \kappa$, $\kappa \in N$. Let $j\colon V \to W \cong \mathrm{Ult}(V, U)$ be the ultrapower embedding induced by U. Consider the structure $\tilde{M}' = j''N \prec j(H_\theta^V)$. $\tilde{M}' \in W$ is closed under $j(f')$, and $\tilde{M}' \cap j(\kappa) \in j(\kappa)$. It follows that $0_{j(\mathrm{Coll}(\omega, <\kappa))}$ forces $\tilde{M}'$ to be closed under $j(\underset{\sim}{f})$. Finally, for every $A \in \tilde{M}' \cap j(U)$, $A = j(\bar{A})$ for some $\bar{A} \in U$ and therefore, $\kappa \in A$.

So $\tilde{M}$ satisfies the conclusion of the lemma in W, and it is closed under $j(f')$. By the elementarity of j, there is $M' \in V$ satisfying the conclusion of the lemma and closed under f'. Thus, $M'[H \cap M'] = M$ satisfies the requirements of the lemma. $\square$

Proposition 3.2. $\mathbb{C}_U$ *is κ-distributive.*

Proof. Since $\kappa = \aleph_1$ in $V[H]$, we need to check that the intersection of a countable family $\{D_n \mid n < \omega\}$ of dense open subsets of $\mathbb{C}_U$ is dense. Pick some regular cardinal $\theta > \kappa^+$ such that $\mathbb{C}_U, \{D_n \mid n < \omega\} \in H_\theta$. By lemma 3.1, for every condition $x \in \mathbb{C}_U$, there exists an elementary substructure $M \prec H_\theta$ of size $|M| = \aleph_0$, with $x, \mathbb{P}, \mathbb{C}_U, \{D_n \mid n < \omega\} \in M$ and further satisfies that $\sup(M \cap \kappa) \in A$ for every $A \in M \cap \kappa$. We may also assume that $M = M'[H \cap M']$ for $M' \in V$, $M' \prec H_\theta^V$.

Denote $\sup(M \cap \kappa)$ by α and pick a cofinal sequence $\langle \alpha_n \mid n < \omega \rangle$ in α. We can construct an increasing sequence of extensions $\langle x_n \mid n < \omega \rangle \subseteq M$ of x, $x_n = \langle c_n, A_n \rangle$ such that $x_{n+1} \in D_n$ and $\max(c_n) \geq \alpha_n$ for every $n < \omega$. Without loss of generality, we may assume that for every $A \in U \cap M$, there is $n < \omega$ such that $A_n \subseteq A$.

Since $x_n = \langle c_n, A_n \rangle \in M$, then $\alpha \in A_n$ for all $n < \omega$. It follows that $x^* = \langle \{\alpha\} \cup (\bigcup_n c_n), \bigcap_n A_n \rangle$ is a condition in $\mathbb{C}_U$, which is clearly an upper bound of $\langle x_n \mid n < \omega \rangle$.[5] We conclude that there exists x^* extending our given condition x such that $x^* \in \bigcap_n D_n$. □

Lemma 3.3. $\mathbb{C}_U$ *is cone homogeneous.*

Proof. Let $x_1 = \langle c_1, A_1 \rangle$, $x_2 = \langle c_2, A_2 \rangle$ be two conditions of $\mathbb{C}_U$. Take $\nu \in A_1 \cap A_2$ above $\max(c_1), \max(c_2)$ and consider the extensions $y_1 = \langle c_1 \cup \{\nu\}, (A_1 \cap A_2) \setminus (\nu + 1) \rangle$, $y_2 = \langle c_2 \cup \{\nu\}, (A_1 \cap A_2) \setminus (\nu + 1) \rangle$ of x_1 Define a cone isomorphism $\sigma \colon \mathbb{C}_U/y_1 \to \mathbb{C}_U/y_2$ by

$$\sigma(\langle c, A \rangle) = \langle c_2 \cup (c \setminus \nu), A \rangle$$

σ is clearly an order preserving map onto $\mathbb{C}_U/y_1$ and has an order preserving inverse which is given by

$$\sigma^{-1}(\langle c, A \rangle) = \langle c_1 \cup (c \setminus \nu), A \rangle$$

□

□

Theorem 3.4. *Suppose $C \subseteq \mathbb{C}_U$ is a $V[H]$-generic filter. Then, in $V[H * C]$, $\kappa = \aleph_1^{V[H*C]}$ is strongly measurable.*

Proof. By Lemma 8.3, $\mathrm{Coll}(\omega, < \kappa) * \mathbb{C}_U$ is cone homogeneous, and therefore, $\mathrm{HOD}^{V[H*C]} \subseteq V$. It is clear from the definition of $\mathbb{C}_U$ that for every subset $S \subseteq \kappa$ in V, S is stationary in $V[H * C]$ if and only if $S \in U$. It follows that the closed unbounded filter on $\kappa = \aleph_1^{V[H*C]}$ in $V[H * C]$ is a HOD-ultrafilter. Therefore $V[H * C] \models \kappa$ is strongly measurable. □

4. Strongly measurable successor of a regular cardinal

In this section, we would like to force a successor of an uncountable regular cardinal, $\kappa = \lambda^+$, to be strongly measurable. There are a few difficulties that arise. First, there is a definable splitting of the ordinals below κ, according to the cofinalities, so the club filter cannot be an ultrafilter but rather an intersection of a few normal measures. This means that we should fix a collection of normal measures so that their intersection is indented to become the club filter. Moreover, when killing a stationary set which is small with respect to the designated filter, we are forcing a club through the previous regulars, which are now going to change cofinalities to various possibilities. This means that a Levy collapse by itself would not provide all the cofinality changes that we need, and we must use a more complicated method of changing cofinalities in a homogeneous way.

Suppose that $\lambda < \kappa$ are two cardinals such that λ is regular and κ is measurable with $o(\kappa) = \lambda + 1$. Let $\mathcal{U} = \langle U_{\alpha, \tau} \mid \lambda < \alpha \leq \kappa, \tau < o^{\mathcal{U}}(\alpha) \rangle$ be a coherent sequence of normal measures with $o^{\mathcal{U}}(\kappa) = \lambda + 1$.

Let $\mathbb{P}_\kappa^{\mathcal{U}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ be the homogeneous iteration of subsection 8.2. In the next two sections, $\mathbb{P}$ stands for $\mathbb{P}_\kappa^{\mathcal{U}}$. For the main properties of $\mathbb{P}$, we refer the reader to Fact 8.6. We will explicitly need the following additional property of the iteration.

Remark 4.1. We note that it follows at once from the definition of $\mathbb{Q}_\alpha$ that every V-set $A \in \bigcap_{i < o^{\mathcal{U}}(\alpha)} U_{\alpha, i}$ contains a tail of the cofinal sequence b_α. This is because every condition $q = \langle t, T \rangle \in \mathbb{Q}_\alpha$ has a direct extension $q^A = \langle t, T^A \rangle$ of q, which satisfies that $\mathrm{succ}_T(s) \subseteq A$ for all $s \in T$.

Definition 4.2. Let $G \subseteq \mathbb{P}$ be a V-generic filter, and let $H \subseteq \mathrm{Coll}(\lambda, < \kappa)$ be the Levy collapse generic over $V[G]$. Working in $V[G * H]$, we consider the filter $\mathcal{F}_\kappa$ generated by $\bigcap_{i \leq \lambda} U_{\kappa, i}$,

$$\mathcal{F}_\kappa = \{A \subseteq \kappa \mid \exists B \in \bigcap_{i \leq \lambda} U_{\kappa, i}, B \subseteq A\}.$$

The filter $\mathcal{F}_\kappa$ is going to generate the club filter in HOD in the generic extension.

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[5]Note that $\bigcap A_n = \bigcap_{A \in M' \cap U} A$.

Lemma 4.3. $\mathcal{F}_\kappa$ *is a κ-complete filter in $V[G * H]$.*

Proof. Suppose that $\langle A_\nu \mid \nu < \beta \rangle \in V[G * H]$ is a sequence of $\beta < \kappa$ many sets of $\mathcal{F}_\kappa$. We would like to show that $\bigcap_{\nu < \beta} A_\nu$ belongs to $\mathcal{F}_\kappa$. We may assume that $A_\nu \in \bigcap_{i \leq \lambda} U_{\kappa i}$ for all $\nu < \beta$.

In order to prove the claim, we move from $V[G * H]$ to $V[G]$, and then to V. Working in $V[G]$, let A_ν be a Coll$(\lambda, < \kappa)$-name for the V-set in $\bigcap_{i \leq \lambda} U_{\kappa,i}$. Since Coll$(\lambda, < \kappa)$ satisfies κ-c.c., there exists a family of V-sets $X_\nu \subseteq \bigcap_{i \leq \lambda} U_{\kappa,i}$, $X_\nu \in V[G]$, of size $< \kappa$ such that $\Vdash A_\nu \in \check{X}_\nu$. Fix in V a $\mathbb{P}$-name $\underset{\sim}{X}_\nu$ for each X_ν.

Let $p \in G$ be a condition forcing the above. Moving back to V, the fusion lemma for nonstationary support iteration of Prikry type forcings [5, Lemma 3.6] guarantees that there exists some $q \in G$ and a sequence of sets $\langle Y_\nu \mid \nu < \beta \rangle$ in V, so that for each $\nu < \beta$, $Y_\nu \subseteq \bigcap_{i \leq \lambda} U_{\kappa,i}$ has size $|Y_\nu| < \kappa$, and $q \Vdash \underset{\sim}{X}_\nu \subseteq \check{Y}_\nu$. For each ν, let $A'_\nu = \bigcap Y_\nu$, and $A' = \bigcap_{\nu < \beta} A'_\nu$. Since $U_{\kappa,i}$ is κ-complete for all $i \leq \lambda$, we have in $V[G * H]$ that $A' \in \mathcal{F}_\kappa$ and $A' \subseteq \bigcap_{\nu < \beta} A_\nu$.   □

To produce a model where κ is ω-strongly measurable, we will force over $V[G * H]$ to add a closed unbounded set $C \subseteq \kappa$ which is almost contained in every set $A \in \mathcal{F}_\kappa$.

Definition 4.4. Working in a V-generic extension $V[G * H]$ by $G * H \subseteq \mathbb{P} * \text{Coll}(\lambda, < \kappa)$, we define the forcing $\mathbb{C}_{\mathcal{F}_\kappa}$. Conditions $x \in \mathbb{C}_{\mathcal{F}_\kappa}$ are pairs $x = \langle c, A \rangle$ where c is a closed and bounded subset of κ and $A \in \mathcal{F}_\kappa$. A condition $x' = \langle c', A' \rangle \in \mathbb{C}_{\mathcal{F}_\kappa}$ extends x (denoted $x' \geq x$) if

(i) $c' \cap \max(c) = c$,
(ii) $A' \subseteq A$, and
(iii) $c' \setminus c \subseteq A$.

For conditions $x = \langle c, A \rangle \in \mathbb{C}_{\mathcal{F}_\kappa}$, we will frequently denote c and A by c^x and A^x, respectively. It is clear that if $R \subseteq \mathbb{C}_{\mathcal{F}_\kappa}$ is generic, then the union $C = \bigcup \{c^x \mid x \in R\}$ is a closed and unbounded subset of κ which is almost contained in every $A \in \mathcal{F}_\kappa$. Since $\mathcal{F}_\kappa$ is a filter and $\kappa^{<\kappa} = \kappa$, the forcing $\mathbb{C}_{\mathcal{F}_\kappa}$ is κ-centered and therefore satisfies κ^+-chain condition.

The following lemma is a parallel of Lemma 3.1. From this lemma we will infer the distributivity of the forcing $\mathbb{C}_{\mathcal{F}_\kappa}$.

Lemma 4.5. *Working in $V[G * H]$, for any regular cardinal $\theta > \kappa^+$ and $\tau \leq \lambda$, there exists a stationary set of structures $M \prec H_\theta$ with $\sup(M \cap \kappa) = \alpha$ which satisfy*

(i) $M^{<\tau} \subseteq M$;
(ii) $o^{\mathcal{U}}(\alpha) = \tau$;
(iii) *For every $A \in \mathcal{F}_\kappa \cap M$, $\alpha \in A$ and, moreover, $b_\alpha \subseteq^* A$ (namely $b_\alpha \setminus A$ is bounded in α).*

Proof. Fix a function $f : [H_\theta]^{<\omega} \to H_\theta$ in $V[G * H]$. Back in the ground model V, let $\underset{\sim}{f}$ be a $\mathbb{P} * \text{Coll}(\lambda, < \kappa)$-name for f. Since Coll$(\lambda, < \kappa)$ is κ-c.c., there exists a $\mathbb{P}$-name function $\underset{\sim}{F} : [H_\theta^V]^{<\omega} \to [H_\theta^V]^{<\kappa}$ such that $\underset{\sim}{f}(x)$ is forced to be a member of $\underset{\sim}{F}(x)$ for every $x \in H_\theta^V$.

Let us consider our ability to approximate $\underset{\sim}{F}$ in V. Let $N \prec H_\theta^V$ be an elementary substructure of size κ with $N^{<\kappa} \subseteq N$ and $\kappa, \mathbb{P}, \underset{\sim}{F} \in N$.

Claim 4.6. *Let N be as above and $p \in \mathbb{P} \cap N$. Then, there is $p^* \leq p$ which is N-generic, namely, for every name for an ordinal $\underset{\sim}{\sigma} \in N$, there is set of ordinals $S \in N$ such that $S \subseteq N$ and $p^* \Vdash \underset{\sim}{\sigma} \in S$.*

Proof. By a standard argument concerning capturing dense open sets in Prikry-type forcings and fat-trees (e.g., see [13]) for every dense open set D of $\mathbb{P}$, $p \in \mathbb{P}$, there exists a direct extension $p' \geq^* p$ which reduces capturing D to a dense subset of $\mathbb{P}_\mu$ for some $\mu < \kappa$, namely, the set of all $r \in \mathbb{P}_\mu$ such that $r^\frown p' \upharpoonright [\mu, \kappa) \in D$ is dense below $p' \upharpoonright \mu$. Moreover, given $\nu < \kappa$, we can also make the direct extension p' to agree with p up to $\nu + 1$ (i.e., $p' \upharpoonright \nu + 1 = p \upharpoonright \nu + 1$), in which case $\mu > \nu$.

Given an initial condition $p \in \mathbb{P}$, we can list the dense open sets in N, $\langle D_i \mid i < \kappa \rangle$, and form an increasing sequence of direct extensions of p, $\langle p^i \mid i < \kappa \rangle$, together with a closed unbounded set $C^* = \langle \nu_i \mid i < \kappa \rangle$ such that for every successor ordinal $i = i' + 1$, $p^i \in N$ reduces the dense set $D_{i'}$ of $\mathbb{P}$

to a bounded dense set $D'_{i'}$ of $\mathbb{P}_{\mu_i}$ for some $\nu_i < \mu_i < \kappa$, and $p^i \upharpoonright \nu_i + 1 = p^{i'} \upharpoonright \nu_i + 1$. By a standard argument concerning nonstationary support iterations (e.g., see the fusion argument in the proof of [5, Lemma 2.2]), the sequence of direct extensions $\langle p^i \mid i < \kappa \rangle$ has an upper bound $p^* \geq^* p$. It follows that for every $\mathbb{P}$-name $\underline{\sigma} \in N$ of an element of H_θ^V, there exists some $\mu < \kappa$ and a $\mathbb{P}_\mu$-name $\underline{\sigma}' \in N$ such that $p^* \Vdash \underline{\sigma} = \underline{\sigma}'$.

In particular, for each such name $\underline{\sigma}$, p^* forces that it can take $< \kappa$ many values in H_θ^V, all of which are in N. This follows from the elementarity of N in H_θ and the fact $\kappa + 1 \subseteq N$. $\quad\square$

Let $j_\tau : V \to M_\tau$ be the ultrapower embedding by $U_{\kappa,\tau}$ and $M' = j_\tau"N \prec j_\tau(H_\theta^V)$, $M' \in M_\tau$.

Claim 4.7. $j_\tau(p^*)$ *forces that* M' *is closed under* $j(\underline{F})$.

Proof. Indeed, if $G^* \subseteq j_\tau(\mathbb{P})$ is M_τ-generic with $j_\tau(p^*) \in G^*$, then for each $\mu < \kappa$, $G^*_\mu = \{ p \upharpoonright \mu \mid p \in G^* \}$ is a V-generic filter for $\mathbb{P}_\mu$. For every $y = j_\tau(\underline{\sigma})_{G^*} \in M' \cap j_\tau(H_\theta^V)$ and $F^* = j_\tau(\underline{F})_{G^*}$, $F^*(y) = j_\tau(\underline{F})(j_\tau(\underline{\sigma}))_{G^*}$ is the G^*-generic interpretation of the $j_\tau(\mathbb{P})$-name $j_\tau(\underline{F}(\underline{\sigma}))$. As p^* forces $\underline{F}(\underline{\sigma}) = \underline{\sigma}'$ for some $\underline{\sigma}' \in N$ which is a $\mathbb{P}_\mu$-name for some $\mu < \kappa$, we see that $j_\tau(p^*)$ forces $j_\tau(\underline{F}(\underline{\sigma})) = j_\tau(\underline{\sigma}')$, where $j_\tau(\underline{\sigma}')$ is a $j_\tau(\mathbb{P}_\mu) = \mathbb{P}_\mu$-name. If $q \in G_\mu$ and $z \in N \cap H_\theta^V$ are such that $q \Vdash_{\mathbb{P}_\mu} \underline{\sigma}' = \check{z}$, then $j_\tau(q) = q \Vdash j_\tau(\underline{\sigma}') = j_\tau(\check{z})$. We conclude that $F^*(y) = z \in M'$. $\quad\square$

We now return to prove the statement of the lemma. It is sufficient to prove that in $V[G]$ there exists some $M' \subseteq H_\theta^V$ which is closed under F and satisfies requirements (i)–(iii). Let $p \in \mathbb{P}$ be a condition. By a standard density argument, there are $N \prec H_\theta^V$ and $p^* \in G$ which is N-generic, with $p^* \geq^* p$.[6] By Claim 4.7, $j_\tau(p^*)$ forces that $M' = j_\tau"N$ is closed under $j_\tau(F)$. It is now clear that M' satisfies condition (ii) in the ultrapower, as $o^{j_\tau(\mathcal{U})}(\kappa) = \tau$ and $M' \cap j_\tau(\kappa) = \kappa$. Condition (i) holds as well, since $j(\mathbb{P}_\kappa)/\mathbb{P}_\kappa$ does not introduce new $<\tau$-sequences to $j_\tau"N$. Therefore, it remains to verify that $j_\tau(p^*)$ forces M' to satisfy condition (iii). For every $A \in M' \cap j_\tau(\mathcal{F}_\kappa)$, there is some $B \in \mathcal{F}_\kappa$ such that $A = j_\tau(B)$. In particular, $A \cap \kappa = B \in \mathcal{F}_\kappa$ and $\kappa \in A$. Since $\mathcal{F}_\kappa \subseteq \bigcap_{i \leq \tau} U_{\kappa,i}$ (which is $\mathcal{F}_\kappa^{M_\tau}$), it follows form remark 4.1 that for every generic filter $G^* \subseteq j_\tau(\mathbb{P})$ over M_τ, if b_κ is the G^*-induced $\mathbb{Q}_\kappa^\tau$ cofinal generic sequence, then it is almost contained in $B = A \cap \kappa$. $\quad\square$

Proposition 4.8. $\mathbb{C}_{\mathcal{F}_\kappa}$ *is* κ-*distributive.*

Proof. Since $\kappa = \lambda^+$ in $V[G * H]$, we need to check that the intersection of every set $\{D_i \mid i < \lambda\}$ of λ-many dense open subsets of $\mathbb{C}_{\mathcal{F}_\kappa}$ is dense. Pick some regular cardinal $\theta > \kappa^+$ such that $\mathbb{P}, \mathbb{C}_{\mathcal{F}_\kappa}, \{D_i \mid i < \lambda\} \in H_\theta$. By Lemma 4.5, for every condition $x \in \mathbb{C}_{\mathcal{F}}$, there exists an elementary substructure $M \prec H_\theta$ of cardinality $< \kappa$, with $x, \mathbb{P}, \mathbb{C}_{\mathcal{F}_\kappa}, \{D_i \mid i < \lambda\} \in M$ and which further satisfies (i) $M^{<\lambda} \subseteq M$; (ii) $\sup(M \cap \kappa) = \alpha$ has $o^{\mathcal{U}}(\alpha) = \lambda$; and (iii) $\alpha \in A$ and b_α is almost contained in A for every $A \in \mathcal{F}_\kappa \cap M$.

Let $\langle \alpha_i \mid i < \lambda \rangle$ be an increasing enumeration of b_α. We construct by induction an increasing sequence of extensions $\langle x_j \mid j < \lambda \rangle$ of x, together with an increasing subsequence $\langle \alpha_{i_j} \mid j < \lambda \rangle$ of b_α such that $x_{j+1} \in D_j$ for every $j < \lambda$, and $\{\alpha\} \cup \{\alpha_{i_j} \mid j > j^*\} \subseteq A^{x_{j^*}}$ for all $j^* < \lambda$. For notational simplicity, denote x by x_{-1}. Given a condition $x_j \in M$ with a suitable α_j as above, we take $x_{j+1} \in D_{j+1}$ to be an extension of x_j with $\max(c^{x_{j+1}}) > \alpha_{i_j}$. Since $A^{x_{j+1}} \in M \cap \mathcal{F}_\kappa$, we can use (iii) and get that $\alpha \in A^{x_{j+1}}$ and there exist some $i' > i_j$ such that $\{\alpha_i \mid i > i'\} \subseteq A^{x_{j+1}}$. Take $i_{j+1} < \lambda$ to be the minimal such $i' > i_j$. It remains to show that the construction goes through at limit stages $\delta \leq \lambda$. Given $\langle x_j \mid j < \delta \rangle$, we define $i_\delta = \sup_{j < \delta} \alpha_{i_j}$. It is clear from our construction at successor steps that $\alpha_{i_\delta} = \sup_{j < \delta} \max(c^{x_j})$ and $\alpha_{i_\delta} \in A^{x_j}$ for every $j < \delta$. It follows that the condition $x_\delta = \langle \{\alpha_\delta\} \cup \bigcup_{j < \delta} c^{x_j}, \bigcap_{j < \delta} A^{x_j} \rangle$ satisfies the desirable conditions. Moreover, if $\delta < \lambda$, then $x_\delta \in M$ since M is closed under $< \lambda$-sequences.

Since the limit construction goes through at stage λ as well (although not producing a condition in M), the limit condition x_λ is an extension of x and belongs to $\bigcap_{j < \lambda} D_j$. $\quad\square$

The argument of the proof of lemma 3.3 for $\mathbb{C}_U$ applies to $\mathbb{C}_{\mathcal{F}_\kappa}$ as well.

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[6]This is true, since for every $q \geq p$ there is $r \geq^* p$ such that $q \geq r$ is a finite Prikry extension. Let $q^* \geq^* q$ be N-generic. Then, there is α such that $q \upharpoonright [\alpha, \kappa) = r \upharpoonright [\alpha, \kappa)$. So, the condition $p^* = r \upharpoonright \alpha \cup q^* \upharpoonright [\alpha, \kappa)$ is an N-generic direct extension of p, from G.

Lemma 4.9. $\mathbb{C}_{\mathcal{F}_\kappa}$ *is cone homogeneous.*

Theorem 4.10. *In the generic extension by* $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}_{\mathcal{F}_\kappa}$, κ *is strongly measurable.*

Proof. Suppose $G(\mathbb{C}_{\mathcal{F}_\kappa}) \subseteq \mathbb{C}_{\mathcal{F}_\kappa}$ is a generic filter over $V[G * H]$. We may identify $G(\mathbb{C}_{\mathcal{F}_\kappa})$ with its derived generic closed and unbounded set

$$C = \bigcup \{c \mid \exists A \langle c, A \rangle \in G(\mathbb{C}_{\mathcal{F}_\kappa})\}.$$

By a standard density argument, we have that for every set $X \subseteq \kappa$ in V, if $X \notin U_{\kappa, \tau}$ for all $\tau \leq \lambda$, then $|C \cap X| < \kappa$.

We conclude that for $X \subseteq \kappa$ in V to be stationary in $V[G * H * C]$, it must belong to $U_{\kappa, \tau}$ for some $\tau \leq \lambda$. It follows that if $\langle S_i \mid i < \eta \rangle \subseteq V$ is a partition of κ into disjoint sets which are stationary in $V[G * H * C]$, then $|\eta| \leq \lambda$. Moreover, since κ is inaccessible in V, we have $(2^\eta)^V < \kappa$.

Finally, we know that each poset $\mathbb{P}$, $\mathrm{Coll}(\lambda, < \kappa)$, and $\mathbb{C}_{\mathcal{F}_\kappa}$ is forced in turn to be cone homogeneous and clearly definable using parameters from the ground model. Therefore, $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}_{\mathcal{F}_\kappa}$ is cone homogeneous, and therefore, $\mathrm{HOD}^{V[G*H*C]} \subseteq V$. The claim follows. □

The result in this section is weaker than the result of section 3 since the club filter is not an ultrafilter in HOD. Since the club filter restricted to $S_\omega^{\omega_2}$ is an ultrafilter in a model of $AD + V = L(\mathbb{R})$, one can force with the $\mathbb{P}_{max}$ forcing and obtain a generic extension in which the club filter restricted to $S_\omega^{\omega_2}$ is an ultrafilter in HOD.[7]

Question 4.11. Is it consistent that the club filter restricted to S_ω^λ is an ultrafilter in HOD for a regular cardinal $\lambda > \aleph_2$?

By the general behavior of covering arguments, it is possible that the consistency strength of ω_2 being ω-strongly measurable in HOD might be much lower than the same property for successors of higher regular cardinals, and possibly as low as a single measurable cardinal.

Question 4.12. What is the consistency strength of ω_2 being ω-strongly measurable in HOD?

5. Many ω-strongly measurable cardinals

Suppose that $\mathcal{U}$ is a coherent sequence of normal measures so that $\lambda < \kappa$ are regular cardinals and $o^{\mathcal{U}}(\kappa) = \lambda + 1$ and that the first measure in $\mathcal{U}$ is on a cardinal strictly greater than λ. Let $\mathbb{P}^{\mathcal{U}}$ be the nonstationary support iteration of Prikry/Magidor forcing from [5], and $\mathbb{C}_{\mathcal{F}^{\mathcal{U}}}$ be the $\mathbb{P}^{\mathcal{U}} * \mathrm{Coll}(\lambda, < \kappa)$-name of the associated diagonalizing club forcing for the filter $\mathcal{F}_\kappa^{\mathcal{U}} = \bigcap_{\tau \leq \lambda} U_{\kappa, \lambda}$ on κ. In the next section, we construct a $\mathbb{P}^{\mathcal{U}} * \mathrm{Coll}(\lambda, < \kappa)$-name of a Prikry-type forcing notion $\bar{\mathbb{C}}_{\mathcal{F}^{\mathcal{U}}}$, which is equivalent to $\mathbb{C}_{\mathcal{F}_\kappa}$, and its direct extension order is λ-closed. We will use that as a black box in this section.

Definition 5.1. Denote the post $\mathbb{P}^{\mathcal{U}} * \mathrm{Coll}(\lambda, < \kappa) * \bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ by $\mathbb{Q}[\mathcal{U}]$.

We have shown in the previous section that $\mathbb{Q}[\mathcal{U}]$ is cone homogeneous and equivalent as a forcing notion to the iteration $\mathbb{P}^{\mathcal{U}} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}_{\mathcal{F}_\kappa}$. By theorem 4.10, we conclude that κ is strongly measurable in the generic extension by $\mathbb{Q}[\mathcal{U}]$.

In what follows, we would like to view $\mathbb{Q}[\mathcal{U}]$ as a Prikry-type forcing whose direct extension order is λ-closed. This is easily possible since $\mathbb{Q}[\mathcal{U}]$ is an iteration of three posets, each of which can be seen as a Prikry-forcing whose direct extension order is λ-closed (for $\mathrm{Coll}(\lambda, < \kappa)$, we identify the direct extension order with the standard order of the poset).

We finally turn to prove our main result.

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[7]We would like to thank the referee for pointing us to this fact.

Proof (Theorem 1.4). To simplify our arguments, we work over a minimal Mitchell model $V = L[\mathcal{U}]$ with a coherent sequence of measures $\mathcal{U}$ witnessing the assumed large cardinal assumption. Therefore, θ is the least inaccessible cardinal in V for which $\{o(\kappa) \mid \kappa < \theta\}$ is unbounded in θ. We note that all normal measures in this model appear on the main sequence $\mathcal{U}$, in particular, $o(\kappa) = o^{\mathcal{U}}(\kappa)$ for all κ. We also record here that by the Mitchell Covering Theorem and the fact θ is not measurable, there is no generic extension of $V = L[\mathcal{U}]$ which preserves the cardinals below θ and changes the cofinality of θ. Similarly, the Mitchell Covering Theorem guarantees that generic extensions of $V = L[\mathcal{U}]$ satisfy the Weak Covering Lemma with respect to V, which implies that successors of singular cardinals cannot be collapsed.

Let $\langle \kappa_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of cardinals below θ, which satisfies the following conditions:

1. $\kappa_0 = \omega$, κ_1 is the least measurable,
2. for a limit ordinal α, $\kappa_\alpha = (\sup_{\beta < \alpha} \kappa_\beta)^+$,
3. for a successor ordinal $\alpha + 1$, let $\kappa_{\alpha+1}$ be the least cardinal such that $o^{\mathcal{U}^\alpha}(\kappa_{\alpha+1}) = \kappa_\alpha + 1$, for the coherent sequence of measures $\mathcal{U}^\alpha = \mathcal{U} \restriction_{(\kappa_\alpha, \kappa_{\alpha+1}]}$. In particular, the first measure of the sequence $\mathcal{U}^\alpha$ has critical point $> \kappa_\alpha$.

We define by induction on $\alpha < \theta$ a Magidor iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \theta \rangle$ of Prikry type forcings. Our description of the Magidor style iteration follows Gitik's handbook chapter [13]. We recall that conditions are sequences of the form $\langle q_\alpha \mid \alpha < \theta \rangle$ where only finitely many coordinates are not a direct extension of the weakest condition $0_{\mathbb{Q}_\alpha}$. Let $\mathbb{Q}_0$ be $\mathrm{Coll}(\omega, < \kappa_1) * \mathbb{C}^*_{\mathcal{F}_{\kappa_1}}$, where $\mathcal{F}_{\kappa_1}$ is the filter generated from the normal measure on κ_1. For $\alpha > 0$, we define $\mathbb{Q}_\alpha = \mathbb{Q}[\mathcal{U}^\alpha]$.

The coherent sequence $\mathcal{U}^\alpha$ from $L[\mathcal{U}]$ uniquely extends in a generic extension by $\mathbb{P}_\alpha$ and can therefore be used to force with $\mathbb{Q}[\mathcal{U}^\alpha]$. This is because as $L[\mathcal{U}]$ satisfies the GCH, we have that $|\mathbb{P}_\alpha| \leq \kappa_\alpha$ and all measures of $\mathcal{U}^\alpha$ are assumed to have critical points strictly above κ_α. It is clear from our definitions that $\mathbb{Q}_\alpha$ satisfies the Prikry Property, that its direct extension order is κ_α-closed and that $\mathbb{Q}_\alpha$ is forced to be cone homogeneous.

By the general theory of Magidor iteration of Prikry type posets, the iteration $\mathbb{P}_\theta / \mathbb{P}_1$ also satisfies the Prikry Property. Moreover, for every $\alpha < \theta$, $\mathbb{P}_\theta / \mathbb{P}_\alpha$ has the Prikry Property in the generic extension by $\mathbb{P}_\alpha$, and its direct extension order is κ_α-closed (see [13] for details).

Claim 5.2. *Every bounded subset of κ_α is introduced by $\mathbb{P}_\alpha$. Moreover, in the generic extension by $\mathbb{P}_\theta$, κ_α is a regular cardinal for all $\alpha < \theta$.*

Proof. The first assertion is an immediate consequence of the fact that $\mathbb{P}_\theta / \mathbb{P}_\alpha$ satisfies the Prikry Property and its direct extension order is κ_α closed. It follows that, in order to show that all cardinal κ_α remain regular in a generic extension by $\mathbb{P}_\theta$, it suffices to show that κ_α remains regular in the intermediate generic extension by $\mathbb{P}_\alpha$. We prove the last assertion by induction on $\alpha < \theta$.

For a limit ordinal α, the assertion follows from the fact that the generic extension by $\mathbb{P}_\alpha$ satisfies the Weak Covering property with respect to the ground model $V = L[\mathcal{U}]$. Indeed, $\kappa_\alpha = (\sup_{\beta < \alpha} \kappa_\beta)^+$ cannot be collapsed without collapsing a tail of the cardinals κ_β, $\beta < \alpha$, which would contradict our inductive assumption.

Suppose now that α is a successor ordinal. Then the forcing of $\mathbb{P}_\alpha$ naturally breaks into two parts $\mathbb{P}_\alpha \cong \mathbb{P}_{\alpha-1} * \mathbb{Q}_{\alpha-1}$. The size of $\mathbb{P}_{\alpha-1}$ is $(2^{\kappa_{\alpha-1}})^V < \kappa_\alpha$ and cannot singularize κ_α. The second poset $\mathbb{Q}_{\alpha-1}$ does not collapse κ_α by Proposition 4.8. Note that, in order to apply the result of Proposition 4.8, we use our inductive hypothesis that $\kappa_{\alpha-1}$ remains regular in a generic extension by $\mathbb{P}_{\alpha-1}$. □

Claim 5.3. *In the generic extension, θ is regular.*

Proof. This follows from the Mitchell Covering Theorem and the smallness assumption of θ, as was mentioned at the beginning of the proof. □

Claim 5.4. $\mathbb{P}$ *is cone homogeneous.*

Proof. It suffices to verify conditions (i) and (ii) of Lemma 8.5 hold for every $\alpha < \kappa$. (i) holds for $\mathbb{Q}_\alpha = \mathbb{Q}[\mathcal{U} \restriction_{(\kappa_\alpha, \kappa_{\alpha+1}]}]$ since $\mathbb{Q}_\alpha, \leq_{\mathbb{Q}_\alpha}, \leq^*_{\mathbb{Q}_\alpha}$ are clearly definable in $V = L[\mathcal{U}]$ from $\mathcal{U}$, κ_α, and $\kappa_{\alpha+1}$. The fact $(\mathbb{Q}_\alpha, \leq_{\mathbb{Q}_\alpha}, \leq^*_{\mathbb{Q}_\alpha})$ is an immediate consequence of Lemma 6.12.   □

Let $G_\theta \subseteq \mathbb{P}_\theta$ be a generic filter over V. We conclude that $\text{HOD}^{V[G_\theta]} \subseteq V$.[8] Moreover, for each $\alpha < \kappa$, the $\mathbb{Q}[\mathcal{U}^\alpha]$ generic filter induced by G_θ guarantees that $\kappa_{\alpha+1} = (\kappa^+_\alpha)^{V[G_\theta]}$ and that κ_α is strongly measurable in $\text{HOD}^{V[G_\theta]}$. It follows that all successors of regular cardinals below θ in $V[G_\theta]$ are strongly measurable in HOD. Since θ remains strongly inaccessible in $V[G_\theta]$ and all the relevant witnessing objects clearly belong to $V^{V[G_\theta]}_\theta$, we conclude that in $V^{V[G_\theta]}_\theta$, all successors of regular cardinals are strongly measurable.   □

6. Embedding $\mathbb{C}_{\mathcal{F}_\kappa}$ in suitable Prikry-type forcings

The method of Section 4 can be iterated finitely many times in order to get finitely many successive ω-strongly measurable cardinals. In order to get a global result (or even just infinitely many ω-strongly measurables), we need to have a preservation of distributivity under iterations.

This is, in general, a difficult task. One way to obtain this is by shifting our goal from preserving distributivity into preserving the Prikry Property. There are several ways to iterate Prikry-type forcings and preserve the Prikry Property as well as the closure properties of the direct extension. Thus, embedding the distributive forcings into a Prikry-type forcing can be used in order to get a suitable distributivity of the iteration. Usually, in order to achieve this, some strong compactness assumption is made that enables one to embed any sufficiently distributive forcing into a Prikry-type forcing. See [14, 6] for some examples for the consistency strength of such constructions.

Our goal is to embed $\mathbb{C}_{\mathcal{F}_\kappa}$ into a Prikry-type forcing without increasing our large cardinal hypothesis from $o(\kappa) = \lambda + 1$. For this, our approach follows the finer technique, introduced by Gitik in [12].

This section is devoted to prove the following technical lemma:

Proposition 6.1. *Let us assume that $\mathcal{U}$ is a coherent measure sequence witnessing $o(\kappa) = \lambda + 1$. Let $\mathbb{P}^{\mathcal{U}}$ be the nonstationary support iteration of Subsection 8.2. Then, in $\mathbb{P}^{\mathcal{U}} * \text{Coll}(\lambda, < \kappa)$, there is a Prikry-type forcing notion $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$, whose direct extension order is λ-closed, and it has a dense subset isomorphic to $\mathbb{C}_{\mathcal{F}_\kappa}$.*

Moreover, both orders $\leq$ and $\leq^*$ witness the forcing $\mathbb{P}^{\mathcal{U}} * \text{Coll}(\lambda, < \kappa) * \bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ to be cone homogeneous.

The proof of the first part of the proposition is given in Corollary 6.10, and the proof of the second part appears in Lemma 6.12. Let us sketch the main ideas behind the proof of the proposition. In order to construct a Prikry-type forcing that projects onto $\mathbb{C}_{\mathcal{F}_\kappa}$, we first work in the generic extension in which κ is singularized to be of cofinality λ. In this model, the Magidor sequence is already a closed unbounded set that diagonalizes the filter $\mathcal{F}_\kappa$, so we can use it as a guide to the generic of $\mathbb{C}_{\mathcal{F}_\kappa}$. This means that there is a projection from the generic extension by the singularizing forcing iterated by a λ-closed forcing onto $\mathbb{C}_{\mathcal{F}_\kappa}$. $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ is obtained by 'forgetting' the Magidor sequence and keeping the diagonalizing club. A technical issue that arises when trying to pull up this strategy is that the singularizing forcing must be defined *after* the cardinals between λ and κ were collapsed, and a major part of this section is devoted to developing this forcing.

The rest of this section is organized as follows. In subsection 6.1, we review the basic construction and properties of the tree Prikry-type forcing notions of the form $\mathbb{Q}^\tau_\kappa$, $\tau \leq o^{\mathcal{U}}(\kappa)$, which is defined in the generic extension by $\mathbb{P}$. Then, we introduce a filter-based variant $\mathbb{Q}^*_{\kappa, \tau}$ to be forced over a generic extension $V[G * H]$ by $\mathbb{P} * \text{Coll}(\lambda, < \kappa)$ extension $V[G * H]$ of V, where $\kappa = \lambda^+$ is no longer measurable.

In subsection 6.2, we use the posets $\mathbb{Q}^*_{\kappa, \tau}$, $\tau \leq \lambda$ in order to introduce a forcing equivalent $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ of $\mathbb{C}_{\mathcal{F}_\kappa}$ with a dense Prikry-type subforcing $\mathbb{C}^*_{\mathcal{F}_\kappa}$ whose direct extension order is λ-closed.

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[8]As a matter of fact, $\text{HOD}^{V[G_\theta]} = V$ since $V = L[\mathcal{U}]$ is ordinal definable in $V[G_\theta]$. We will not used this fact here.

This completes the proof of Section 5, as the posets $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ can be iterated on different cardinals to construct models with many ω-strongly measurable cardinals.

6.1. The forcing $\mathbb{Q}^*_{\kappa,\tau}$

We turn back to consider our forcing scenario with $\mathcal{F}_\kappa$ over $V[G * H]$, where $G \subseteq \mathbb{P}$ is generic over V, and $H \subseteq \mathrm{Coll}(\lambda, < \kappa)$ is generic over $V[G]$. Recall that $o^{\mathcal{U}}(\kappa) = \lambda + 1$ and that each measure $U_{\kappa,\tau}$, $\tau \le \lambda$ in V extends in $V[G]$ to $U_{\kappa,\tau}(t)$, where t is τ-coherent. For each such $\tau \le \lambda$ and t, let $j_{\kappa,\tau,t} : V[G] \to M_{\kappa,\tau,t}$ be the ultrapower embedding of $V[G]$ by $U_{\kappa,\tau}(t)$.

Moving to the further generic extension $V[G * H]$ of $V[G]$, κ is no longer measurable. Let $F_{\kappa,\tau}(t)$ denote the filter generated by $U_{\kappa,\tau}(t)$ on $\mathcal{P}(\kappa)^{V[G]}$, and let $F_{\kappa,\tau}(t)^+$ denote the poset on $\mathcal{P}(\kappa)^{V[G]}$ of $F_{\kappa,\tau}(t)$ positive sets where a set A is stronger than B if $A \setminus B$ belongs to the dual ideal of $F_{\kappa,\tau}(t)$.

By further forcing with the collapse quotient

$$\mathbb{R}_\tau = \mathrm{Coll}(\lambda, < j_{\kappa,\tau,t}(\kappa))/H \cong \mathrm{Coll}(\lambda, [\kappa, j_{\kappa,\tau,t}(\kappa)),$$

over $V[G * H]$, producing a generic filter $H^*_\tau \subseteq \mathrm{Coll}(\lambda, < j_{\kappa,\tau,t}(\kappa))$, with $H^*_\tau \upharpoonright \mathrm{Coll}(\lambda, < \kappa) = H$, the elementary embedding $j_{\kappa,\tau,t}$ extends into

$$j^*_{\kappa,\tau,t} : V[G * H] \to M_{\kappa,\tau,t}[H^*_\tau].$$

In turn, the embedding $j^*_{\kappa,\tau,t}$ generates a $V[G * H]$ ultrafilter $U_{\kappa,\tau}(t)^* \subseteq F_{\kappa,\tau}(t)^+$, which is an $F_{\kappa,\tau}(t)^+$-generic ultrafilter over $V[G * H]$, by standard arguments connecting forcing with positive sets and generic ultrapowers.[9] Since the poset $\mathbb{R}_\tau = \mathrm{Coll}(\lambda, < j_{\kappa,\tau,t}(\kappa))/H$ is λ-closed in $V[G * H]$, we have that $F_{\kappa,\tau}(t)^+$ has a λ-closed dense subforcing $D_{\kappa,\tau,t}$. Other examples of applications of Prikry-type forcings generated by ideals can be found in [4] and [1].

It would be useful for our purposes to work with a concrete description of the sets in $D_{\kappa,\tau,t}$. We proceed to introduce the relevant notions.

Definition 6.2. Let $G \subseteq \mathbb{P}$ be a generic filter over V. For each cardinal $\nu < \kappa$ with $o^{\mathcal{U}}(\nu) > 0$, let b_ν be the G-induced generic cofinal sequence in ν.

1. Recall that every finite coherent sequence $t = \langle \nu_0, \dots, \nu_{k-1} \rangle \in [\kappa]^{<\omega}$ in $V[G]$ has an assigned closed unbounded set $b_t = \cup_{i<k}(b_{\nu_i} \cup \{\nu_i\})$. For a coherent sequence t and a finite set of ordinals $s \in [\min(t)]^{<\omega}$, we define $\pi^s(t) = \min(b_t \setminus (\max(s) + 1))$. When $t = \langle \nu \rangle$ has a single element, we will often abuse this definition and write $\pi^s(\nu)$ for $\pi^s(\langle \nu \rangle)$.

   For every η, the function

   $$\pi^s_\eta(\nu) = \min(\{\mu \in b_\nu \setminus (\max(s) + 1) \mid o^{\mathcal{U}}(\mu) = \eta\})$$

   defines a Rudin-Keisler projection from $U_{\kappa,\tau}(s)$ to $U_{\kappa,\eta}(s)$, for all $\tau > \eta$. In particular, $\pi^s = \pi^s_0 : \kappa \to \kappa$ is a Rudin-Keisler projection of $U_{\kappa,\tau}(s)$ to its normal projected measure $U_{\kappa,0}(s)$, for every $\tau \ge 0$.

2. Let $T \subseteq [\kappa]^{<\omega}$ be a tree, $t \in [\kappa]^{<\omega}$, and $Q : T \to \mathrm{Coll}(\lambda, < \kappa)$ be a function. We say that Q is (T, t)-**suitable** if for every $s \in T$, we have
   ○ $Q(s) \in \mathrm{Coll}(\lambda, < \kappa)$, and
   ○ for every $s' \in T$ that extends s, $Q(s') \upharpoonright \lambda \times \pi^{t \frown s}_0(s') = Q(s)$.

   For every $s \in T$, we define Q_s to be the induced function on $T_s = \{r \in [\kappa]^{<\omega} \mid s \frown r \in T\}$, given by

   $$Q_s(r) = Q(s \frown r).$$

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[9]Indeed, one can verify that the trivial condition in $\mathrm{Coll}(\lambda, [\kappa, j_{\kappa,\tau,t}(\kappa))$ forces $\kappa \in j(\dot{X})$ if and only if there is a subset of $\dot{X}$ in $U_{\kappa,\tau}(t)$.

3. Suppose that $H \subseteq \mathrm{Coll}(\lambda, < \kappa)$ is generic over $V[G]$, $T, Q \in V[G]$ as above, and let $A = \mathrm{succ}_T(\emptyset)$. We define in $V[G][H]$ the set Q-generic restriction of A with respect to H to be the set

$$A_Q^H = \{v \in A \mid Q(v) \in H\}.$$

Let $G \subseteq \mathbb{P}$ be generic over V and $H \subseteq \mathrm{Coll}(\lambda, < \kappa)$ be generic over $V[G]$. In $V[G]$, for all $\tau \leq \lambda$ and $t \in [\kappa]^{<\omega}$, the function $\pi^t = \pi_0^t$ represents κ in the ultrapower by $U_{\kappa,\tau}(t)$. It is therefore immediate from our definition of $D_{\kappa,\tau,t} \subseteq F_{\kappa,\tau}(t)^+$ that sets in $D_{\kappa,\tau,t}$ are of the form $A_r^H = \{v \in A \mid r(v) \in H\}$ where $A \in U_{\kappa,\tau}(t)$, and $r : A \to \mathrm{Coll}(\lambda, < \kappa)$ satisfying $r(v) \in \mathrm{Coll}(\lambda, [\pi^t(v), \kappa))$ for all $v \in A$.[10]

We use these facts to introduce a variant of Gitik's forcing $\mathbb{Q}_{\kappa,\tau}$ in $V[G * H]$. The following poset $\mathbb{Q}_{\kappa,\tau}^*$ collapses cardinals up to κ^+ to λ and adds a cofinal Magidor sequence b_κ of length ω^τ to κ, which diagonalizes the filter $\bigcap_{\tau' < \tau} U_{\kappa,\tau'}$ (i.e., b_κ is almost contained in each filter set).

Definition 6.3. In $V[G * H]$, the forcing $\mathbb{Q}_{\kappa,\tau}^*$ consists of all $(t, T, Q) \in V[G]$ such that:

1. t is a τ-coherent finite sequence of ordinals below κ,
2. T is a tree of τ-coherent finite sequences with stem t,
3. Q is a (T, t)-suitable function,
4. $Q(\emptyset) \in H$ and
5. For every $s, s' \in T$, if $b_{t^\frown s} = b_{t^\frown s'}$, then $Q(s) = Q(s')$.

As in $\mathbb{Q}_{\kappa,\tau}$, we identify two conditions $(t, T, Q), (t', T, Q) \in \mathbb{Q}_{\kappa,\tau}^*$, whenever $b_t = b_{t'}$.

The **direct extension** ordering of $\mathbb{Q}_{\kappa,\tau}^*$ naturally extends the direct extension ordering of $\mathbb{Q}_{\kappa,\tau}$. Namely, for two conditions $(t, T, Q), (t^*, T^*, Q^*)$ of $\mathbb{Q}_{\kappa,\tau}^*$, we have $(t, T, Q) \leq^* (t^*, T^*, Q^*)$ if $(t, T) \leq_{\mathbb{Q}_{\kappa,\tau}}^* (t^*, T^*)$ and $Q^*(s) \geq Q(s)$ for every $s \in T^*$.

We observe that the direct extension order $\leq^*$ is λ-closed in $V[G * H]$. For this, note that it is immediate from the definition above that the partial order $\tilde{\leq} \in V[G]$, obtained from $\leq^*$ by removing the requirement $Q(\emptyset) \in H$, belongs to $V[G]$ and is clearly λ-closed in both $V[G]$ and $V[G * H]$ (note that the two generic extensions agree on sequences of length $< \lambda$). Then, as $\leq^*$ is equivalent to the restriction of $\tilde{\leq}$ to a λ-closed set (which is essentially H), it remains λ-closed in $V[G * H]$.

The **end-extension** ordering of $\mathbb{Q}_{\kappa,\tau}^*$ is based on the restriction of the end-extension of $\mathbb{Q}_{\kappa,\tau}$ to the Q-generic restriction of T with respect to H. Namely, for a condition $p = (t, T, Q)$, **the only values $v \in \mathrm{succ}_T(\emptyset)$ which are allowed to be used when taking a one-point extension, are $v \in \mathrm{succ}_T(\emptyset)_Q^H$**.[11] In this case, the resulting one-point extension is defined to be $p^\frown \langle v \rangle = (t \cup \{v\}, T_{\langle v \rangle}, Q_{\langle v \rangle})$. In general, for a sequence $r = \langle v_0, \ldots, v_{k-1} \rangle \in T$, the end extension of p by r, denoted $p^\frown r$, is the one obtained by taking a sequence of one-point extensions by $v_0, \ldots, v_{k-1}$, in turn.

We note that although $\mathbb{Q}_{\kappa,\tau}^*$ depends on the collapse generic H, and is fully defined only in $V[G * H]$, we still have that $\mathbb{Q}_{\kappa,\tau}^* \subseteq V[G]$. Moreover, dropping the requirement $Q(\emptyset) \in H$ in the definition of conditions $p = (t, T, Q) \in \mathbb{Q}_{\kappa,\tau}^*$ allows us to examine conditions (t, T, Q) and evaluate possible direct extensions and one-point extensions in $V[G]$. For example, working in $V[G]$, we can consider possible one-point extensions $p^\frown \langle v \rangle$ of a condition $p = (t, T, Q)$ by an arbitrary $v \in \mathrm{succ}_T(\emptyset)$. Although eventually, in $V[G * H]$, p will be a valid condition only if $Q(\emptyset) \in H$, and $p^\frown \langle v \rangle$ will form valid extensions of p only for a $D_{\kappa,\tau,t}$-positive set of ordinals $v \in \mathrm{succ}_T(\emptyset)$, it is still possible to decide certain properties of such extensions on a measure one set of $U_{\kappa,\tau}(t)$ in $V[G]$. This approach of arguing from $V[G]$ about the poset $\mathbb{Q}_{\kappa,\tau}^*$ in $V[G * H]$ plays a significant role in our proof below, showing that $\mathbb{Q}_{\kappa,\tau}^*$ satisfies the Prikry Property.

Remark 6.4. The forcing $\mathrm{Coll}(\kappa < \lambda) * \mathbb{Q}_{\kappa,\tau}^*$ is isomorphic to the collection of all (t, T, Q) that satisfy all requirements of Definition 6.3 except the fourth one, $Q(\emptyset) \in H$. Nevertheless, the decomposition into the collapse part and the singularization part would be more appropriate for our construction, as eventually $\mathbb{Q}_{\kappa,\tau}^*$ is used as merely an auxiliary forcing.

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[10]i.e., $\mathrm{dom}(r(v)) \subseteq \lambda \times (\kappa \setminus \pi^t(v))$.
[11]i.e., $v \in A_Q^H$ for $A = \mathrm{succ}_T(\emptyset)$.

The coherency requirements in Definition 6.3 allow us to obtain a natural amalgamation property, similar to the one satisfied by the $V[G]$ poset $\mathbb{Q}_{\kappa,\tau}$.

Lemma 6.5. *Work in $V[G]$. Let $p = (t, T, Q)$, forced to be a condition in $\mathbb{Q}_{\kappa,\tau}^*$ by $q = Q(\emptyset)$, and $A = \operatorname{succ}_T(\emptyset)$. For each $\eta < \tau$, denote*

$$A(\eta) = A \cap \{v \in A \mid o^{\mathcal{U}}(v) = \eta\} \in U_{\kappa,\eta}(t).$$

Suppose that there are $\bar{\eta} < \tau$, a set $A'(\bar{\eta}) \subseteq A(\bar{\eta})$ with $A'(\bar{\eta}) \in U_{\kappa,\bar{\eta}}(t)$ and a sequence of conditions $\langle (t \cup \{v\}, T^v, Q^v) \mid v \in A'(\bar{\eta}) \rangle$, such that $(t \cup \{v\}, T^v, Q^v)$ is a direct extension of $p^\frown \langle v \rangle$ for all $v \in A'(\bar{\eta})$. Then, there exists a direct extension $p^* \geq^* p$, $p^* = (t, T^*, Q^*)$, such that the set $\{(t^\frown \langle v \rangle, T^v, Q^v) \mid v \in A'(\bar{\eta})\}$ is forced by $Q^*(\emptyset)$ to be predense above p^*.

Remark 6.6. In the proof of the lemma, we make use of several construction arguments involving trees T associated to conditions (t, T) in the poset $\mathbb{Q}_{\kappa,\tau}$ from [11]. We list these arguments and refer the reader to [11] for proofs.

For a finite sequence s, we write $o(s) = \max(\{o(v) \mid v \in s\})$.

1. Suppose that (t, T) is a condition of $\mathbb{Q}_{\kappa,\tau}$ and $A'(\eta) \subseteq \operatorname{succ}_T(\emptyset)$ belongs to $U_{\kappa,\eta}(t)$ for some $\eta < \tau$. Then there exists a subtree T' of T, so that $(t, T') \in \mathbb{Q}_{\kappa,\tau}$ is a direct extension of (t, T), and

$$\{v \in \operatorname{succ}_{T'}(\emptyset) \mid o(v) = \eta\} \subseteq A'(\eta).$$

   Similarly, for every $s \in T$ and $A'_s(\eta) \subseteq \operatorname{succ}_T(s)$ which belongs to $U_{\kappa,\eta}(t^\frown s)$ there is a direct extension (t, T') of (t, T), which only requires shrinking the tree T above s (i.e., shrinking T_s) and in particular $s \in T'$, so that $\{v \in \operatorname{succ}_{T'}(s) \mid o(v) = \eta\} \subseteq A'_s(\eta)$.

   Furthermore, this construction can be naturally combined over different values $s \in T$. Namely, given a family $\{A'_s(\eta) \mid s \in T\}$ of sets as above we can apply the same procedure, level by level, to the tree T and obtain a subtree $T' \subseteq T$ with the property that $(t, T') \in \mathbb{Q}_{\kappa,\tau}$ and for every $s \in T'$,

$$\{v \in \operatorname{succ}_{T'}(s) \mid o(v) = \eta\} \subseteq A'_s(\eta).$$

2. For a condition (t, T), $s \in T$, and $\eta < \tau$, there exists a direct extension $(t, T') \geq^* (t, T)$, which only requires shrinking the tree T above s such that for all $s' \in T$ which end extends s, if there exists $v \in b_{s'} \setminus b_s$ such that $o(v) = \eta$,[12] then $v' = \pi_\eta^{t^\frown s}(s')$ (the minimal such v) belongs to $\operatorname{succ}_{T'}(s)$. Repeating this construction, level by level, produces a direct extension (t, T') of (t, T) satisfying that for every $s \in T'$ and $s' \in T'$ which extends s, $v' = \pi_\eta^{t^\frown s}(s') \in \operatorname{succ}_{T'}(s) \cap \{v < \kappa \mid o(v) = \eta\}$.

   We note that if $s \in T'$ satisfies that $o(\mu) < \eta$ for all $\mu \in s$, then for every $\mu \in \operatorname{succ}_{T'}(s)$ with $o(\mu) \geq \eta$, we have $\pi_\eta^{t^\frown s}(\mu) = \pi_\eta^t(s^\frown \langle \mu \rangle)$, which by our assumption of T' (applied to $s' = s^\frown \langle \mu \rangle$), implies that $\pi_\eta^t(s^\frown \langle \mu \rangle) \in \operatorname{succ}_{T'}(\emptyset)$. It follows that

$$\operatorname{succ}_{T'}(s) \cap \{v \mid o(v) = \eta\} \subseteq \operatorname{succ}_{T'}(\emptyset) \cap \{v \mid o(v) = \eta\}.$$

   Since the former set belongs to $U_{\kappa,\eta}(t^\frown s)$, we conclude that

$$\operatorname{succ}_{T'}(\emptyset) \cap \{v \mid o(v) = \eta\} \in U_{\kappa,\eta}(t^\frown s)$$

as well.

   The same consideration applies to any $s \in T'$ and $s' \in T'$ which extends s, and for which $o(v) < \eta$ for every $v \in s' \setminus s$, and implies that

$$\operatorname{succ}_{T'}(s) \cap \{v \mid o(v) = \eta\} \in U_{\kappa,\eta}(t^\frown s').$$

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[12]this is equivalent to the existence of $\mu \in s' \setminus s$ such that $o(\mu) \geq \eta$.

3. Let (t, T') be a condition as in the previous clause. There exists a direct extension (t, T^*) of (t, T') such that for every $s' \in T^*$ for which $v' = \pi_\eta^t(s') \in b_{s'} \setminus b_t$ is defined, not only that

$$v' \in \mathrm{succ}_{T^*}(\emptyset) \cap \{v \mid o(v) = \eta\},$$

but further, there is some $s'' \in T^*$ which extends $\langle v' \rangle$ such that $b_{t \frown s''} = b_{t \frown s'}$ and $T_{s'}^* \subseteq T_{s''}^*$. We note that it implies that the set

$$\{(t \cup \{v\}, T_{\langle v \rangle}^*) \mid v \in \mathrm{succ}_{T^*}(\emptyset), o(v) = \eta\}$$

is predense above (t, T^*).

We turn to the proof of Lemma 6.5.

Proof (Lemma 6.5). For $s \in T$, define $o(s) = \max(\{o(v) \mid v \in s\})$, and for a tree $T \subseteq [\kappa]^{<\omega}$, and η, $T(< \eta) = \{s \in T \mid o(s) < \eta\}$. Let $p = (t, T, Q), A'(\bar\eta)$, and $\langle (t \cup \{v\}, T^v, Q^v) \mid v \in A'(\bar\eta) \rangle$, as in the statement of the Lemma. By part (1) of Remark 6.6 above, we may assume (by reducing to a suitable subtree) that $\mathrm{succ}_T(\emptyset) \cap \{v \mid o(v) = \bar\eta\} \subseteq A'(\bar\eta)$. Furthermore, by part (2) of the remark, we may further assume that $A'(\bar\eta) \in U_{\kappa, \bar\eta}(t \frown s)$ for every $s \in T$ with $o(s) < \bar\eta$.

Recall that for each sequence $s \in T$, $o(s) < \bar\eta$, the function $\pi^{t \frown s}(v)$ is a normal projection of $U_{\kappa, \bar\eta}(t \frown s)$ to $U_{\kappa, 0}(t \frown s)$. Since $Q^v(\emptyset) \upharpoonright \lambda \times \pi^{t \frown s}(v)$ is bounded in $\pi^{t \frown s}(v)$,[13] we can press down on its value and find a subset $A_s'(\bar\eta) \in U_{\kappa, \bar\eta}(t \frown s)$ of $A'(\bar\eta)$ and a collapse condition $Q'(s)$ such that $Q^v(\emptyset) \upharpoonright \pi^{t \frown s}(v) = Q'(s)$ for all $v \in A_s'(\bar\eta)$.

By applying the construction arguments of Remark 6.6, we may find direct extension (t, T') of (t, T) having both properties from parts (1) and (2) of the remark, where (1) is applied with respect to the sets $A_s'(\bar\eta), s \in T', o(s) < \bar\eta$, given by the pressing down process above, by which $Q'(s)$ is defined. We note that, as mentioned at the end of part (2) of the remark, for every $s' \in T'(< \bar\eta) = \{s \in T' \mid o(s) < \bar\eta\}$ which end extends s,

$$\mathrm{succ}_{T'}(s) \cap \{v \mid o(v) = \bar\eta\} \in U_{\kappa, \bar\eta}(t \frown s').$$

Moreover, since

$$\mathrm{succ}_{T'}(s) \cap \{v \mid o(v) = \bar\eta\} \subseteq A_s'(\bar\eta),$$

we conclude that $A_s'(\bar\eta) \in U_{\kappa, \bar\eta}(s')$. It follows that $A_s'(\bar\eta) \cap A_{s'}'(\bar\eta) \neq \emptyset$, which in turn, implies that

$$Q'(s) = Q'(s') \upharpoonright \lambda \times \pi^{t \frown s}(s')$$

(as witnessed by $Q^v(\emptyset)$ for any $v \in A_s'(\bar\eta) \cap A_{s'}'(\bar\eta)$).

Finally, we form a subtree T'' of T' by intersecting $T_{\langle v \rangle}'$ with T^v, for each $v \in \mathrm{succ}_{T'}(\emptyset) \cap \{v \mid o(v) = \bar\eta\} \subseteq A'(\bar\eta)$. By appealing to part (3) of the previous remark, we can find a direct extension (t, T^*) of (t, T'') which further satisfies that for every $s \in T^*$ for which $v_s := \pi_\eta^t(s) \in b_s \setminus b_t$ is defined, $v_s \in \mathrm{succ}_{T^*}(\emptyset) \cap \{v \mid o(v) = \eta\}$, and there exists some $s' \in T^*$ which end extends $\langle v_s \rangle$, such that $b_{t \frown s'} = b_{t \frown s}$ and $T_s^* \subseteq T_{s'}^*$. Let $\bar s = s' \setminus v_s + 1$.

$Q^{v_s}(\bar s)$ is defined, since $T_{\langle v_s \rangle}^* \subseteq T^{v_s} = \mathrm{dom}(Q^{v_s})$. Moreover, since $(t \cup \{v_s\}, T^{v_s}, Q^{v_s})$ is assumed to be a condition in $\mathbb{Q}_{\kappa, \tau}^*$, the value $Q^{v_s}(\bar s)$ does not depend on the choice of a sequence s' and its associated subsequence $\bar s \in T^{v_s}$ satisfying $b_{t \frown \langle v_s \rangle \frown \bar s} = b_{t \frown s'} = b_{t \frown s}$.

We turn to define the function Q^* on T^*. We follow the convention from the last paragraph, where for $s \in T^*$ with $o(s) \geq \bar\eta$, we denote $v_s = \pi_{\bar\eta}^t(s)$. We set

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[13] i.e., $Q^v(\emptyset) \upharpoonright \lambda \times \pi^{t \frown s}(v) \in V_{\pi^{t \frown s}(v)}$ and $\pi^{t \frown s}(v)$ is an inaccessible cardinal

$$Q^*(s) = \begin{cases} Q^{\nu_s}(\bar{s}) & \text{if } o(s) \geq \bar{\eta}, \bar{s} \in T^{\nu_s} \text{ satisfies } b_{t^\frown \langle \nu_s \rangle^\frown \bar{s}} = b_{t^\frown s} \\ Q'(s) & \text{if } o(s) < \bar{\eta} \end{cases}$$

We claim that $Q^*(\emptyset)$ forces that (t, T^*, Q^*) is a condition that extends (t, T, Q). We show first that $Q^*(\emptyset)$ forces (t, T^*, Q^*) is a condition of $\mathbb{Q}^*_{\kappa,\tau}$, which requires verifying the first three conditions in the definition of the poset. Conditions (i) and (ii) are clearly satisfied as $(t, T^*) \in \mathbb{Q}_{\kappa,\tau}$. To verify condition (iii), we need to check that for every $s, s' \in T^*$, if s' extends s, then $Q^*(s) = Q^*(s') \upharpoonright \lambda \times \pi^{t^\frown s}(s')$. The verification breaks down to three cases.

Case I: If $o(s), o(s') < \bar{\eta}$, then $s, s' \in T^*(< \bar{\eta})$ and, as described above, the result is an immediate consequence of the fact that $s' \in T'(< \bar{\eta})$ end extends s.

Case II: If $o(s) < \bar{\eta}$ and $o(s') \geq \bar{\eta}$, then $\nu' = \pi^{t^\frown s}_{\bar{\eta}}(s') \in A'_s(\bar{\eta})$. As $\pi^{t^\frown s}(s') = \pi^{t^\frown s}(\nu')$ and $(t \cup \{\nu'\}, T^\nu, Q^\nu) \in \mathbb{Q}^*_{\kappa,\tau}$, it follows that

$$Q^*(s') \upharpoonright \lambda \times \pi^{t^\frown s}(s') =$$
$$Q^{\nu'}(\bar{s'}) \upharpoonright \lambda \times \pi^{t^\frown s}(s') =$$
$$Q^{\nu'}(\emptyset) \upharpoonright \lambda \times \pi^{t^\frown s}(\nu') = Q'(s) = Q^*(s).$$

Case III: If $o(s) \geq \bar{\eta}$, then there exists some $\bar{s} \in T^*_{\nu_s} \subseteq T^{\nu_s}$ such that $b_{t^\frown \langle \nu_s \rangle^\frown \bar{s}} = b_{t^\frown s}$ and $T^*_s \subseteq T^*_{\langle \nu_s \rangle^\frown \bar{s}}$. In particular, $Q^*(s) = Q^*(\bar{s})$ and $s' \in T^*_{\langle \nu_s \rangle^\frown \bar{s}} \subseteq T^{\nu_s}_{\bar{s}}$. Since Q^{ν_s} is $(t \cup \{\nu_s\}, T^{\nu_s})$-coherent, we conclude that

$$Q^*(s') \upharpoonright \lambda \times \pi^{t^\frown s}(s') =$$
$$Q^{\nu_s}(s') \upharpoonright \lambda \times \pi^{t^\frown s}(s') = Q^{\nu_s}(\bar{s}) = Q^*(s).$$

This concludes the proof that $p^* = (t, T^*, Q^*)$ satisfies the property (iii) of the definition of $\mathbb{Q}^*_{\kappa,\tau}$, and thus, that $Q^*(\emptyset) \in \mathrm{Coll}(\lambda, < \kappa)$ forces it as a condition of $\mathbb{Q}^*_{\kappa,\tau}$. It is immediate from its definition that p^* is a direct extension of p. Finally, our choice of the tree T^* above, obtained from T' using fact (3) from Remark 6.6 above, implies at once that $Q^*(\emptyset)$ forces $\{(t^\frown \langle \nu \rangle, T^\nu, Q^\nu) \mid \nu \in A'(\bar{\eta})\}$ to be predense above p^*. $\square$

Lemma 6.7. $\mathbb{Q}^*_{\kappa,\tau}$ *satisfies the Prikry Property.*

Proof. Suppose otherwise. Working in $V[G * H]$, let $(t, T, Q), \sigma$ be condition and statement of $\mathbb{Q}^*_{\kappa,\tau}$, respectively, such that no direct extension of (t, T, Q) decides σ. Back in $V[G]$, let $q \in H$ be a condition which forces this statement about (t, T, Q) and σ. Since q forces (t, T, Q) to be a condition of $\mathbb{Q}^*_{\kappa,\tau}$, we have that $q \geq Q(\emptyset)$. Therefore, by moving to a direct extension of (t, T, Q), we may assume that $q = Q(\emptyset)$. For notational simplicity, we make the assumption that $t = \emptyset$. The proof for an arbitrary sequence t is similar.

Let $A = \mathrm{succ}_T(\emptyset)$. We may assume that $q \in V_{\mu_0}$, where $\mu_0 = \min(\{\pi_0^\emptyset(\nu) \mid \nu \in A\})$. For each $\nu \in A$, we choose a condition $q(\nu) \in \mathrm{Coll}(\lambda, < \kappa)$, extending $q \cup Q(\nu)$, which decides the $\mathrm{Coll}(\lambda, < \kappa)$ statement of whether there exists a direct extension $p^\nu = (\langle \nu \rangle, T^\nu, Q^\nu)$ of $p^\frown \langle \nu \rangle$ which decides σ, and if so, whether p^ν forces σ or $\neg\sigma$. Let A^0 be the sets of $\nu \in A$ for which $q(\nu)$ forces p^ν exists, and '$p^\nu \Vdash \sigma$'. Similarly, let $A^1 \subseteq A$ consist of ν such that $q(\nu)$ forces p^ν exists, and '$p^\nu \Vdash \neg\sigma$' and $A^2 = A \setminus (A^0 \uplus A^1)$. The proof splits now into three main cases:

Case 0: There exists some $\bar{\eta} < \tau$ such that $A^0 \in U_{\kappa,\bar{\eta}}(\emptyset)$.

Let $A'(\bar{\eta}) = A^0 \cap \{\nu \mid o(\nu) = \bar{\eta}\}$. By applying Lemma 6.5 with respect to the family of conditions $\{p^\nu = (\langle \nu \rangle, T^\nu, Q^\nu) \mid \nu \in A'(\bar{\eta})\}$, we can find a $p^* = (\emptyset, T^*, Q^*)$ which is forced by $Q^*(\emptyset)$ to be a direct extension of $(\emptyset, T, Q)$, and to have $\{p^\nu \mid \nu \in A'(\bar{\eta})\}$ be a predense in $\mathbb{Q}^*_{\kappa,\tau}/(\emptyset, T^*, Q^*)$. It follows that

$Q^*(\emptyset) \geq Q(\emptyset) = q$ forces $p^* = (\emptyset, T^*, Q^*)$ is a direct extension of p which decides σ. This contradicts our assumption.

Case 1: There exists some $\bar{\eta} < \tau$ such that $A^1 \in U_{\kappa, \bar{\eta}}(\emptyset)$.

The argument for this case is similar to the previous one and leads to an extension $q^* \geq q$ in $\mathrm{Coll}(\lambda, < \kappa)$ and a direct extension $\bar{p} \geq^* p$, such that q^* forces $\bar{p} \Vdash_{\mathbb{Q}^*_{\kappa, \tau}} \neg \sigma$. Contradiction.

Case 2: $A^2 \in \bigcap_{\eta < \tau} U_{\kappa, \eta}(\emptyset)$.

Let $A^2(0) = A^2 \cap \{\nu \mid o(\nu) = 0\}$, and apply Lemma 6.5 with respect to $A^2(0)$ and $\{p^\frown \langle \nu \rangle \mid \nu \in A^2(0)\}$ to obtain a direct extension $p_0^* = (\emptyset, T_0^*, Q_0^*)$ of p, with $\{p^\frown \langle \nu \rangle \mid \nu \in A^2(0)\}$ being predense in $\mathbb{Q}^*_{\kappa, \tau}/p_0^*$.

Denoting $q_0^* = Q_0^*(\emptyset)$, we define $\bar{A}^2$ to be the set of all $\nu \in A^2$ for which q_0^* forces there is no direct extension $p^\nu \geq^* p'^\frown \nu$ which decides σ. Note that $\bar{A}^2$ must belong to $\bigcap_{\eta < \tau} U_{\kappa, \eta}(\emptyset)$, since otherwise, there would be some $\eta < \tau$, and a set $A'(\eta) \subseteq \bar{A}^2$ consisting of ν for which some $q^*(\nu) \geq q_0^*$ forces there exists a direct extension of $p_0^* \frown \nu$ which decides σ. This, in turn would allow us to repeat the construction of one of the previous cases 0 and 1, to show that there is $q^* \geq^* q_0^*$ which forces some direct extension p^* of p_0^* to force either σ or $\neg \sigma$, contradicting the choice of q_0^*.

Let $q^1 := q_0^*$ and $p^1 = (\emptyset, T^1, Q^1)$ be the direct extension of p_0^* obtained by shrinking $\mathrm{succ}_{T^{p_0^*}}(\emptyset)$ to points in $\bar{A}^2$. It follows from the construction q^1 forces that for all $\nu \in \mathrm{succ}_{T^1}(\emptyset)$, $p^1 \frown \langle \nu \rangle$ does not have a direct extension in $\mathbb{Q}^*_{\kappa, \tau}$ which decides σ.

Next, we move up to the second level of the tree. To each $\mu \in \mathrm{succ}_{T^1}(\emptyset)$, we can repeat the above analysis with respect to $q(\mu) = q^1 \cup Q^1(\mu)$ and $p^1 \frown \langle \mu \rangle = (\langle \mu \rangle, T^1_{\langle \mu \rangle}, Q^1_{\langle \mu \rangle})$. Accordingly, we split $B = \mathrm{succ}_{T^1}(\emptyset)$ into three sets, B^0, B^1, B^2, based on whether the analysis for $q(\mu)$ and $p^1 \frown \langle \mu \rangle$ has produced an extension $q^*(\mu) \geq q(\mu)$ which forces some direct extension $p^{1, \mu} \geq^* p^1 \frown \langle \mu \rangle$ to decide σ (B^0 and B^1 for forcing σ and $\neg \sigma$, respectively). The argument above shows that if B^0 or B^1 belongs to $U_{\kappa, \eta}(\emptyset)$ for some $\eta < \tau$, then there exists some $q^* \geq q^1$ which forces that p^1 has a direct extension which decides σ, contradicting our assumptions.

It follows that $B^2 \in \bigcap_{\eta < \tau} U_{\kappa, \eta}(\emptyset)$, and by repeating the argument from the beginning of Case 2 for each $p^1 \frown \langle \mu \rangle$, $\mu \in B^2$, we can find for each $\mu \in B^2$, conditions $q^*(\mu) \geq q^1(\mu)$, $q^*(\mu) \in \mathrm{Coll}(\lambda, \pi_0^\emptyset(\mu))$, and $p_\mu^* = (\langle \mu \rangle, T_\mu^*, Q_\mu^1) \geq^* p^1 \frown \mu$, such that $q^*(\mu)$ forces there is no direct extension of $p_\mu^* \frown \nu$ which decides σ, for any $\nu \in \mathrm{succ}_{T_\mu^*}(\emptyset)$. We may assume $q^*(\mu) = Q_\mu^1(\emptyset)$ and apply Lemma 6.5 with respect to $B^2(0)$ and $\{p_\mu^* \mid \mu \in B^2(0)\}$, to conclude, similarly to the above, that there are extensions $q^2 \geq q^1$ and $p^2 = (\emptyset, T^2, Q^2) \geq^* p^1$, such that q^2 forces that for any pair $\langle \nu_0, \nu_1 \rangle \in T^2$, $p^2_{\langle \nu_0, \nu_1 \rangle}$ does not have a direct extension which decides σ.

The construction is now repeated level by level, for all $n < \omega$. This produces sequences of extensions $q = q^0 \leq q^1 \leq \cdots \leq q^n \cdots$ in $\mathrm{Coll}(\lambda, < \kappa)$ and $p = p^0 \leq^* p^1 \leq^* \cdots \leq^* p^n \cdots$ in $\mathbb{Q}^*_{\kappa, \tau}$, such that for each $n < \omega$, writing $p^n = (\emptyset, T^n, Q^n)$, we have that q^n forces that for all sequences $s \in T^n$ of length $|s| \leq n$, $p^n \frown s$ does not have a direct extension which decides σ. Finally, let $q^\omega \in \mathrm{Coll}(\lambda, < \kappa)$ be a union of all q^n, $n < \omega$, and $p^* \in \mathbb{Q}^*_{\kappa, \tau}$ be a direct extension of p^n for all $n < \omega$. Writing $p^* = (\emptyset, T^*, Q^*)$, it follows from the construction that p^ω forces that for no $s \in T^*$ such that $p^* \frown s$ has a direct extension which decides σ. This is, of course, absurd. □

We conclude that $(\mathbb{Q}^*_{\kappa, \tau}, \leq, \leq^*)$ is a Prikry-type forcing whose direct extension order $\leq^*$ is λ-closed. In particular, it does not add bounded subsets to λ. Moreover, like $\mathbb{Q}^\tau_\kappa$, $(\mathbb{Q}^*_{\kappa, \tau}, \leq)$ introduces a generic club $b_\kappa^\tau \subseteq \kappa$ of order-type $\mathrm{ot}(b_\kappa^\tau) = \omega^\tau$. Finally, since $U_{\kappa, \tau'} \subseteq F_{\kappa, \tau'}(t)$ is clearly contained in $D_{\kappa, \tau', t}$ for every coherent sequence t, it follows from a standard density argument that if b_κ^τ is a $\mathbb{Q}^*_{\kappa, \tau}$-generic sequence over $V[G * H]$, then for every V-set $A \in \mathcal{F}_\kappa$, there exists some $\beta < \kappa$ such that $b_\kappa^\tau \setminus \beta \subseteq A$.

6.2. The forcing $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$

Our goal now is to introduce a poset $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ which is equivalent to $\mathbb{C}_{\mathcal{F}_\kappa}$, and further has a dense subforcing $\mathbb{C}^*_{\mathcal{F}_\kappa}$, which is of the Prikry-type, and its direct extension order is λ-closed. We first introduce the poset $\mathbb{C}^*_{\mathcal{F}_\kappa}$, obtained from $\mathbb{Q}^*_{\kappa,\lambda}$ and $\mathbb{C}_{\mathcal{F}_\kappa}$.

Recall that $G * H$ is V generic for $\mathbb{P} * \mathrm{Coll}(\lambda, <\kappa)$. Working in $V[G*H]$, we consider the two-step iterations $\mathbb{Q}^*_{\kappa,\lambda} * \mathbb{C}_{\mathcal{F}_\kappa}$ consisting of conditions $(q, \underset{\sim}{x})$ so that $q \Vdash_{\mathbb{Q}^*_{\kappa,\lambda}} \underset{\sim}{x} \in \check{\mathbb{C}}_{\mathcal{F}_\kappa}$. Note that when forcing with $\mathbb{C}_{\mathcal{F}_\kappa}$ over a $V[G*H]$-generic extension by $\mathbb{Q}^*_{\kappa,\lambda}$, we require that the bounded closed sets $c \subseteq \kappa$ in the conditions $x = \langle c, A \rangle \in \mathbb{C}_{\mathcal{F}_\kappa}$ are actually ground model sets, from $V[G*H]$. In particular, for every condition $(q, \underset{\sim}{x})$, there exists an extension $q' \geq q$ and a pair $x \in \mathbb{C}_{\mathcal{F}_\kappa}$ so that $q' \Vdash \underset{\sim}{x} = \check{x}$.

Let $b^\lambda_\kappa \subseteq \kappa$ be a $\mathbb{Q}^*_{\kappa,\lambda}$ generic club in κ. We know that $\mathrm{ot}(b^\lambda_\kappa) = \lambda$ and that b^λ_κ is almost contained in every set $A \in \mathcal{F}_\kappa$. Working in a $\mathbb{Q}^*_{\kappa,\lambda}$ generic extension $V[G*H*b^\lambda_\kappa]$ of $V[G*H]$, we see that for every condition $x = \langle c, A \rangle$ in $\mathbb{C}_{\mathcal{F}_\kappa}$, there exists some $\beta \in A \setminus (\max c + 1)$ such that $x' = \langle c', A \rangle$, with $c' = c \cup \{\beta\}$, extends x and satisfies that $b^\lambda_\kappa \setminus (\max c' + 1) \subseteq A$.

Definition 6.8 ($\mathbb{C}^\lambda_{\mathcal{F}_\kappa}$). Working in a $\mathbb{Q}^*_{\kappa,\lambda}$ generic extension $V[G*H*b^\lambda_\kappa]$ of $V[G*H]$, let $\mathbb{C}^\lambda_{\mathcal{F}_\kappa}$ denote the subset of $\mathbb{C}_{\mathcal{F}_\kappa}$, consisting of conditions $x' = \langle c', A' \rangle$ so that $\max(c') \in b^\lambda_\kappa$, and $b^\lambda_\kappa \setminus (\max(c') + 1) \subseteq A'$.

It follows from the above that $\mathbb{C}^\lambda_{\mathcal{F}_\kappa}$ is a dense subset of $\mathbb{C}_{\mathcal{F}_\kappa}$. Since $b^\lambda_\kappa \subseteq \kappa$ is closed of order-type $\lambda = \mathrm{cf}(\kappa)^{V[G*H*b^\lambda_\kappa]}$, and no sequences of ordinals of length $< \lambda$ are introduced by b^λ_κ, it follows that the restriction of the $\mathbb{C}_{\mathcal{F}_\kappa}$ order to $\mathbb{C}^\lambda_{\mathcal{F}_\kappa}$ is λ-closed.

With this observation, we move back to $V[G*H]$ to define the poset $\mathbb{C}^*_{\mathcal{F}_\kappa}$.

Definition 6.9 ($\mathbb{C}^*_{\mathcal{F}_\kappa}$). Let $\mathbb{C}^*_{\mathcal{F}_\kappa}$ be the two step iteration $\mathbb{C}^*_{\mathcal{F}_\kappa} = \mathbb{Q}^*_{\kappa,\lambda} * \mathbb{C}^\lambda_{\mathcal{F}_\kappa}$. We define the direct extension ordering $\leq^*$ of $\mathbb{C}^*_{\mathcal{F}_\kappa}$ to be the extension of the usual direct extension order of $\mathbb{Q}^\lambda_\kappa$ with the standard order on the second $\mathbb{C}^\lambda_{\mathcal{F}_\kappa}$ component.

Corollary 6.10. $\mathbb{C}^*_{\mathcal{F}_\kappa}$ *is a dense subforcing of* $\mathbb{Q}^*_{\kappa,\lambda} * \mathbb{C}_{\mathcal{F}_\kappa}$ *which satisfies the Prikry Property, and its direct extension order is* λ-*closed.*

Note that for every dense subset D of $\mathbb{C}_{\mathcal{F}_\kappa}$ and a condition $(q, \underset{\sim}{x}) \in \mathbb{C}^*_{\mathcal{F}_\kappa}$, there exists a direct extension $(q^*, \underset{\sim}{x}^*) \geq^* (q, \underset{\sim}{x})$ such that $q^* \Vdash \underset{\sim}{x}^* \in \check{D}$. Similarly, it is clear that the set of conditions $(q', \check{x}') \in \mathbb{C}^*_{\mathcal{F}_\kappa}$, for which the second component is a canonical name $\check{x}'$ of a condition $x' \in \mathbb{C}_{\mathcal{F}_\kappa}$, is dense in $\mathbb{C}^*_{\mathcal{F}_\kappa}$. The map $(q', \check{x}') \mapsto x'$ defined on this dense set naturally induces a forcing projection π from $\mathbb{C}^*_{\mathcal{F}_\kappa}$ to the boolean completion of $\mathbb{C}_{\mathcal{F}_\kappa}$. This projection sends a condition of the form $\langle q, \underset{\sim}{x} \rangle$ to the join of the collection of all $y \in \mathbb{C}_{\mathcal{F}_\kappa}$ such that there is some extension of q, q' that forces $\underset{\sim}{x} = \check{y}$.

Next, we follow Gitik's machinery from [13], to form a Prikry-type forcing notion $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ which is equivalent to $\mathbb{C}_{\mathcal{F}_\kappa}$, from $\mathbb{C}^*_{\mathcal{F}_\kappa}$.

Definition 6.11 ($\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$). We define a Prikry-type forcing notion $(\bar{\mathbb{C}}_{\mathcal{F}_\kappa}, \leq', \leq^*)$ as follows.

○ $\bar{\mathbb{C}}_{\mathcal{F}_\kappa} = \mathbb{C}^*_{\mathcal{F}_\kappa}$,
○ the partial ordering $\leq'$ is defined by $p' \geq' p$ if $\pi(p') \geq \pi(p)$ and
○ $\leq^*$ is taken to be the same direct extension order of $\mathbb{C}^*_{\mathcal{F}_\kappa}$

It is immediate from the definition that $(\bar{\mathbb{C}}_{\mathcal{F}_\kappa}, \leq')$ is equivalent as a forcing notion to $(\mathbb{C}_{\mathcal{F}_\kappa}, \leq)$ and that the direct extension order $\leq^*$ of $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ is λ-closed. To show that $(\bar{\mathbb{C}}_{\mathcal{F}_\kappa}, \leq', \leq^*)$ satisfies the Prikry Property, it suffices to verify that for every statement σ in the forcing language of $\mathbb{C}_{\mathcal{F}_\kappa}$ and every condition $p \in \bar{\mathbb{C}}_{\mathcal{F}_\kappa}$, there is a direct extension $p^* \geq^* p$ such that $\pi(p^*)$ decides σ. Indeed, defining $D_0 = \{p' \in \mathbb{C}^*_{\mathcal{F}_\kappa} \mid \pi(p') \Vdash \sigma\}$ and $D_1 = \{p' \in \mathbb{C}^*_{\mathcal{F}_\kappa} \mid \pi(p') \Vdash \neg\sigma\}$, it is clear that $D_0 \cup D_1$ is dense in $\mathbb{C}^*_{\mathcal{F}_\kappa}$ and that a generic filter G^* of $\mathbb{C}^*_{\mathcal{F}_\kappa}$ will have a nontrivial intersection with exactly one of the two sets. Let $\sigma^* \colon G^* \cap D_0 \neq \emptyset$. Then σ^* is a statement for the forcing language of $\mathbb{C}^*_{\mathcal{F}_\kappa}$. Moreover, it is clear from our construction that for a condition $p^* \in \mathbb{C}^*_{\mathcal{F}_\kappa}$ which decides σ^*, we have that $p^* \Vdash \sigma^*$ implies

that $\pi(p^*) \Vdash \sigma$, and $p^* \Vdash \neg\sigma^*$ implies that $\pi(p^*) \Vdash \sigma$. Since $\mathbb{C}^*_{\mathcal{F}_\kappa}$ satisfies the Prikry Property, every condition p has a direct extension $p^* \geq^* p$ which decides σ^*.

We note that, similarly to $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}_{\mathcal{F}_\kappa}$, the forcing $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}^*_{\mathcal{F}_\kappa}$ is cone homogeneous. In most applications of homogeneity, moving to an equivalent forcing does not change the main properties of its iterations. In order to apply the results from section 8.1 and argue that the iteration $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ is cone homogeneous, we need to verify that this poset meets the assumptions of Lemma 8.5.

Lemma 6.12. *Denote $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}^*_{\mathcal{F}_\kappa}$ by $\mathbb{W}$, and its regular and direct extension orders by $\leq_\mathbb{W}$ and $\leq^*_\mathbb{W}$, respectively.*

For every $w_0, w_1 \in \mathbb{W}$ there are direct extensions w_0^*, w_1^* of w_0, w_1, respectively, and a cone isomorphism $\varphi \colon \mathbb{W}/w_0^* \to \mathbb{W}/w_1^*$ which respects the direct extension order $\leq^*_\mathbb{W}$.

In particular, the forcing $\mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \bar{\mathbb{C}}_{\mathcal{F}_\kappa}$ is cone homogeneous.

We observe that assuming the coherent sequence $\mathcal{U}$ (by which $\mathbb{W} = \mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}^*_{\mathcal{F}_\kappa}$ is defined) is ordinal definable in V, then the statement of the lemma guarantees that $\mathbb{W}$ satisfies the requirements of the iterated poset $\mathbb{Q}_\alpha$ from Lemma 8.5.

Proof. Let us start with the last assertion. Since the identity is a projection from $\mathbb{C}^*_{\mathcal{F}_\kappa}$ to $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$, an isomorphism of a cone of elements in $\mathbb{C}^*_{\mathcal{F}_\kappa}$ naturally induces an isomorphism of the corresponding cone in $\bar{\mathbb{C}}_{\mathcal{F}_\kappa}$.

Let $w_0 = \langle p_0, c_0, \langle q_0, x_0 \rangle \rangle$ and $w_1 = \langle p_1, c_1, \langle q_1, x_1 \rangle \rangle$, where the conditions $q_0 = \langle t_0, T_0, Q_0 \rangle$ and $q_1 = \langle t_1, T_1, Q_1 \rangle$ belong to $\mathbb{Q}^*_{\kappa, \lambda}$.

By [5, Theorem 4.6] applied to the iteration $\mathbb{P} * \mathbb{Q}_{\kappa, \lambda}$, there are direct extensions $\langle p_0^*, (t_0, T) \rangle$ of $\langle p_0, (t_0, T_0) \rangle$, and $\langle p_1^*, (t_1, T) \rangle$ of $\langle p_1, (t_1, T_1) \rangle$, with a common top tree T, and an isomorphism ψ between the cone below $\langle p_0^*, (t_0, T) \rangle$ and the cone below $\langle p_1^*, (t_1, T) \rangle$. We record here that the map ψ constructed in the proof of [5, Theorem 4.6] satisfies two additional properties. First, it does not make any changes to the $\mathcal{F}_\kappa$-trees S appearing in conditions $p^\frown \langle s, S \rangle \in \mathbb{P} * \mathbb{Q}_{\kappa, \lambda}$. Second, it respects the direct extension order of $\mathbb{P} * \mathbb{Q}_{\kappa, \lambda}$.

Next, we move to examine the Levy collapse condition and the suitable functions in the conditions from $\mathbb{W}$. Since the collapsing forcing $\mathrm{Coll}(\lambda, < \kappa)$ is evaluated in the generic extension by $\mathbb{P}$, ψ naturally acts also on the $\mathbb{P}$-names c_0 and Q_0 that appear in w_0. As usual, we denote the resulting names by c_0^ψ and Q_0^ψ. Let $\tau_\emptyset$ be a $\mathbb{P}$-name of an automorphism of the Levy collapse poset which maps an extension c_0' of c_0^ψ to an extension c_1^* of c_1, and define $c_0^* = (c_0')^{\psi^{-1}}$. Note that since c_i forces $Q_i(\emptyset) \in \underset{\sim}{H}$, we may extend $Q_0(\emptyset), Q_1(\emptyset)$ to $Q_0^*(\emptyset), Q_1^*(\emptyset)$ so that $c_i = Q_i^*(\emptyset)$ for $i = 0, 1$. Next, for each $s \in T$ and $i = 0, 1$, define $\mathrm{dom}_1(Q_i(s)) = \{\alpha < \kappa \mid Q_i(s) \restriction \lambda \times \{\alpha\} \neq \emptyset\}$ and $\rho_s^i = \sup(\mathrm{dom}_1(Q_i(s)))$. Set $\rho_s = \max(\rho_s^0, \rho_s^1)$. By moving to a direct extension tree T^* of T, we may assume that for every $s \in T^*$ and $\nu \in \mathrm{succ}_{T^*}(s)$, $\pi_0^{t^\frown s}(\nu) > \rho_s$ where the projection is computed in both generic extensions. This leaves enough space between the conditions $Q_i(s), Q_i(s^\frown \langle \nu \rangle), i = 0, 1$, to define automorphisms taking an extension $Q_0^*(s)$ of $Q_0(s)$ to an extension $Q_1^*(s)$ of $Q_1(s)$, without conflicting with $Q_i(s^\frown \langle \nu \rangle), i = 0, 1$. We can therefore define by induction on the lexicographic order $<_{lex}$ on T^* (where two sequences are compared from their top elements down) automorphisms $\tau_s, s \in T^*$, of $\mathrm{Coll}(\lambda, < \kappa)$, and collapse extensions $Q_i^*(s) \geq Q_i(s)$, $i = 0, 1$, with the following properties. For all $s \in T^*$,

○ τ_s is supported in $\mathrm{Coll}(\lambda, < \rho_s)$,[14]
○ $\tau_s((Q_0^*)^\psi(s)) = Q_1^*(s)$,
○ If $s' \in T_s^*$, then $\tau_{s'} \restriction \mathrm{Coll}(\lambda, < \pi_0^{t^\frown s}(s)) = \tau_s$,
○ If $\bar{s} \in T^*$ and $b_{t_0^\frown s} = b_{t_0^\frown \bar{s}}$, then $\tau_s = \tau_{\bar{s}}$.

Let $\bar{\mathbb{W}} = \mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{Q}^*_{\kappa, \lambda}$ be the initial forcing iteration of

$$\mathbb{W} = \mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * \mathbb{C}^*_{\mathcal{F}_\kappa} = \mathbb{P} * \mathrm{Coll}(\lambda, < \kappa) * (\mathbb{Q}^*_{\kappa, \lambda} * \mathbb{C}^\lambda_\kappa).$$

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[14]i.e., for every $p \in \mathrm{Coll}(\lambda, < \kappa)$, if $p = p_0 \cup p_1$ where $p_0 = p \restriction \lambda \times \rho_s$, then $\tau_s(p) = \tau_s(p_0) \cup p_1$.

Let $w_0 \upharpoonright \bar{\mathbb{W}} = \langle p_0, c_0, \langle t_0, T_0, Q_0 \rangle \rangle$ and $w_1 \upharpoonright \bar{\mathbb{W}} = \langle p_1, c_1, \langle t_1, T_1, Q_1 \rangle \rangle$ be the restrictions of w_0, w_1 to $\bar{\mathbb{W}}$, and consider their direct extensions $\bar{w}_o, \bar{w}_1$ in $\bar{\mathbb{W}}$, defined by $\bar{w}_i = \langle p_0^*, c_0^*, \langle t_0, T^*, Q_0^* \rangle \rangle$, $i = 0, 1$.

Our choice of cone isomorphism ψ for $\mathbb{P} * \mathbb{Q}_{\kappa,\lambda}$, together with the collection of Levy-Collapse automorphisms $\vec{\tau} = \{\tau_s\}_{s \in T^*}$, naturally induces a function $\bar{\varphi}$ on the cone $\bar{\mathbb{W}}/\bar{w}_0$, defined as follows. For a condition $\bar{w} = \langle p, c, \langle s, S, Q \rangle \rangle$, we set $\bar{\varphi}(\bar{w}) = \langle p', c', \langle s', S, Q' \rangle \rangle$ to be:

$$\langle p', \langle s', S \rangle \rangle = \psi(\langle p', \langle s, S \rangle \rangle), \quad c' = \tau_s(c^\psi), \quad Q'(s') = \tau_{s \frown s'}(Q(s'))$$

We claim that $\langle p', c', \langle s', S, Q' \rangle \rangle$ is a condition in $\bar{\mathbb{W}}$. First, it is immediate from our choice of p, c that $\langle p', c' \rangle \in \mathbb{P} * \mathrm{Coll}(\lambda, < \kappa)$. It therefore remains to verify that $\langle s', S, Q' \rangle$ is forced by $\langle p', c' \rangle$ to be a condition in $\mathbb{Q}^*_{\lambda, <\kappa}$. The fact that $\langle s', S, Q' \rangle$ satisfies requirements (1) and (2) of definition 6.3 is immediate. To verify the coherence requirement (3) of definition 6.3, we note that Q' is forced by p' to be (s, S)-suitable. Indeed, it follows from our choice of $\tau_{s'}$ that its support is bounded below the projection $\pi_0^{s \frown s'}(v)$, for any $v \in \mathrm{succ}_{T^*}(s')$. Property (4) follows from the fact that the statement "$c \Vdash_{\mathrm{Coll}(\lambda,<\kappa)} Q(\emptyset) \in \underline{H}$" is forced by $p \in \mathbb{P}$, which implies that $\tau_\emptyset(c^\psi) \Vdash \tau_\emptyset(Q(\emptyset)^\psi) \in \tau_\emptyset(\underline{H}^\psi)$ is forced by p'. This, combined with definition of c' and Q', and the fact that the name $\underline{H}$ is a fixed point of both $\tau_\emptyset$ and ψ, guarantees that requirement (4) is satisfied. Next, (s', S, Q') satisfies requirement (5) of definition 6.3 by a similar argument to the previous one, using the fact that (s, S, Q) satisfies property (5) together with the last property listed above for $\{\tau_s\}_{s \in T}$. Having verified that $\langle p', c', \langle s', S, Q' \rangle \rangle$ is a condition in $\bar{\mathbb{W}}$, it is straightforward to check that it extends $\bar{w}_1$ and thus $\bar{\varphi}: \bar{\mathbb{W}}/\bar{w}_0 \to \bar{\mathbb{W}}/\bar{w}_1$ is a well defined function. In order to show that it is cone isomorphism, we need to show that it is order-preserving.

Let us remark that the automorphism ψ modifies the values of b_s for $s \in T^*$ by changing the value of their initial segments. Since those initial segments do not affect the definition of τ_{b_s}, we will ignore it and write always b_s instead b_s^ψ.

Let $\bar{w}_1 = \langle p_1, c_1, \langle s_1, S_1, Q_1 \rangle \rangle$, $\bar{w}_2 = \langle p_2, c_2, \langle s_2, S_2, Q_2 \rangle \rangle$ be a pair of conditions in the cone above $\bar{w}_0$. We need to show that $\bar{\varphi}(\bar{w}_1) \le \bar{\varphi}(\bar{w}_2)$ if and only if $\bar{w}_1 \le \bar{w}_2$.

For direct extensions, this is clear, as the tree S_1 does not move under $\bar{\varphi}$. Let us assume that $\bar{w}_2$ is a one-point extension of $\bar{w}_1$, by the point $\langle v \rangle$. By moving to a dense subset, $c_2 \ge c_1, Q_1(\langle v \rangle)$ and $b_{s_2} = b_{s_1 \frown \langle v \rangle}$. Let us apply $\bar{\varphi}$ on $\bar{w}_1, \bar{w}_2$. The trees S_1 and S_2 do not move, so we must verify that $\langle v \rangle$ is still a legitimate choice for an one-point extension of $\bar{\varphi}(\bar{w}_1)$. Indeed, $\tau_{b_{s_2}}(c_2)$ is (by the definition of $\tau_{b_{s_2}}$) stronger than $\tau_{\langle v \rangle}(Q_1(\langle v \rangle))$. Thus, we conclude that $\bar{\varphi}(\bar{w}_2)$ is an one-point extension of $\bar{\varphi}(\bar{w}_2)$ by $\langle v \rangle$. The other direction is the same.

Finally, to obtain a desirable cone isomorphism φ for $\mathbb{W} = \bar{\mathbb{W}} * \mathbb{C}_\kappa^\lambda$, it remains to extend $\bar{\varphi}$ to the final additional components $\underline{x}_0, \underline{x}_1$ of $\mathbb{C}_{\mathcal{F}_\kappa}^\lambda$. The proof Lemma 4.9 shows that there are $\bar{\mathbb{W}}$-names $\underline{y}_0', \underline{y}_1$ of extensions of $\underline{x}_0^{\bar{\varphi}}, \underline{x}_1$ respectively, and a name of a cone isomorphism $\sigma: \mathbb{C}_\kappa^\lambda/\underline{y}_0' \to \mathbb{C}_\kappa^\lambda/\underline{y}_1$. Accordingly, we set $\underline{y}_0 = (\underline{y}_0')^{\bar{\varphi}^{-1}}$ and define direct extensions $w_0^* \ge^* w_0, w_1^* \ge^* w_1$ and a map $\varphi: \mathbb{W}/w_0^* \to \mathbb{W}/w_1^*$ by $w_i^* = \bar{w}_i \frown \underline{y}_i$ and

$$\varphi(\langle \bar{w}, \underline{y} \rangle) = \langle \bar{\varphi}(\bar{w}), \sigma(\underline{y}^{\bar{\varphi}}) \rangle.$$

The fact that w_0^*, w_1^*, φ satisfies the result stated in the lemma is an immediate consequence of the fact that $\bar{w}_0, \bar{w}_1, \bar{\varphi}$ satisfies similar properties for $\bar{\mathbb{W}}$ and our choice of φ, y_0, y_1. $\qquad\square$

7. Strong measurability at successors of singulars

Suppose that $V = \mathrm{HOD}$, κ is a supercompact cardinal and $\lambda > \kappa$ is a measurable cardinal with a normal measure $\mathcal{U}$. We would like to construct a cone homogeneous poset in V which will collapse λ to be the successor of κ, change the cofinality of κ to ω and add a closed unbounded subset of λ whose restriction to the set of $\{\alpha < \lambda \mid \alpha$ is regular in $V\}$ is almost contained in every set $A \in \mathcal{U}$.

It is natural to attempt obtaining this result by starting with an indestructible supercompact cardinal κ, and forcing with a Levy collapse of λ to κ^+ followed by a Prikry forcing at κ and a club forcing at λ.

The difficulty with this approach is in its second step, where the choice of the measure on κ depends on the generic filter for the Levy collapse and might lead to a Prikry generic sequence which will introduce to HOD information about the collapse of λ to κ^+, and in particular prevent from HOD to witness that λ is a measurable cardinal.

Instead, our approach will be based on recent use of the supercompact extender based forcing, introduced by Merimovich ([21]). Given a supercompact cardinal κ, we derive a (κ, λ)-supercompact extender E from a supercompact embedding $j : V \to M$ for which $^\lambda M \subseteq M$. Let $\mathbb{P}_E$ be the supercompact extender based forcing associated to the extender E of [21]. The conditions of $\mathbb{P}_E$ are pairs of the form $\langle f, T \rangle$ where f is roughly a condition in the Cohen forcing and T is a tree, with large splittings. We denote by $\mathbb{P}_E^*$ the Cohen part.

The forcing $\mathbb{P}_E$ preserves λ and singularizes all the regular cardinals in the interval $[\kappa, \lambda)$. We will follow the definitions and notations of [21]. In [15], Gitik and Merimovich show that this forcing is weakly homogeneous (and therefore cone homogeneous).

Let $\mathcal{U}$ be a normal measure on λ in the ground model. We would like to force a club to diagonalize $\mathcal{U}$ relative to the set of V-regular cardinals below λ. Note that the ordinals of uncountable cofinality below λ in the extender based forcing extension are of measure zero in $\mathcal{U}$. Therefore, our club shooting poset has to allow V-singular ordinals as well. Moreover, since the set of previous inaccessible cardinals below λ does not reflect at its complement, it is impossible for the generic club to avoid ground model singular cardinals of countable cofinality. Thus, we restrict our club forcing poset to diagonalize $\mathcal{U}$ only relative to the set of the regular cardinals in V. To make this precise, we denote by Sing the set of all ground model singular cardinals below λ and define $\bar{\mathcal{U}} = \{\text{Sing} \cup A \mid A \in \mathcal{U}\}$. We force with the poset $\mathbb{C}_{\bar{\mathcal{U}}}$, consisting of pairs (c, B) where $c \subseteq \lambda$ is a closed bounded set and $B \in \bar{\mathcal{U}}$. The extension order is as in the previous section.

We start by recalling a fundamental and useful fact, which lies in the heart of the proof of the Prikry Property of $\mathbb{P}_E$.

Lemma 7.1. *Let M be an elementary submodel of H_χ for some large χ such that $M \cap \lambda = \delta \in \lambda$ is inaccessible cardinal and $M^{<\delta} \subseteq M$. Let $p \in M \cap \mathbb{P}_E$.*

Then, there is a condition $f^* \in \mathbb{P}_E^*$ which is M-generic (namely, it belongs to every dense open subset of $\mathbb{P}_E^*$ in M) and $\text{dom } f^* = M \cap \lambda$. Moreover, if $p^* = \langle f^*, T \rangle$ is a condition in $\mathbb{P}_E$, then there is $T^* \subseteq T$, $E(f^*)$-large such that $T^* \subseteq M$ and $D \in M$ is a dense open subset of $\mathbb{P}_E$, then there is a natural number n such that for every $\langle v_0, \ldots, v_{n-1} \rangle$ in the n-th level of $T*$, $p^*_{\langle v_0, \ldots, v_{n-1} \rangle} \in D$.

Proof. The first claim follows from the closure of $\mathbb{P}_E^*$. Let us focus on the second part.

Let f^* be as in the lemma. Let $D \in M$ be dense open. For each $\langle v_0, \ldots, v_{n-1} \rangle \in M$ and for each $g \in \mathbb{P}_E^* \cap M$, we can ask whether there is a condition $q \in D$ of the form $\langle h, S \rangle \in D$ such that $h \geq^* g_{\langle v_0, \ldots, v_{n-1} \rangle}$. The set of conditions that decide this statement is dense open and definable in M, and thus f^* decides whether there is such extension or not (for each possible $\langle v_0, \ldots, v_{n-1} \rangle$). Let $D_{\vec{v}}$ be this set, and let us split it into two parts $D_{\vec{v}}^0 \cup D_{\vec{v}}^1$ according to the decision, where conditions in $D_{\vec{v}}^0$ are direct extensions that enter D after the nondirect extension.

Let $p^* = \langle f^*, T \rangle$. Since a typical point v in a measure one tree T, associate with the measures $E(f^*)$ is a finite sequence of elements contained in M each has size $|v| < \kappa$, we may assume that $T \subseteq M$. There is an extension $q \geq p^*$ in D. By the definition of the order of $\mathbb{P}_E$, q is obtained by taking first some Prikry extension and then a direct extension, and therefore the Prikry extension is done using some $\vec{v} \in M$. Thus, for this specific Prikry extension, $f^* \in D_{\vec{v}}^0$. We conclude that already $p_{\vec{v}}^* \in D$.

We can now shrink T in order to stabilize the length of the extensions that enter D. □

Lemma 7.2. *$\bar{\mathcal{U}}$ extends to a λ-complete filter in the generic extension by $\mathbb{P}_E$.*

Proof. Assume that this is not the case. Since κ is singular, the closure of $\bar{\mathcal{U}}$ must drop to some cardinal $\rho < \kappa$. Let $\langle A_i \mid i < \rho \rangle$ be a sequence of names of elements in $\bar{\mathcal{U}}$ which are forced to have non-measure one intersection.

Using the strong Prikry Property, we can find a sequence of direct extensions p_i and natural numbers n_i such that any n_i-length Prikry extension of p_i decides the value of $\underset{\sim}{A}_i$. Since there are fewer than λ many such extensions, we can find a set $B_i \in \bar{\mathcal{U}}$ such that $p_i \Vdash B_i \subseteq \underset{\sim}{A}_i$. In particular, $p_\rho \Vdash \bigcap B_i \subseteq \bigcap \underset{\sim}{A}_i$, but $\bigcap B_i \in \bar{\mathcal{U}}$. □

Lemma 7.3. $\mathbb{C}_{\bar{\mathcal{U}}}$ *is λ-distributive in the generic extension by* $\mathbb{P}_E$.

Proof. Since κ is singular in the extension by $\mathbb{P}_E$, it is enough to show that the forcing $\mathbb{C}_{\bar{\mathcal{U}}}$ is ρ-distributive for every $\rho < \kappa$.

We first work in V. Let $\vec{D} = \langle \underset{\sim}{D}_i \mid i < \rho \rangle$ be a sequence of $\mathbb{P}_E$-names for dense open subsets of $\mathbb{C}_{\bar{\mathcal{U}}}$, $\rho < \kappa$. Let $\langle p, q \rangle$ be a condition in $\mathbb{P}_E * \mathbb{C}_{\bar{\mathcal{U}}}$. Let us define an increasing sequence of models $\langle M_i \mid i < \rho \rangle$ such that:

- $\vec{D}, \mathbb{P}_E, \mathbb{C}_{\bar{\mathcal{U}}} \in M_0$.
- $M_i \prec H_\chi$ for some large χ, $M_i \cap \lambda = \delta_i \in \lambda$ inaccessible.
- $M_i^{<\delta_i} \subseteq M_i$ and $\delta_i \in \bigcap\{A \in \mathcal{U} \cap M_i\}$.
- $\langle M_j \mid j < i \rangle \in M_i$.

This chain of models can be easily obtained using the same argument as in Lemma 3.1.

Next, let us pick by induction, for each $i < \rho$, an M_i-generic condition $f_i^* \in \mathbb{P}_E^*$ such that $f_i^* \in M_{i+1}$ and $f_i^* \subseteq f_j^*$ for $i < j$. We will define a sequence of names $\underset{\sim}{q}_i$ and a sequence of conditions p_i such that:

- $p_i = \langle f_i^*, T_i \rangle \in M_{i+1}$, $\underset{\sim}{q}_i \in M_i$.
- $p_{i+1} \Vdash \underset{\sim}{q}_{i+1} \in \underset{\sim}{D}_i$.
- The sequence of conditions p_i is $\leq^*$-increasing. Let p_ρ be their limit.
- p_ρ forces that the conditions $\underset{\sim}{q}_i$ are increasing and they have a limit $\underset{\sim}{q}_\rho$.

In M_i, let D_i' be the dense open set in $\mathbb{P}_E$ of all extensions of p_i that force for some condition $q = (c^q, B^q) \geq \underset{\sim}{q}_i$ to be in $\underset{\sim}{D}_i$, and decide its maximum and its large set B^q from $\bar{\mathcal{U}}$. By applying Lemma 7.1 inside M_i, we conclude that there is an $E(f_i^*)$-large tree $T_i \subseteq M_i$ and a natural number n_i such that, for the condition $p_i = \langle f_i^*, T_i \rangle$, for every $\vec{v} \in Lev_{n_i}(T_i)$, $(p_i)_{\vec{v}} \in D_i'$. In particular, it picks a condition $q_{i+1,\vec{v}} \geq \underset{\sim}{q}_i$ from $\mathbb{C}_{\bar{\mathcal{U}}}$, which is going to be in M_i. Since this condition is in M_i, it is going to be bounded below δ_i, and its large set belongs to $\bar{\mathcal{U}} \cap M_i$.

Note that the collection of all n-step extensions of a fixed condition in $\mathbb{P}_E$ is always a maximal antichain above this condition and thus, we can define $\underset{\sim}{q}_{i+1}'$ to be equal to $q_{i+1,\vec{v}}$ above $(p_i)_{\vec{v}}$, and trivial below any condition which is incompatible with p_i. Finally, we define $\underset{\sim}{q}_{i+1}$ to be the extension of $\underset{\sim}{q}_{i+1}'$ by the single ordinal δ_i. By the construction, this is indeed an extension, as $\delta_i \in B$ for all $B \in \bar{\mathcal{U}} \cap M_i$.

At limit steps, we define $\underset{\sim}{q}_i$ to be the limit of previous conditions. This is possible since the filter $\bar{\mathcal{U}}$ is still λ-complete and since the maximal element of the closed set in $\underset{\sim}{q}_j$ is forced to be δ_j. Therefore, the maximal element of $\underset{\sim}{q}_i$ is δ_i which is singular strong limit cardinal in the limit case. □

Lemma 7.4. *Let $B' \in \mathcal{U}$. Then, B' is stationary in $\mathbb{P}_E * \mathbb{C}_{\bar{\mathcal{U}}}$.*

Proof. Let $\underset{\sim}{C}$ be a name for a club. We show that every condition $q \in \mathbb{C}_{\bar{\mathcal{U}}}$ has an extension which forces that $\underset{\sim}{C} \cap B' \neq \emptyset$. Working in V, let $M \prec H_\chi$, such that $M \cap \lambda = \delta$, $\mathbb{P}_E, \mathbb{C}_{\bar{\mathcal{U}}}, \underset{\sim}{C}, q, B' \in M$, and $\delta \in B'$ is inaccessible. Moreover, let us assume that M is obtained as a union of a chain of models of length δ, M_i, such that $M_i \cap \lambda = \delta_i$ and $M_{i+1}^{<\delta_{i+1}} \subseteq M_{i+1}$ and $\delta_{i+1} \in \bigcap(\mathcal{U} \cap M_{i+1})$.

For each i, let f_i^* be M_i-generic for $\mathbb{P}_E^*$, such that $f_i^* \subseteq f_j^*$ for $i < j$. Let $f^* = \bigcup f_i^*$.

Let $G \subseteq \mathbb{P}_E$ be a generic filter that contains a condition $p^* = \langle f^*, A \rangle$, $A \subseteq M$. In $V[G]$, cf $\delta = \omega$. Let $\langle \delta_n \mid n < \omega \rangle$ be a cofinal sequence in δ. For each n, for sufficiently large $\xi < \delta$, M_ξ contains the dense set of conditions in $\mathbb{P}_E$ that decide on some condition $q \in \mathbb{C}_{\bar{\mathcal{U}}}$ that forces some ordinal $\gamma_n \geq \delta_n$ to be in $\underset{\sim}{C}$.

Following the same arguments as in the previous lemma, we can define a condition q_n by going over some maximal antichain. The maximum of the closed set of q_n is always δ_{n+1}. Finally, the sequence of conditions q_n^G has an upper bound, by attaching δ on top of the union. Let q_ω be the upper bound. Clearly, q_ω forces $\delta \in \underset{\sim}{C}$, as wanted. □

Finally, the following proposition finishes the proof of Theorem 1.6.

Proposition 7.5. *Let κ be λ-supercompact, where λ is measurable. Then, there is a generic extension in which* cf $\kappa = \omega$, κ *is a cardinal, $\lambda = \kappa^+$ and it is $((S^\lambda_{reg})^V, 1)$-strongly measurable cardinal.*

We can now finish the proof of Theorem 1.6.

Proof of Theorem 1.6. The iteration $\mathbb{P}_E * \mathbb{C}_{\bar{\mathcal{U}}}$ is cone homogeneous as an iteration of two cone homogeneous, ordinal definable forcing notions. Since $\mathbb{P}_E$ preserves cardinals below κ and $\geq \lambda$ and $\mathbb{C}_{\bar{\mathcal{U}}}$ preserves cardinals, the result follows. The set S^λ_{reg} is stationary by Lemma 7.4.   □

The result that we obtain for the successor of a singular cardinal is weaker than the result for a successor of a regular cardinal. The reason is that, in order to get the closed unbounded filter to be sets from the intersection of some ground model normal measures, we will have to obtain a situation in which the regular cardinals between the supercompact cardinal κ and the measurable cardinal λ are going to change cofinalities into values which differ from the cofinality of κ in the generic extension. This is also the reason that such a method cannot work for getting an ω-strongly measurable successor of a singular cardinal of uncountable cofinality.

We remark that Woodin in [39], proved that it is consistent relative to the large cardinal axiom I_0 that a successor of a singular cardinal is ω-strongly measurable.

Question 7.6. Is it consistent that there is an ω-strongly measurable cardinal λ^+, where cf $\lambda > \omega$ is a limit cardinal?

Question 7.7. Is it consistent that there is a cardinal λ^+, where cf $\lambda > \omega$ is a limit cardinal and $(S^{\lambda^+}_{reg})^{\text{HOD}}$ contains a club in V?

8. Appendix - homogeneity

In this section we review some basic facts related to homogeneity and develop some basic tools in order to preserve homogeneity of iterations of Prikry-type forcings.

8.1. *Homogeneity and* HOD

When dealing with HOD, we would like to modify the universe (via forcing) while not adding objects to HOD. The main method to obtain this is to force with posets which satisfy a certain weak homogeneity property. The main results of this work will focus on the notion of cone homogeneous posets.

Definition 8.1. We say that a poset $\mathbb{P}$ is **cone homogeneous** if, for every $p, q \in \mathbb{P}$, there are extensions p^*, q^* of p, q, respectively, and a forcing isomorphism φ from the cone $\mathbb{P}/p^*$ (i.e., of conditions extending p^*) to the cone $\mathbb{P}/q^*$.

This notion can also be found under different names in the literature concerning weak forms of homogeneity. Our terminology follows Dorbinen and Friedman, [9] for the most part. It is easy to see that cone homogeneous posets satisfy most standard properties of homogeneous posets concerning ordinal definability sets. In particular, the following well-known result holds.

Fact 8.2 (Levy, [19]). If $\mathbb{P}$ is cone homogeneous and belongs to HOD, and $G \subseteq \mathbb{P}$ is generic over V, then $\text{HOD}^{V[G]} \subseteq \text{HOD}^V$.

If φ is an isomorphism of two cones $\mathbb{P}/p_0$ and $\mathbb{P}/p_1$, and σ is a $\mathbb{P}/p_0$ name, then by recursively applying φ, we obtain a $\mathbb{P}/p_1$-name, which we denote by σ^φ.

Let $\mathbb{P} = \mathbb{P}_\kappa$ where $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ is an iteration of cone homogeneous posets $\mathbb{Q}_\alpha$, and moreover, let us assume that all cone automorphisms of $\mathbb{P}_\alpha$ do not modify $\mathbb{Q}_\alpha$ as a poset. For simplicity, we may assume that $\mathbb{Q}_\alpha$ and its order are ordinal definable.

Given two conditions $\vec{p} = \langle p_\alpha \mid \alpha < \kappa \rangle, \vec{q} = \langle q_\alpha \mid \alpha < \kappa \rangle$ in $\mathbb{P}$, it is natural to try forming extensions $\vec{p}^* = \langle p_\alpha^* \mid \alpha < \kappa \rangle \geq \vec{p}, \vec{q}^* = \langle q_\alpha^* \mid \alpha < \kappa \rangle \geq \vec{q}$, and an isomorphism $\varphi \colon \mathbb{P}/\vec{p}^* \to \mathbb{P}/\vec{q}^*$ as follows.

By induction on $\beta \leq \kappa$, we attempt defining extensions $\vec{p}^\beta = \langle p_\alpha^* \mid \alpha < \beta \rangle$ of $\vec{p} \restriction \beta$, and $\vec{q}^\beta = \langle q_\alpha^* \mid \alpha < \beta \rangle$ of $\vec{q} \restriction \beta$, and an isomorphism $\varphi_\beta \colon \mathbb{P}_\beta/\vec{p}^\beta \to \mathbb{P}_\beta/\vec{q}^\beta$. Our inductive assumptions further include $\vec{p}^{\beta_1} \restriction \beta_0 = \vec{p}^{\beta_0}, \vec{q}^{\beta_1} \restriction \beta_0 = \vec{q}^{\beta_0}$, and $\varphi_{\beta_1} \restriction \mathbb{P}_{\beta_0}/\vec{p}^{\beta_0} = \varphi_{\beta_0},$[15] for all $\beta_0 < \beta_1$.

For $\beta = 0$, where $\mathbb{P}_0 = \{0_{\mathbb{P}_0}\}$ is a trivial forcing, we take φ_0 to be the identity. At a successor step, assuming $\vec{p}^\beta, \vec{q}^\beta$ and φ_β have been defined, we have that $\varphi_\beta(\vec{p}^\beta) = \vec{q}^\beta$ forces that $p_\beta^{\varphi_\beta}$ and q_β are conditions of the cone homogeneous poset $\mathbb{Q}_\beta$. There are therefore $\mathbb{P}_\beta$-names p_β' and q_β' of extensions of $p_\beta^{\varphi_\beta}$ and q_β, respectively, and a name of a cone isomorphism $\psi_\beta \colon \mathbb{Q}_\beta/p_\beta' \to \mathbb{Q}_\beta/q_\beta'$. We stress that we use the maximality principle and do not extend the conditions $\vec{p}^\beta$ and $\vec{q}^\beta$ in order to determine the values of p_β', q_β' and ψ_β.

Let $p_\beta^* = (p_\beta')^{\varphi_\beta}$ and $q_\beta^* = q_\beta'$. Clearly, $\vec{p}^\beta, \vec{q}^\beta$ force that p_β^*, q_β^* extend p_β, q_β, respectively. We set $\vec{p}^{\beta+1} = \vec{p}^\beta {}^\frown \langle p_\beta^* \rangle, \vec{q}^{\beta+1} = \vec{q}^\beta {}^\frown \langle q_\beta^* \rangle$ and define $\varphi_{\beta+1} \colon \mathbb{P}_{\beta+1}/\vec{p}^{\beta+1} \to \mathbb{P}_{\beta+1}/\vec{q}^{\beta+1}$ by mapping a condition $\vec{r} = \vec{r} \restriction \beta {}^\frown \langle r_\beta \rangle \in \mathbb{P}_{\beta+1}/\vec{p}^{\beta+1}$ to

$$\varphi_{\beta+1}(\vec{r}) = \varphi_\beta(\vec{r} \restriction \beta) {}^\frown \langle \psi_\beta(r_\beta^{\varphi_\beta}) \rangle.$$

It is immediate from our assumption of φ_β and choice of ψ_β that $\varphi_{\beta+1}$ is an isomorphism. Finally, for a limit ordinal $\delta \leq \kappa$, $\vec{p}^\delta$ (similarly $\vec{q}^\delta$) is determined by the requirement $\vec{p}^\delta \restriction \beta = \vec{p}^\beta$ for all $\beta < \delta$ (similarly for $\vec{q}^\delta$), and φ_δ by the requirement $\varphi_\delta \restriction \mathbb{P}_\beta/\vec{p}^\beta = \varphi_\beta$ for all $\beta < \delta$. See [9] for a more detailed proof for the validity of this construction.

We conclude that, for this construction to succeed, the following conditions need to hold for all $\beta \leq \kappa$: (i) $\vec{p}^\beta, \vec{q}^\beta$ are well-defined conditions in $\mathbb{P}_\beta$ which extend $\vec{p} \restriction \beta, \vec{q} \restriction \beta$, respectively, and (ii) φ_β is a well-defined cone isomorphism.

If the construction succeeds throughout all stages $\beta \leq \kappa$, then the final conditions $\vec{p}^* = \vec{p}^\kappa, \vec{q}^* = \vec{q}^\kappa$ and cone isomorphism $\varphi = \varphi_\kappa$ satisfy the required properties. It is easy to see that condition (i) and (ii) may only fail at limit stages $\delta \leq \kappa$, where the precise formation of the iteration (e.g., its support) may prevent $\vec{p}^\delta$ from being a condition in $\mathbb{P}_\delta$. Similarly, the definition of the limit order $\leq_{\mathbb{P}_\delta}$ might prevent the defined map φ_δ from being an isomorphism.

This problem does not occur for finite iteration.

Lemma 8.3 ([9]). *A finite iteration of ordinal definable cone homogeneous forcings is cone homogeneous.*

Since our proof of Theorem 1.4 is based on a construction of a **Magidor Iteration** $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \theta \rangle$ of Prikry-type forcings $(\mathbb{Q}_\alpha, \leq_{\mathbb{Q}_\alpha}, \leq^*_{\mathbb{Q}_\alpha})$, we conclude this section with a description of a specific variant of cone homogeneity for the posets $\mathbb{Q}_\alpha$, which guarantees that the Magidor iteration $\mathbb{P}$ is cone homogeneous as well.

Definition 8.4 (Prikry-type forcing, [13]). $\langle \mathbb{P}, \leq, \leq^* \rangle$ is a Prikry-type forcing if

○ $\leq \supseteq \leq^*$ are partial orders on $\mathbb{P}$ and
○ (the Prikry Property) for every statement σ in the forcing language for $\langle \mathbb{P}, \leq \rangle$, and a condition p, there is a condition $p^*, p \leq^* p^*$ such that $p^* \Vdash \sigma$ or $p^* \Vdash \neg\sigma$.

Conditions in the Magidor iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ of Prikry-type posets $\langle \mathbb{Q}_\alpha, \leq_{\mathbb{Q}_\alpha}, \leq^*_{\mathbb{Q}_\alpha} \rangle$ are sequences $\vec{p} = \langle p_\alpha \mid \alpha < \kappa \rangle$, which beyond the standard requirement of $p \restriction \alpha \Vdash p_\alpha \in \mathbb{Q}_\alpha$, also satisfy that for all but finitely many ordinals $\alpha < \kappa$, $\vec{p} \restriction \alpha \Vdash p_\alpha \geq^*_{\mathbb{Q}_\alpha} 0_{\mathbb{Q}_\alpha}$. We note that, in particular, the definition allows using full-support conditions, as long as almost all components p_α are direct extensions of the trivial conditions. Similarly, for the definition of the ordering $\leq_{\mathbb{P}}$, we have that $\vec{p}' \geq \vec{p}$

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[15]i.e., the restriction $\varphi_{\beta_1} \restriction \mathbb{P}_{\beta_0}/\vec{p}^{\beta_0}$ is obtained by identifying conditions $\vec{r}^{\beta_0} \in \mathbb{P}_{\beta_0}/\vec{p}^{\beta_0}$ with their extension $\vec{r}^{\beta_1} = \vec{r}^{\beta_0} {}^\frown (\vec{p}^{\beta_1} \restriction_{[\beta_0, \beta_1)})$.

requires both that $\vec{p}' \upharpoonright \alpha \Vdash p'_\alpha \geq_{\mathbb{Q}_\alpha} p_\alpha$ for all α and that for all but finitely many ordinals $\alpha < \kappa$, $\vec{p}' \upharpoonright \alpha \Vdash p'_\alpha \geq^* p_\alpha$. See [13] for a comprehensive description of the Magidor iteration style and its main properties.

Lemma 8.5. *Suppose that* $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa \rangle$ *is a Magidor iteration of Prikry-type posets* $\langle \mathbb{Q}_\alpha, \leq_{\mathbb{Q}_\alpha}, \leq^*_{\mathbb{Q}_\alpha} \rangle$ *so that the following conditions hold for each* $\alpha < \kappa$:

(i) $\mathbb{Q}_\alpha$, $\leq_{\mathbb{Q}_\alpha}$ *and* $\leq^*_{\mathbb{Q}_\alpha}$ *are ordinal definable in V, and*
(ii) *it is forced by* $0_{\mathbb{P}_\alpha}$ *that for every two conditions* $p, q \in \mathbb{Q}_\alpha$, *there are* $p^* \geq^*_{\mathbb{Q}_\alpha} p$ *and* $q^* \geq^*_{\mathbb{Q}_\alpha} q$ *and a cone isomorphism* $\psi_\alpha \colon \mathbb{Q}_\alpha / p^* \to \mathbb{Q}_\alpha / q^*$ *which respects the direct extension order* $\leq^*_{\mathbb{Q}_\alpha}$.

Then $\mathbb{P}$ *is cone homogeneous.*

Proof. Let $\vec{p}, \vec{q} \in \mathbb{P}$, and let $(\vec{p}^\beta, \vec{q}^\beta, \varphi_\beta \mid \beta \leq \kappa)$ be the sequence obtained form the procedure described above. It suffices to verify inductively that conditions (i) and (ii) are satisfied by the sequence.

We note that, in the successor step construction of p'_β, $p^*_\beta = (p'_\beta)^{\varphi_\beta^{-1}}$, $q^*_\beta = q'_\beta$, and ψ_β, we may assume that $\varphi_\beta(\vec{p}^\beta) \Vdash p'_\beta \geq^*_{\mathbb{Q}_\beta} p_\beta^{\varphi_\beta}$, $\vec{q}^\beta \Vdash q'_\beta \geq^* q_\beta$, and that ψ_β is $\leq^*_{\mathbb{Q}_\beta}$-preserving. Since $\leq^*_{\mathbb{Q}_\beta}$ is ordinal definable in V, $0_{\mathbb{P}_\beta} \Vdash \leq^*_{\mathbb{Q}_\beta} = (\leq^*_{\mathbb{Q}_\beta})^{\varphi_\beta^{-1}}$, and therefore by applying the automorphism φ_β^{-1} we get $\vec{p}^\beta \Vdash p^*_\beta \geq^*_{\mathbb{Q}_\beta} p_\beta$, and $p \mapsto \psi_\beta(p^{\varphi_\beta})$ is forced by $\vec{p}^\beta$ to be $\leq^*_{\mathbb{Q}_\beta}$-preserving in the cone below p^*_β. In particular, assuming φ_β is order preserving and $\vec{r}^{\beta+1} \geq \vec{s}^{\beta+1} \in \mathbb{P}_{\beta+1} / \vec{p}^{\beta+1}$, $\vec{r}^{\beta+1} = \vec{r}^\beta ^\frown \langle r_\beta \rangle$, $\vec{s}^{\beta+1} = \vec{s}^\beta ^\frown \langle s_\beta \rangle$, we have that if $\vec{r}^\beta \Vdash r_\beta \geq^*_{\mathbb{Q}_\beta} s_\beta$ then $\varphi_\beta(\vec{r}^\beta) \Vdash \psi_\beta(r_\beta^{\varphi_\beta}) \geq^*_{\mathbb{Q}_\beta} \psi_\beta(r_\beta^{\varphi_\beta}) = \varphi_{\beta+1}(\vec{s}^{\beta+1})_\beta$. The same conclusion holds for $\leq_{\mathbb{Q}_\beta}$ which is also ordinal definable in V.

We conclude that, first, p^*_β, q^*_β are forced to be direct extensions of $0_{\mathbb{Q}_\beta}$ whenever p_β, q_β are, which in turn, implies that $\vec{p}^\alpha, \vec{q}^\alpha$ are conditions of $\mathbb{P}_\alpha$ for all $\alpha \leq \kappa$. Hence, (i) is satisfied. Second, for every $\vec{r}^\alpha, \vec{s}^\alpha \in \mathbb{P}_\alpha / \vec{p}^\alpha$ and $\beta < \alpha$, if $\vec{r}^\alpha \upharpoonright \beta \Vdash r_\beta \geq^*_{\mathbb{Q}_\beta} s_\beta$, then $\varphi_\alpha(\vec{r}^\alpha) \upharpoonright \beta \Vdash \varphi_\alpha(\vec{r}^\alpha)_\beta \geq^*_{\mathbb{Q}_\beta} \varphi_\alpha(\vec{s}^\alpha)_\beta$, and similarly, when replacing $\leq^*_{\mathbb{Q}_\beta}$ with $\leq_{\mathbb{Q}_\beta}$. It follows at once from this and the definition of the ordering $\leq_{\mathbb{P}_\alpha}$ of the Magidor iteration that φ_α is a cone isomorphism. Hence, (ii) holds. □

8.2. Homogeneous change of cofinalities

Our approach to construct a model with an ω-strongly measurable cardinal κ is to force over a ground model satisfying $V = \mathrm{HOD}$ with a weakly homogeneous poset (i.e., therefore also cone-homogeneous) to form a generic extension $V[G]$ with a cardinal κ, which satisfies the conditions of Lemma 2.4. In light of Lemma 2.5 above, we see that many regular cardinals in V need to change their cofinality in $V[G]$. The main challenge in that regard is to change the cofinality of many cardinals with a weakly homogeneous forcing.

Fortunately, such forcing has been constructed in [5], where the theory of nonstationary support iteration of Prikry-type forcings is developed and employed to form a weakly homogeneous variant of the Gitik iteration ([11]). We note that, as opposed to an Easton-style version of the Gitik iteration, which has a good chain condition (i.e., κ-c.c. when iterating up to a Mahlo cardinal κ), the nonstationary support variant of [5] has a weaker, fusion-type property.

We briefly describe the construction of the nonstationary support iteration $\mathbb{P}$ of iteration of Prikry-type forcings $\mathbb{Q}_\alpha$ from [5]. The iteration, which is based on the given coherent sequence of measures $\langle U_{\alpha,\tau} \mid \alpha < \kappa, \tau < o^{\mathcal{U}}(\alpha) \rangle$, is nontrivial at each $\alpha < \kappa$, $o^{\mathcal{U}}(\alpha) > 0$. As this α, the forcing $\mathbb{Q}_\alpha$ adds a cofinal closed unbounded set b_α to α of order-type $\omega^{o^{\mathcal{U}}(\alpha)}$ (ordinal exponentiation). More specifically, given a V-generic filter $G_\alpha \subseteq \mathbb{P}_\alpha$, which adds clubs b_β, for $\beta < \alpha$, $o^{\mathcal{U}}(\beta) > 0$, one considers finite sequences $t = \langle v_0, \ldots, v_{k-1} \rangle$ with the property that for every $i < k - 1$, if $o^{\mathcal{U}}(v_i) < o^{\mathcal{U}}(v_{i+1})$ then $v_i \in b_{v_{i+1}}$ and $b_{v_{i+1}} \cap v_i = b_{v_i}$. Such sequences are called coherent (with respect to G_α). If ρ is an ordinal so that $o^{\mathcal{U}}(v_i) < \rho$ for all $i < k$, then we say t is ρ-coherent. Otherwise, we denote by $t \upharpoonright \rho$ to be the subsequence of $v_i \in t$ so that $o^{\mathcal{U}}(v_i) < \rho$.

Working at $V[G_\alpha]$, one constructs posets $\mathbb{Q}_{\alpha,\tau}$, $\tau \leq o^{\mathcal{U}}(\alpha)$ and simultaneously shows by induction on $\tau \leq o^{\mathcal{U}}(\alpha)$ that, for each τ-coherent sequence t, $U_{\alpha,\tau} \in \mathcal{U}$ extends to $U_{\alpha,\tau}(t)$. We define $\mathbb{Q}_\alpha^0$ to be the trivial poset, and given that the measures $U_{\alpha,\tau'}(t')$ have been defined for every $\tau' < \tau$, and every τ'-coherent sequence t', the forcing $\mathbb{Q}_\alpha^\tau$ consists of pairs $q = \langle t, T \rangle$ where t is τ-coherent, $T \subseteq [\alpha]^{<\omega}$ is a tree whose stem is $\emptyset$ and for every $s \in T$, $\mathrm{succ}_T(s) := \{\mu < \kappa \mid s ^\frown \langle \mu \rangle \in T\} \in \bigcap_{\tau' < \tau} U_{\alpha,\tau'}(t ^\frown s \restriction \tau')$. Direct extensions and end extensions of $\mathbb{Q}_\alpha^\tau$ are defined as usual. $\mathbb{Q}_\alpha^\tau$ is a Prikry-type forcing whose direct extension is α-closed. With $\mathbb{Q}_{\alpha,\tau}$ determined, we consider the V-ultrapower by $U_{\alpha,\tau}$, by taking $j_{\alpha,\tau} : V \to M_{\alpha,\tau} \cong Ult(V, U_{\alpha,\tau})$, and define for each τ-coherent sequence t a $V[G_\alpha]$ measure $U_{\alpha,\tau}(t)$ by $X = \underset{\sim}{X}_{G_\alpha} \in U_{\alpha,\tau}(t)$ if there exist $p \in G_\alpha$ and a valid tree T such that

$$p ^\frown \langle t, T \rangle ^\frown j_{\alpha,\tau}(p) \setminus (\alpha + 1) \Vdash_{j_{\alpha,\tau}(\mathbb{P})} \check{\alpha} \in j_{\alpha,\tau}(\underset{\sim}{X}).$$

Fact 8.6.

1. For each α such that $o^{\mathcal{U}}(\alpha) > 0$, b_α is a cofinal sequence at α of order type $\omega^{o^{\mathcal{U}}(\alpha)}$ (ordinal exponentiation).
2. For each $\alpha \leq \kappa$, $(\mathbb{P}_\alpha, \leq, \leq^*)$ is a Prikry-type forcing.
3. For every $\gamma < \alpha \leq \kappa$, the quotient $(\mathbb{P}_\alpha / \mathbb{P}_\gamma, \leq, \leq^*)$ is a Prikry-type forcing whose direct extension order $\leq^*$ is γ-closed. In particular, the quotient $\mathbb{P}_\alpha / \mathbb{P}_\gamma$ does not add new bounded subsets to γ.
4. For every $\gamma < \alpha$, the iteration $\mathbb{P}_\alpha / \mathbb{P}_{\gamma+1}$ is weakly homogeneous.

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