Canad. Math. Bull. Vol. **25** (4), 1982

ABEL TRANSFORMATIONS INTO l^1

BY
J. A. FRIDY

ABSTRACT. Let t be a sequence in $(0, 1)$ that converges to 0, and define the Abel matrix A_t by $a_{nk} = t_n(1-t_n)^k$. The matrix A_t determines a sequence-to-sequence variant of the classical Abel summability method. The purpose of this paper is to study these transformations as l-l summability methods: e.g., A_t maps l^1 into l^1 if and only if t is in l^1. The Abel matrices are shown to be stronger l-l methods than the Euler–Knopp means and the Nörlund means. Indeed, if t is in l^1 and $\sum x_k$ has bounded partial sums, then A_tx is in l^1. Also, the Abel matrix is shown to be translative in an l-l sense, and an l-l Tauberian theorem is proved for A_t.

1. **Introduction.** The well-known Abel summability method is a sequence-to-function transformation which can be described as follows: if x is a complex number sequence such that

$$\lim_{r \to 1-} (1-r) \sum_{k=0}^{\infty} r^k x_k = L,$$

then x is Abel summable to L. This can be modified into a sequence-to-sequence transformation by replacing the continuous parameter r with a sequence $\{1-t_n\}_{n=0}^{\infty}$ that converges to 1 (cf. [3, Theorem 4]). Thus the sequence x is transformed into the sequence A_tx whose nth term is given by

$$(A_tx) = t_n \sum_{k=0}^{\infty} (1-t_n)^k x_k.$$

In order to ensure that $1-t_n$ approaches 1 from the left (as in $r \to 1^-$), we shall assume throughout that $0 < t_n < 1$ for all n and $\lim_n t_n = 0$. This transformation is determined by the matrix A_t whose nkth term is given by

$$a_{nk} = t_n(1-t_n)^k.$$

The matrix A_t is called an Abel matrix.

The summability matrix A is said to be an l-l method provided that Ax is in l^1 whenever x is in l^1. The summability field $A^{-1}[l^1]$ is denoted by l_A. In [6] Knopp and Lorentz characterized l-l matrices by the property $\sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty$. Since the appearance of [6], there have been numerous

Received by the editors August 4, 1980 and, in revised form, December 18, 1980.
AMS Subject classification numbers: Primary—40D05, 40D25, 40E05, 40G10

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

studies of general properties of l-l methods, but there are relatively few results about specific l-l methods. This shortage of examples of l-l methods has motivated the present study.

The purpose of this paper is to study the above Abel matrices as l-l matrices. In the next section we determine when A_t is an l-l matrix, and then examine the strength of this method by comparing its summability field l_{A_t} with some general sequence spaces as well as the summability fields of the methods of Euler–Knopp and Nörlund. In the third section we prove that A_t is translative in the l-l setting and also prove an l-l Tauberian theorem for the Abel matrices.

2. **The strength of the A_t method.** If s is a subsequence of t, then the matrix A_s is obtained by deleting certain rows from A_t. Therefore, A_sx will be a subsequence of A_tx provided that x is in the domain of A_t. Thus the following observation is an immediate consequence of the definition.

PROPOSITION 1. *If s is a subsequence of t, then $l_{A_t} \subseteq l_{A_s}$.*

We can also observe that in the setting of ordinary convergence, A_s includes A_t whenever s is a subsequence of t. Similarly, every Abel matrix includes the classical Abel summability method.

The sequence x is in the domain of A_t if and only if the series $\sum_k (1-t_n)^k x_k$ is convergent for each n. Since $\lim_n t_n = 0$, this is equivalent to the assertion that $\sum_k x_k z^k$ is convergent for $|z|<1$. Therefore, we can state a simple description of the domain of A_t.

PROPOSITION 2. *The sequence x is in the domain of the Abel matrix A_t if and only if* $\lim_k |x_k|^{1/k} \leq 1$.

The first of the main results gives a simple way of determining if A_t is an l-l matrix.

THEOREM 1. *The Abel matrix A_t is an l-l matrix if and only if t is in l^1.*

Proof. Since $0<t_n<1$, we have

$$\sum_{n=0}^{\infty} |a_{nk}| = \sum_{n=0}^{\infty} t_n(1-t_n)^k \leq \sum_{n=0}^{\infty} t_n,$$

for every k. Thus if t is in l^1, the Knopp–Lorentz Theorem guarantees that A_t is an l-l matrix. Conversely, if t is not in l^1, then we consider the sum of the first column of A_t:

$$\sum_{n=0}^{\infty} |a_{n,0}| = \sum_{n=0}^{\infty} t_n = \infty,$$

which shows that A_t is not an l-l matrix.

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

The classical Abel summability method is a rather strong method, and the Abel matrices are similarly strong in the l-l setting. The next result gives an indication of how large l_{A_t} must be.

THEOREM 2. *If A_t is an l-l matrix and the series $\sum_k x_k$ has bounded partial sums, then x is in l_{A_t}.*

Proof. In order to apply Abel's summation by parts technique, we define $s_k = \sum_{j=0}^{k} x_j$, $s_{-1} = 0$, and $\tau_n = 1 - t_n$. Then

$$\begin{aligned}\left|\sum_{k=0}^{\infty} (1-t_n)^k x_k\right| &= \left|\sum_{k=0}^{\infty} (s_k - s_{k-1})\tau_n^k\right| \\ &= \left|\sum_{k=0}^{\infty} s_k(\tau_n^k - \tau_n^{k+1})\right| \\ &\le \sup_k |s_k|.\end{aligned}$$

Hence,

$$|(A_t x)_n| \le t_n \sup_k |s_k|,$$

so $A_t x$ is in l^1 whenever t is in l^1.

COROLLARY. *If A_t is an l-l matrix, then l_{A_t} contains all sequences x such that $\sum x_k$ is conditionally convergent.*

We can give a further indication of the size of l_{A_t} by showing that if A_t is an l-l matrix then l_{A_t} contains an unbounded sequence. Consider the sequence x given by $x_k = (-1)^k(k+1)$. Differentiation of the power series $\sum_k (-z)^k$ yields

$$\sum_{k=0}^{\infty} (-1)^k(k+1)z^k = (1+z)^{-2}, \quad \text{if} \quad |z|<1.$$

Therefore

$$(A_t x)_n = t_n(2-t_n)^{-2} \le t_n.$$

Hence, if A_t is an l-l matrix, then t is in l^1, so x is in l_{A_t}.

The Euler–Knopp mean of order r (see [5, pp. 56–60]) is given by the matrix E_r whose nkth entry is

$$E_r[n,k] = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k, & \text{if} \quad k \le n, \\ 0, & \text{if} \quad k > n. \end{cases}$$

In [2, Theorem 4] it was shown that E_r is an l-l matrix if and only if $0 < r \le 1$. The next result compares the l-l strength of E_r with that of A_t.

THEOREM 3. *If A_t is an l-l matrix, then $l_{E_r} \subset l_{A_t}$ if and only if $r \ge 1/2$.*

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

Proof. The asserted inclusion is equivalent to the statement that $A_tE_r^{-1}$ is an l-l matrix. In order to simplify typography, let $s=1/r$ and consider $A_tE_s = A_tE_r^{-1}$; the nkth entry is given by

$$(*)\qquad (A_tE_s)[n,k] = t_n \sum_{j=k}^{\infty} (1-t_n)^j \binom{j}{k} (1-s)^{j-k} s^k$$
$$= t_n s^k (1-t_n)^k [1-(1-t_n)(1-s)]^{-k-1},$$

provided that $|(1-t_n)(1-s)|<1$. This proviso is equivalent to

$$1+\frac{-1}{1-t_n} < s < 1 + \frac{1}{1-t_n}.$$

Since $\lim_n t_n = 0$, we conclude that A_tE_s exists if and only if $0<s\leq 2$, i.e., $r\geq 1/2$. Once it is guaranteed that A_tE_s exists, we prove that it is an l-l matrix by showing that the coefficient of t_n in $(*)$ is bounded; thus A_tE_s will satisfy the Knopp–Lorentz property. Consider the following:

$$s^k(1-t_n)^k[1-(1-t_n)(1-s)]^{-k-1} = \left[\frac{s(1-t_n)}{t_n+s(1-t_n)}\right]^k \frac{1}{t_n+s(1-t_n)}$$
$$< \frac{1}{t_n+s(1-t_n)}$$

because $0<t_n<1$ and $s>0$. Hence, $l_{E_r} \subseteq l_{A_t}$ if and only if $r\geq 1/2$. To show that $l_{E_r} \neq l_{A_t}$, we show the existence of a sequence x such that $\sum x_k$ is conditionally convergent and $\sum_{k=0}^{\infty} |(\Delta x)_k|\sqrt{k}<\infty$. Then Theorem 2 ensures that A_tx is in l^1, and the Tauberian result in [4, Theorem 4] implies that E_rx cannot be in l^1 since x is not in l^1. We wish to have x_k positive throughout a block B_i of consecutive terms and then alternate to negative values in the next block. Also, $|(\Delta x)_k|$ is constant throughout the ith block and Δx_k changes sign only at the "middle term" of the block, say $k=m(i)$. Therefore in the ith block, $|x_k|$ increases from 0 to $|x_{m(i)}|$, then decreases to 0. If the block contains $2l_i$ terms, it follows that

$$A_i = \sum_{k\in B_i} |x_k| = l_i^2 |(\Delta x)_{m(i)}|.$$

Also, the middle of the ith block can be located by

$$m(i) = 2\sum_{j=1}^{i} l_j - l_i.$$

Now choose l_i and $(\Delta x)_k$ satisfying

$$l_i \sim \tfrac{3}{2} i^2 \quad\text{and}\quad |(\Delta x)_k| \sim k^{-5/3}.$$

Then $m(i)\sim i^3$ and

$$A_i = (\tfrac{3}{2}i^2)^2\, |(\Delta x_{i^3}| \sim \tfrac{9}{4} i^{-1}.$$

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

Also, $|(\Delta x)_k k^{1/2}| \sim k^{-7/6}$, so $\sum_{k=1}^{\infty} |(\Delta x)_k| \sqrt{k} < \infty$ and $\sum x_k$ is conditionally convergent.

The strength of the Abel matrices can also be demonstrated by comparing them with the Nörlund matrices:

$$N_p[n, k] = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{if} \quad k \le n, \\ 0, & \text{if} \quad k > n, \end{cases}$$

where p is a non-negative number sequence with $p_0 > 0$. In [2, Theorem 2] it was proved that N_p is an l-l matrix if and only if p is in l^1. Using techniques developed by J. DeFranza [1], one can show that if A_t and N_p are l-l matrices, then $l_{N_p} \subseteq l_{A_t}$. The proof of this result will appear elsewhere with DeFranza's work.

3. **Translativity and Tauberian Theorems.** Following the concept of translativity in ordinary summability, we say that the matrix A is l-translative provided that each of the sequences Tx and Sx is in l_A whenever x is l_A, where $Tx = \{x_1, x_2, x_3, \ldots\}$ and $Sx = \{0, x_0, x_1, \ldots\}$.

THEOREM 4. *Every l-l Abel matrix is l-translative.*

Proof. Consider the calculation

$$\begin{aligned}(A_t Tx)_n &= \sum_{k=0}^{\infty} t_n(1-t_n)^k x_{k+1} \\ &= \frac{1}{1-t_n}\left\{\sum_{i=0}^{\infty} t_n(1-t_n)^i x_i - t_n x_0\right\} \\ &= \frac{1}{1-t_n}\{(A_t x)_n - t_n x_0\}.\end{aligned}$$

It is clear that the last expression represents a sequence in l^1 whenever t and $A_t x$ are in l^1. Therefore, $l_{A_t} \subseteq l_{A_t T}$. Similarly,

$$(A_t Sx)_n = (1-t_n)(Ax)_n,$$

which shows that $l_A \subseteq l_{A_t S}$. Hence, A_t is l-translative.

The final result is an l-l Tauberian theorem for the Abel matrices. The concept of an l-l Tauberian theorem was introduced in [4], where such results were proved for Euler–Knopp and Borel matrices. The original Tauberian theorem [7] can be stated (in matrix form) as follows:

> if x is a sequence such that $A_t x$ is convergent and $\{j(\Delta x)_j\}_{j=0}^{\infty}$ is in c_0, then x itself is convergent.

We now prove that l-l analogue of this statement.

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

THEOREM 5. *Let A_t be an l-l Abel matrix; if x is a sequence such that $A_t x$ and* $\{j(\Delta x)_j\}_{j=0}^{\infty}$ are in l^1, then x itself is in l^1.

Proof. In order to show that $A_t x - x$ is in l^1 we write

$$(A_t x)_n - x_n = \sum_{k=0}^{\infty} t_n(1-t_n)^k(x_k - x_n).$$

Letting $a_{nk} = t_n(1-t_n)^k$, we shall prove that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{nk}\,|x_k - x_n| < \infty.$$

Proceeding by exactly the same steps as in the proof of Theorem 3 of [4], we deal with this sum in two parts:

$$C = \sum_{n=0}^{\infty}\sum_{k=0}^{n-1} a_{nk}\,|x_k - x_n|$$

and

$$D = \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} a_{nk}\,|x_k - x_n|.$$

This leads to

$$C \le \sum_{j=0}^{\infty} |(\Delta x)_j|\, C_j \quad \text{and} \quad D \le \sum_{j=0}^{\infty} |(\Delta x)_j|\, D_j,$$

where

$$C_j = \sum_{n=j+1}^{\infty}\sum_{k=0}^{j} a_{nk} \quad \text{and} \quad D_j = \sum_{n=0}^{j}\sum_{k=j+1}^{\infty} a_{nk}.$$

By showing that $C_j = 0(j)$ and $D_j = 0(j)$, we will prove that $\sum_{j=0}^{\infty} |(\Delta x)_j|\, j < \infty$ implies that $A_t x - x$ is in l^1. These $0(j)$ assertions are easily verified since A_t is both l-l and regular; for

$$C_j = \sum_{k=0}^{j}\sum_{n=j+1}^{\infty} a_{nk} \le (j+1)\sup_k \sum_{n=1}^{\infty} |a_{nk}| = 0(j),$$

and

$$D_j = \sum_{n=0}^{j}\sum_{k=j+1}^{\infty} a_{nk} \le \sum_{n=0}^{j} \sup_n \sum_{k=0}^{\infty} |a_{nk}|$$
$$= \sum_{n=0}^{j} 1 = j+1 = 0(j).$$

Thus the proof is complete.

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press

References

1. J. DeFranza, *Nörlund methods of summability that map the set of absolutely convergent series into itself*, Ph.D. Dissertation, Kent State Univ., Kent, Ohio, 1979.

2. J. A. Fridy, *Absolute summability matrices that are stronger than the identity mapping*, Proc. Amer. Math. Soc. **47** (1975), pp. 112–118.

3. ——, *Tauberian theorems via block dominated matrices*, Pacific J. Math. **81** (1979), pp. 81–91.

4. J. A. Fridy and K. L. Roberts, *Some Tauberian theorems for Euler and Borel summability*, Internat. J. Math. & Math. Sci., **3** (1980) pp. 731–738.

5. R. L. Powell and S. M. Shah, *Summability Theory and Its Applications*. Van Nostrand Reinhold Co., London, 1972.

6. K. Knopp and G. G. Lorentz, *Beiträge zur absoluten Limitierung*, Arch Math. **2** (1949), pp. 10–16.

7. A. Tauber, *Ein Satz aus der Theorie der Unendlichen Reihen*, Monatash. Math. Phys. **8** (1897), pp. 273–277.

Kent State University
Kent, Ohio

https://doi.org/10.4153/CMB-1982-060-5 Published online by Cambridge University Press