INVARIANT SUBSPACE THEOREMS FOR FINITE RIEMANN SURFACES

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1. Introduction. The purpose of this paper is to extend various invariant subspace theorems for the circle group to multiply connected domains. Such attempts are not new. Actually, Sarason **(4)** studied the invariant subspaces of annulus operators acting on L^2 and showed certain parallelisms between the unit disk case and the annulus case. Voichick **(8)** observed analytic functions on a finite Riemann surface and generalized the Beurling theorem on the closed invariant subspaces of H^2 as well as the Beurling–Rudin theorem on the closed ideals of the disk algebra. Here we shall consider $L^p(\Gamma)$ and $C(\Gamma)$ defined on the boundary Γ of a finite orientable Riemann surface R. We wish to find the subspaces of $L^p(\Gamma)$ and $C(\Gamma)$ that are closed and invariant under multiplication by every function analytic on R and continuous on $\bar{R}$.

In §2, we gather some known facts about finite Riemann surfaces and certain analytic functions defined on them. As Voichick **(8)** pointed out, multiple-valued inner functions on R play a very important role in the determination of invariant subspaces of $H^2(\Gamma)$. If we want to find invariant subspaces of $L^p(\Gamma)$, then it turns out that we need certain non-analytic analogues of the multiple-valued inner functions. Such new functions are defined only on the boundary Γ of the surface R. In this paper, we call them i-functions. It is seen that our i-functions can be captured as single-valued functions, subject to certain restriction, defined on the product space $\Gamma \times \mathfrak{g}$, where $\mathfrak{g}$ denotes the integral homology group of the 1-cycles of R. This is quite natural because the multiple-valuedness of analytic functions on R is due to the connectivity of the surface. The i-functions are defined in §3. Once we get the concept of i-functions, the whole theory is quite parallel to the well-known one for the circle group. In §4, we prove the invariant subspace theorem for $L^p(\Gamma)$, which corresponds to some results in **(6; 7)**. In §§5 and 6, we discuss closed invariant subspaces of $C(\Gamma)$ and $M(\Gamma)$, where $M(\Gamma)$ is the space of Radon measures on Γ. Our theorems in these sections extend our earlier results in **(2)** for the circle group. Finally we shall show that the theorems obtained by Sarason **(4)** and Voichick **(8)** follow quickly from our theorems. Moreover, we shall prove that $A(R)$, the algebra of functions analytic on R and continuous on $\bar{R}$, is a maximal closed subalgebra of $C(\Gamma)$. A special case of this theorem, in which Γ is topologically a circle, was proved by Wermer **(9)**.

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2. Preliminaries. Let R be a finite orientable Riemann surface in the sense of **(5)** with non-empty boundary Γ. The boundary Γ consists of a finite number of components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, each of which is a closed analytic curve. For each $j = 1, 2, \ldots, k$, there exists an open annulus U_j of R such that Γ_j is one of the boundaries of U_j. Each U_j can be mapped conformally by a function z_j onto an annulus $r_j < |z_j| < 1$ so that the continuous extension of z_j to Γ_j maps Γ_j onto $|z_j| = 1$. The functions z_j are called boundary uniformizers.

It is well known that, for any divisor d on $\bar{R}$ $(= R \cup \Gamma)$, there is a meromorphic function (or a meromorphic differential) whose divisor is exactly d. In particular, there is an analytic function on $\bar{R}$ that has a simple zero at an arbitrarily prescribed point in R and vanishes nowhere else on $\bar{R}$. There is also a non-vanishing analytic differential ω on $\bar{R}$. We denote by ω^* the tangential component of ω along Γ, i.e. $\omega^* = b(z)d\theta$ on Γ_j if $\omega = a(z)dr + b(z)d\theta$ in terms of a boundary uniformizer $z_j = re^{i\theta}$ near Γ_j. We fix such a differential through our discussion.

Let $G(\zeta, t)$ be the Green function of R and let ζ_0 be a point in R, which is chosen once for all. We set

$$dm = \frac{1}{2\pi} \frac{\partial G(\zeta_0, t)}{\partial n_t} ds_t$$

for $t \in \Gamma$, where the right-hand side is computed by means of boundary uniformizers for Γ. Then m is positive and $\int f\, dm = f(\zeta_0)$ for any $f \in A$, so that m is multiplicative on A. Here $A = A(R)$ is the algebra of functions analytic on R and continuous on $\bar{R}$. It is clear that m is equivalent to ω^*. We write $L^p(\Gamma)$ instead of $L^p(\Gamma, dm)$ and let $H^p(\Gamma)$ be the subspace of $L^p(\Gamma)$ consisting of functions extendable to R as analytic functions. Each function in $H^p(\Gamma)$ can be regarded as the boundary value of an analytic function h on R such that $|h|^p$ has a harmonic majorant. The following theorem proved by Royden **(3**, Theorem 2**)** (see also Voichick **(8**, Corollary 5.3**)**) has fundamental importance and is a generalization of the F. and M. Riesz theorem on measures on the circle group.

THEOREM 1. *If a Borel measure μ on Γ is orthogonal to $A(R)$, then $\mu = h\omega^*$ for an $h \in H^1(\Gamma)$, and conversely.*

The open unit disk D is a universal covering surface of R, so that there exists a covering map T from D onto R, which is analytic and a local homeomorphism. We choose T in such a way that $T(0) = \zeta_0$. Let $\mathfrak{G}$ be the group of linear transformations τ of D onto itself such that $T \circ \tau = T$. Then it is known that there exists a fundamental region Δ of $\mathfrak{G}$, which has exactly k free sides γ_j with $T(\gamma_j) = \Gamma_j$. If we put $\gamma = \cup \gamma_j$ and $\Omega = \cup\{\tau(\gamma) : \tau \in \mathfrak{G}\}$, then Ω is an open dense subset of the unit circumference X and T can be extended to be analytic and locally one-to-one in a neighbourhood of $D \cup \Omega$.

If f is a single-valued function on R, then $f \circ T$ is an analytic function on D that is invariant under the group $\mathfrak{G}$, meaning that $\tilde{f}(\tau z) = \tilde{f}(z)$ for any $z \in D$ and $\tau \in \mathfrak{G}$ with $\tilde{f} = f \circ T$. In what follows, we use $\tilde{f}$ in place of $f \circ T$. Now we say that a multiple-valued function h on R is *multiplicative* if h is analytic and $|h|$ is single-valued. If h is multiplicative, then $\tilde{h}$ is *modulus invariant*, i.e. $|\tilde{h}|$ is invariant under $\mathfrak{G}$. An analytic function F on D is modulus invariant if and only if, for each $\tau \in \mathfrak{G}$, there exists a constant c_τ of modulus one such that $F(\tau z) = c_\tau F(z)$ for any $z \in D$. Let $\mathfrak{H}^p(R)$ $(1 \leqslant p < +\infty)$ be the space of multiplicative functions h on R such that $|h|^p$ has a harmonic majorant on R. For $p = +\infty$, $\mathfrak{H}^\infty(R)$ denotes the space of all bounded multiplicative functions on R. It is known that, for any $h \in \mathfrak{H}^p(R)$, $|h|$ has non-tangential limits a.e. on Γ, which form a function in $L^p(\Gamma)$. We note that the spaces $\mathfrak{H}^p(R)$ are not necessarily linear.

Every function $h \in \mathfrak{H}^1(R)$ can be factored into its inner and outer factors. Suppose that h is not identically zero. Then $\log |h|$ is subharmonic on R and has non-tangential limits $\log |h(t)|$, $t \in \Gamma$, a.e. on Γ, which form an integrable function. We define a multiplicative function h_0 by

$$\log |h_0(\zeta)| = \frac{1}{2\pi} \int_\Gamma \frac{\partial G(\zeta, t)}{\partial n_t} \log |h(t)| \, ds_t.$$

h_0 is determined uniquely up to a constant factor of modulus one. h_0 has no zero and $|h_0| = |h|$ a.e. on Γ. Therefore we have $|h(\zeta)| \leqslant |h_0(\zeta)|$ for any $\zeta \in R$. Now let $h_i = h_0^{-1}h$. Then h_i is also a multiplicative function on R. Clearly $|h_i(\zeta)| \leqslant 1$ on R and $|h_i| = 1$ a.e. on Γ. After Voichick **(8)**, we say that a bounded multiplicative function h is *inner* if $|h| = 1$ a.e. on Γ. We also say that a multiplicative function $g \in \mathfrak{H}^1(R)$ is *outer* if

$$\log |g(\zeta)| = \frac{1}{2\pi} \int_\Gamma \frac{\partial G(\zeta, t)}{\partial n_t} \log |g(t)| \, ds_t.$$

So we have shown that every non-zero function $h \in \mathfrak{H}^1(R)$ is factored into an inner function and an outer function.

Let us assume that h is an inner function. Then $\tilde{h}$ is a modulus-invariant inner function on D. Let $\tilde{h} = F_b F_s$ be a factorization of $\tilde{h}$ into a Blaschke product F_b and a singular function F_s.

LEMMA 1. *F_b and F_s are modulus invariant.*

Proof. Let $\tau \in \mathfrak{G}$. Then $\tilde{h}(\tau z) = c_\tau \tilde{h}(z)$ for a constant c_τ of modulus one. So $F_b(\tau z) F_s(\tau z) = c_\tau F_b(z) F_s(z)$. Since $F_s(\tau z)$ vanishes nowhere on D, $F_b(\tau z)$ has the same zeros as $F_b(z)$. Hence F_b is the Blaschke factor of $F_b(\tau z)$ so that $F_b(\tau z) = F_b(z) C(z)$ for some inner function $C(z)$. We also have

$$F_b(\tau^{-1} z) = F_b(z) D(z),$$

where $D(z)$ is another inner function. Thus we have $C(z)D(z) = 1$ everywhere on D. Since $|C(z)| \leqslant 1$ and $|D(z)| \leqslant 1$ on D, we conclude that C and D must be constant. Hence $F_b(z)$ and, consequently, $F_s(z)$ are modulus invariant, as was to be proved.

We put $h_b = F_b \circ T^{-1}$ and $h_s = F_s \circ T^{-1}$. Then h_b and h_s are multiplicative functions and $h = h_b h_s$. h_b has the same zeros as h, and h_s has no zero in R. Since $|h_s| \leqslant 1$ and h_s never vanishes on R, $-\log |h_s|$ is a positive harmonic function on R. By a theorem of Royden (**3**, Proposition 8), there exists a unique positive measure ν on Γ such that

$$-\log |h_s(\zeta)| = \frac{1}{2\pi} \int_\Gamma \frac{\partial G(\zeta, t)}{\partial n_t} d\nu(t).$$

Since $|h_s| = 1$ a.e. on Γ, ν must be a singular measure. h_s is determined uniquely by ν up to a constant factor of modulus one.

Finally we note that, for any prescribed periods along a homology basis of 1-cycles of R, there exists an analytic function on $\bar{R}$ that does not vanish on $\bar{R}$ and whose periods are exactly the given ones; cf. (**5**, Chapter IV).

3. m-functions and i-functions. Voichick (**8**) has shown that multiplicative functions play an important role in the description of invariant subspaces of $H^2(\Gamma)$. Every multiplicative function $h \in \mathfrak{H}^1(R)$ has non-tangential limits a.e. on Γ. The boundary values then form a multiple-valued function on Γ. We shall first define a class of functions on Γ including such multiple-valued functions. Since the multiple-valuedness of analytic functions on R is caused by the connectivity of the surface R, it is natural to introduce the integral homology group $\mathfrak{g}$ of 1-cycles of R.

Definition 1. A single-valued numerical function Q on $\Gamma \times \mathfrak{g}$ is called an *m-function* if (a) for any $\alpha \in \mathfrak{g}$, $Q(t, \alpha)$ is defined a.e. for $t \in \Gamma$ and measurable, and (b) for each $\alpha \in \mathfrak{g}$, there exists a constant c_α of modulus one such that $Q(t, \alpha + \beta) = c_\alpha c_\beta Q(t, 0)$ for any α, $\beta \in \mathfrak{g}$. An m-function Q is called an *i-function* if $|Q| = 1$ a.e. on Γ.

Definition 2. Two m-functions Q_1 and Q_2 are said to be *equivalent* if there exists an element $\alpha_0 \in \mathfrak{g}$ such that $Q_2(t, \alpha + \alpha_0) = Q_1(t, \alpha)$ a.e. for any $t \in \Gamma$ and $\alpha \in \mathfrak{g}$. If Q_1 and Q_2 are equivalent, then we write $Q_1 \equiv Q_2$.

We shall show that every multiplicative function h in $\mathfrak{H}^1(R)$ gives rise to an m-function, which we denote by $h(t, \alpha)$ for $t \in \Gamma$, $\alpha \in \mathfrak{g}$. If R is simply connected, then R is equivalent to the unit disk and there is nothing to prove. So we assume that R is multiply connected. By making cuts along suitable curves on R, we can construct a simply connected domain—a normal form of R—so that Γ forms a part of the boundary. Let us fix such a normal form of R, for which ζ_0 is an interior point, and call it R_0. Now let h be any multiplicative

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function $\in \mathfrak{H}^1(R)$ on R. First we choose a branch of h at ζ_0 and call it $h(\zeta, 0)$, where 0 denotes the zero of the group $\mathfrak{g}$. $h(\zeta, 0)$ defines a single-valued analytic function on R_0. $h(\zeta, 0)$ has non-tangential limits a.e. on Γ, which form a function $h(t, 0)$, $t \in \Gamma$. For any $\alpha \in \mathfrak{g}$, let $\tilde{\alpha}$ be a 1-cycle that starts from ζ_0 and represents α. We continue our branch $h(\zeta, 0)$ along $\tilde{\alpha}$ on R and get another branch of h at ζ_0, which we denote by $h(\zeta, \alpha)$. $h(\zeta, \alpha)$ then defines a single-valued analytic function on R_0 and consequently defines a function on Γ a.e., which we call $h(t, \alpha)$. It is clear that $h(t, \alpha)$ depends only on α and not on the representative $\tilde{\alpha}$ of α. Since h is multiplicative, $|h(\zeta, 0)| = |h(\zeta, \alpha)|$ for $\zeta \in R_0$ so that $h(\zeta, \alpha) = c_\alpha h(\zeta, 0)$ for a constant c_α of modulus one. Therefore

$$h(t, \alpha) = c_\alpha h(t, 0)$$

on Γ. Obviously, the correspondence $\alpha \to c_\alpha$ is a representation of the group $\mathfrak{g}$ onto the circle group. Hence $h(t, \alpha)$ is an m-function. So we have the following

LEMMA 2. *Any multiplicative function $h \in \mathfrak{H}^1(R)$ defines an m-function $h(t, \alpha)$, $t \in \Gamma$, $\alpha \in \mathfrak{g}$, by means of the normal form R_0 of R. $h(t, \alpha)$ is uniquely determined by h up to equivalence in the sense of Definition* 2.

It is clear from our construction that the constants c_α depend only on the multiplicative function h and not on the normal form R_0. Now we fix, once and for all, a normal form R_0 of R and a fundamental region Δ of the group $\mathfrak{G}$ in such a way that the covering map T maps Δ onto R_0, i.e. the boundary of Δ is mapped onto the cuts and the boundary Γ which define R_0. We may also assume without loss of generality that the origin 0 is in Δ and $T(0) = \zeta_0$.

We note, for later use, the relation between the homology group $\mathfrak{g}$ of 1-cycles of R and the transformation group $\mathfrak{G}$. For any $\tau \in \mathfrak{G}$, we draw a (smooth) curve L joining 0 with $\tau(0)$ within D. Then $T(L)$ is a 1-cycle starting from ζ_0. Clearly any two such curves define homologous cycles of R. Therefore $T(L)$ determines an element α in $\mathfrak{g}$. The correspondence $\tau \to \alpha$ preserves group operations so that it gives a homomorphism of $\mathfrak{G}$ onto $\mathfrak{g}$. We call it the canonical homomorphism of $\mathfrak{G}$ onto $\mathfrak{g}$.

4. Invariant subspaces of $L^p(\Gamma)$. We shall determine closed invariant subspaces of $L^p(\Gamma)$. Let $A_0(R)$ be the subalgebra of $A(R)$ consisting of functions in $A(R)$ that vanish at ζ_0.

Definition 3. Let $\mathfrak{M}$ be a closed subspace of $L^p(\Gamma)$, $1 \leqslant p < +\infty$. Then $\mathfrak{M}$ is called *doubly* (*simply*) *invariant* if the L^p-closure $[A_0(R)\mathfrak{M}]_p$ of $A_0(R)\mathfrak{M}$ is equal to (strictly contained in) $\mathfrak{M}$. If $p = +\infty$, then we replace the closedness by the weak* closedness in $L^\infty(\Gamma)$ and the L^p-closure by the weak* closure.

If R is the unit disk D, then the structure of closed invariant subspaces of $L^p(X)$ on the unit circumference X is well known. So we shall transfer our problem to the circle group X by means of the covering map T. Although T does not preserve the measure m and in fact $m \circ T$ may sometimes be an infinite measure, it preserves measurability as well as null sets. The following lemma can be found in Voichick (**8**, Lemma 6.1):

LEMMA 3. *If* $f \in L^p[\Gamma]$, *then* $f \circ T \in L^p(X)$.

THEOREM 2. (a) $\mathfrak{M}$ *is a closed (weakly* closed, if* $p = +\infty$*) doubly invariant subspace of* $L^p(\Gamma)$ *if and only if* $\mathfrak{M} = C_S L^p(\Gamma)$ *for some measurable subset* S *of* Γ, *where* C_S *is the characteristic function of* S. S *is determined uniquely by* $\mathfrak{M}$ *up to a null set.*

(b) $\mathfrak{M}$ *is a closed (weakly* closed, if* $p = +\infty$*) simply invariant subspace of* $L^p(\Gamma)$ *if and only if* $\mathfrak{M} = I^p(Q)$ *for some i-function* Q *on* $\Gamma \times \mathfrak{g}$, *where* $I^p(Q)$ *denotes the totality of functions* $f \in L^p(\Gamma)$ *such that* f/Q *is equivalent to an m-function defined by some function in* $\mathfrak{H}^p(R)$. Q *is determined uniquely by* $\mathfrak{M}$ *up to equivalence and a constant factor of modulus one.*

Proof. (i) $1 \leqslant p < +\infty$. Let $\mathfrak{M}$ be a closed invariant subspace of $L^p(\Gamma)$, i.e. $A_0(R)\mathfrak{M} \subseteq \mathfrak{M}$. By Lemma 3, $f \circ T \in L^p(X)$ for any $f \in \mathfrak{M}$. Let $\{\mathfrak{M}\}_p$ be the closed invariant subspace of $L^p(X)$ generated by $\{f \circ T: f \in \mathfrak{M}\}$. Then $\{\mathfrak{M}\}_p$ is either doubly invariant (i.e. $z\{\mathfrak{M}\}_p = \{\mathfrak{M}\}_p$) or simply invariant (i.e. $z\{\mathfrak{M}\}_p < \{\mathfrak{M}\}_p$).

Suppose first of all that $\{\mathfrak{M}\}_p$ is doubly invariant. Then a theorem in (**6**) shows that $\{\mathfrak{M}\}_p = C_{S'} L^p(X)$, where S' is a measurable subset of X. Let $S = T(S')$. Then S is a measurable subset of Γ and any f in $\mathfrak{M}$ vanishes off S, i.e. $\mathfrak{M} \subseteq C_S L^p(\Gamma)$. To show the converse inclusion, we note that, for any $g \in L^{p'}(\Gamma)$ $(p^{-1} + p'^{-1} = 1)$, there exists a $g_1 \in L^{p'}(\Gamma)$ such that $g\, dm = g_1 \omega^*$. This is possible because ω^* and dm are equivalent and the Radon–Nikodym derivatives ω^*/dm and dm/ω^* are bounded. Suppose $g \perp \mathfrak{M}$. Then $g_1 \omega^* \perp \mathfrak{M}$. Since $\mathfrak{M}$ is invariant, we have $\int \phi f g_1 \omega^* = 0$ for any $\phi \in A(R)$ and $f \in \mathfrak{M}$. By Theorem 1, $fg_1 \in H^1(\Gamma)$. Going to X by means of T, we see that $\tilde{f}\tilde{g}_1 \in H^1(X)$. So $\int u\tilde{f}\tilde{g}_1\, dz = 0$ for any $u \in A = A(D)$, where D is the unit disk. By taking L^p-limits in $u\tilde{f}$, we see that $\int v\tilde{g}_1\, dz = 0$ for any $v \in \{\mathfrak{M}\}_p$. As $\{\mathfrak{M}\}_p = C_{S'}L^p(X)$, $\tilde{g}_1$ must vanish on S' so that g_1 vanishes on S. This proves that $g_1 \perp C_S L^p(\Gamma)$. Consequently $g \perp C_S L^p(\Gamma)$, and hence $\mathfrak{M} = C_S L^p(\Gamma)$.

Now suppose that $\{\mathfrak{M}\}_p$ is simply invariant. Then, by (**7**), $\{\mathfrak{M}\}_p = qH^p(X)$ for some $q \in L^\infty(X)$ with $|q| = 1$ a.e. on X. Since it is clear that $\{\mathfrak{M}\}_p$ is invariant under the group $\mathfrak{G}$, q is modulus invariant on X, meaning that, for any $\tau \in \mathfrak{G}$, there exists a constant c_τ of modulus one satisfying $q(\tau z) = c_\tau q(z)$ for $z \in X$.

We consider the fundamental region Δ of the group $\mathfrak{G}$ mentioned in §2 and define an i-function Q as follows. For the zero element 0 of the homology group $\mathfrak{g}$ of 1-cycles of R, we put

$$Q(t, 0) = q(z), \qquad \text{if } t = Tz,\ z \in \gamma.$$

Then $Q(t, 0)$ is defined a.e. on Γ. For any $\alpha \in \mathfrak{g}$, take any τ that is mapped to α under the canonical homomorphism of $\mathfrak{G}$ onto $\mathfrak{g}$. Then we set

$$Q(t, \alpha) = q(\tau z), \qquad \text{if } t = Tz,\ z \in \gamma.$$

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Then Q is an i-function. Indeed, we have

$$Q(t, \alpha) = q(\tau z) = c_\tau q(z) = c_\tau Q(t, 0).$$

Since the mapping $\tau \rightarrow c_\tau$ is a representation of $\mathfrak{G}$ onto the circle group and since the circle group is commutative, the mapping induces a representation of $\mathfrak{g}$ onto the circle group and therefore c_τ depends only on the homology class to which τ corresponds under the canonical homomorphism. So Q is an i-function.

We want to show that $\mathfrak{M} = I^p(Q)$. Let $f \in \mathfrak{M}$. Then $\tilde{f} = f \circ T = q\psi$ for some $\psi \in H^p(X)$. Since f is invariant, ψ is a modulus invariant function. Let $h = \psi \circ T^{-1}$. Since R_0 is defined as the image of Δ under T, it is easy to see that $f(t)/Q(t, \alpha) = h(t, \alpha)$ for $t \in \Gamma$, $\alpha \in \mathfrak{g}$. Since h is a multiplicative function $\in \mathfrak{H}^p(R)$ on R, we have seen that $f \in I^p(Q)$. Hence $\mathfrak{M} \subseteq I^p(Q)$. To see the converse inclusion, we take any $g \in L^{p'}(\Gamma)$, $p^{-1} + p'^{-1} = 1$, such that $g \perp \mathfrak{M}$. As before, we define $g_1 \in L^{p'}(\Gamma)$ by $g_1 \omega^* = g\, dm$. Then

$$\int \phi f g_1 \omega^* = \int \phi f g\, dm = 0$$

for any $\phi \in A(R)$ and $f \in \mathfrak{M}$. By Theorem 1, $fg_1 \in H(\Gamma)$. Thus $\tilde{f}\tilde{g}_1 \in H^1(X)$ and consequently $\int u\tilde{f}\tilde{g}_1\, dz = 0$ for any $u \in A = A(D)$. By taking L^p-limits in $u\tilde{f}$, we have that $\int v\tilde{g}_1\, dz = 0$ for any $v \in \{\mathfrak{M}\}_p$. Since $\{\mathfrak{M}\}_p = qH^p(X)$, we have $\int qF\tilde{g}_1\, dz = 0$ for any $F \in H^p(X)$. This shows that $q\tilde{g}_1 \in H^{p'}(X)$. If $f \in I^p(Q)$, then $f/Q \in \mathfrak{H}^p(R)$ so that $\tilde{f}/q \in H^p(X)$. Thus

$$\widetilde{fg_1} = \tilde{f}\tilde{g}_1 = (\tilde{f}/q)(q\tilde{g}_1) \in H^1(X).$$

Therefore $fg_1 \in H^1(\Gamma)$. From this follows immediately that

$$\int fg\, dm = \int fg_1 \omega^* = 0.$$

Hence $g \perp I^p(Q)$ and so $\mathfrak{M} = I^p(Q)$.

(ii) $p = +\infty$. Let $\mathfrak{M}$ be a weakly* closed invariant subspace of $L^\infty(\Gamma)$. We define $\mathfrak{N} \subseteq L^1(\Gamma)$ by setting

$$\mathfrak{N} = \{f \in L^1(\Gamma) : \int fg\omega^* = 0 \text{ for all } g \in \mathfrak{M}\}.$$

Then $\mathfrak{N}$ is closed and invariant. By (i) we then have two cases: either $\mathfrak{N} = C_{S'} L^1(\Gamma)$ for some measurable subset S' of Γ or $\mathfrak{N} = I^1(Q')$ for some i-function Q'. In the first case, it is obvious that $\mathfrak{M} = C_S L^\infty(\Gamma)$, where $S = \Gamma - S'$. In the second case, $\mathfrak{M} = I^\infty(Q)$ for $Q = \bar{Q}'$. Indeed, if $f \in \mathfrak{M}$, then $\int \phi f g\omega^* = 0$ for any $g \in \mathfrak{N}$ and $\phi \in A(R)$. So $fg \in H^1(\Gamma)$. Since

$$g \in \mathfrak{N} = I^1(Q'),$$

then $g/Q' = g\bar{Q}' = gQ \in \mathfrak{H}^1(R)$, i.e. $g(t)Q(t, \alpha) = h(t, \alpha)$ for an $h \in \mathfrak{H}^1(R)$. We have

$$f(t)/Q(t, \alpha) = (f(t)g(t))/(g(t)Q(t, \alpha)) = f(t)g(t)/h(t, \alpha).$$

Since we can always find a non-vanishing h, this shows that $f(t)/Q(t, \alpha)$ can be extended to a bounded multiplicative function on R. So $f \in I^\infty(Q)$. Hence

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$\mathfrak{M} \subseteq I^\infty(Q)$. Now we take any $g \in L^1(\Gamma)$ ssch that $g\omega^* \perp \mathfrak{M}$. We have $g \in \mathfrak{N} = I^1(Q')$. Take any $f \in I^\infty(Q)$. Then $f(t)/Q(t, \alpha) = h(t, \alpha)$ for $h \in \mathfrak{H}^\infty(R)$. So

$$\begin{aligned}\int f(t)g(t)\omega^*(t) &= \int f(t)Q(t,\alpha)^{-1}Q(t,\alpha)g(t)\omega^*(t)\\ &= \int h(t,\alpha)(Q(t,\alpha)g(t))\omega^*(t) = 0\end{aligned}$$

since both $h(t, \alpha)$ and $Q(t, \alpha)g(t)$ are analytic. Thus $g\omega^* \perp I^\infty(Q)$. Hence $\mathfrak{M} = I^\infty(Q)$, as was to be proved.

(iii) Clearly the spaces $C_S L^p(\Gamma)$ are doubly invariant. So we have only to show that $I^p(Q)$, $1 \leqslant p \leqslant +\infty$, are simply invariant. Let χ be any non-vanishing multiplicative function analytic on $\bar{R}$ such that $Q\chi$ is single-valued. For any $f \in I^p(Q)$,

$$f(t)/Q(t, \alpha) = h(t, \alpha) \in \mathfrak{H}^p(R),$$

so that

$$f(t)Q(t,\alpha)^{-1}\chi(t,\alpha)^{-1} = h(t,\alpha)\chi(t,\alpha)^{-1} \in H^p(\Gamma).$$

If follows that $I^p(Q) = (Q\chi)H^p(\Gamma)$. On the other hand, we see, by Theorem 1, that $H^p(\Gamma) = [A(R)]_p$ for $1 \leqslant p < +\infty$. For $1 \leqslant p < +\infty$,

$$[A_0(R)I^p(Q)]_p = (Q\chi)[A_0(R)H^p(\Gamma)]_p = (Q\chi)H_0^p(\Gamma) < (Q\chi)H^p(\Gamma).$$

Hence $I^p(Q)$ is simply invariant. The closedness of $I^p(Q)$ is obvious. If $p = +\infty$, then we get a similar conclusion by replacing the norm-closure by the weak* closure.

Finally, since each closed (or weakly* closed, if $p = +\infty$) invariant subspace of $L^p(\Gamma)$ is either doubly or simply invariant, the theorem is established.

5. Invariant subspaces of $C(\Gamma)$ and $M(\Gamma)$. We shall study closed invariant subspaces of the space $C(\Gamma)$ of continuous functions on Γ as well as weakly* closed invariant subspaces of the space $M(\Gamma)$ of Radon measures on Γ.

Definition 4. A uniformly closed subspace B of $C(\Gamma)$ is called *doubly* (*simply*) *invariant* if $[A_0(R)B]_\infty = B$ ($[A_0(R)B]_\infty < B$), where $[\ \ ]_\infty$ denotes the uniform closure.

A weakly* closed subspace N of $M(\Gamma)$ is called *doubly* (*simply*) *invariant* if $[A_0(R)\,N]_* = N$ ($[A_0(R)N]_* < N$), where $[\ \ ]_*$ denotes the weak* closure in $M(\Gamma)$.

For the circle group X, invariant subspaces of $C(X)$ and $M(X)$ have been studied in **(2)** in detail. It turns out that a similar argument works in our general case, once we know the structure of the invariant subspaces of $L^p(\Gamma)$, especially for $p = 1, +\infty$.

Let K be a subset of Γ and let $Z(K)$ be the subspace of functions in $C(\Gamma)$ that vanish on K. Furthermore, $M(K)$ denotes the subspace of $M(\Gamma)$ consisting of all Radon measures supported on K. Then we have

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THEOREM 3. *Let B be a uniformly closed subspace of $C(\Gamma)$. Then:*

(a) *B is doubly invariant if and only if $B = Z(K)$ for some subset K of Γ.*

(b) *If B is simply invariant, then $B = I^\infty(Q) \cap Z(K)$, where Q is an i-function and K is a compact set in Γ of measure zero.*

(c) *If $B = I^\infty(Q) \cap Z(K)$ for some i-function Q and some compact set $K \subseteq \Gamma$ and if B is non-trivial, then B is simply invariant.*

THEOREM 4. *Let N be a weakly* closed subspace of $M(\Gamma)$. Then:*

(a) *N is doubly invariant if and only if $N = M(K)$ for some compact subset K of Γ.*

(b) *If N is simply invariant, then $N = I^1(P)\omega^* + M(K)$, where P is an i-function and K is a compact set in Γ of measure zero.*

We wish to present a combined proof of Theorems 3 and 4, as we already did in (2). Let B be a closed invariant subspace of $C(\Gamma)$ and $N = B^\perp$. Then N is weakly* closed and invariant. It is clear that every weakly* closed invariant subspace of $M(\Gamma)$ can be obtained in this way.

Let K be the set of the common zeros of the functions in B. Then K is closed. Let μ be any Radon measure on Γ that is orthogonal to B and let $\mu = F\omega^* + \nu$ be the Lebesgue decomposition of μ with respect to ω^*, where $F \in L^1(\Gamma)$. Since B is invariant, we have $\int \phi f\, d\mu = 0$ for any $\phi \in A(R)$ and $f \in B$. By Theorem 1, $f\, d\mu = h\omega^*$ for some $h \in H^1(\Gamma)$. So we get $fF = h$ and $f\nu = 0$. Thus ν is supported on K and of course orthogonal to B. Consequently $F\omega^*$ is also orthogonal to B and hence we have shown that

$$N = B^\perp = (N \cap L^1(\Gamma)\omega^*) + M(K).$$

We define a subspace $\mathfrak{N}$ of $L^1(\Gamma)$ by $N \cap L^1(\Gamma)\omega^* = \mathfrak{N}\omega^*$. Then it is easy to see that $\mathfrak{N}$ is a closed invariant subspace of $L^1(\Gamma)$. By Theorem 2, we have either $\mathfrak{N} = C_{S'} L^1(\Gamma)$ for some measurable set S' or $\mathfrak{N} = I^1(P)$ for some i-function P. Now we divide our argument into three parts.

(i) Suppose first that $\mathfrak{N} = C_{S'} L^1(\Gamma)$ and therefore

$$N = C_{S'} L^1(\Gamma)\omega^* + M(K).$$

Then $m(S' - K) = 0$. Suppose, on the contrary, that $m(S' - K) > 0$. Then there exists a compact subset K' of $S' - K$ such that $m(K') > 0$. It follows from the definition of K that there exists a function $f \in B$ such that

$$\int_{K'} |f|\, dm > 0.$$

As $K' \subseteq S'$, f cannot be orthogonal to $C_{S'} L^1(\Gamma)\omega^*$. This contradiction shows that $m(S' - K) = 0$. Therefore

$$C_{S'}L^1(\Gamma)\omega^* \subseteq C_K L^1(\Gamma)\omega^* \subseteq M(K).$$

Hence $N = M(K)$ and consequently $B = Z(K)$. In this case, both B and N are doubly invariant.

(ii) Suppose now that $\mathfrak{N} = I^1(P)$. Then $N = I^1(P)\omega^* + M(K)$ and $B = (I^1(P)\omega^*)^\perp \cap Z(K)$. As shown in (ii) of the proof of Theorem 2,

$$(I^1(P)\omega^*)^\perp = I^\infty(Q)$$

with $Q = \bar{P}$. Hence $B = I^\infty(Q) \cap Z(K)$. In this case, $m(K) = 0$, because a function in $I^\infty(Q)$ cannot vanish on a set of positive measure without vanishing identically.

We want to show that both B and N are simply invariant. We take a non-vanishing multiplicative function χ continuous on $\bar{R}$ such that $P\chi$ is single-valued. Then $I^1(P) = (P\chi)H^1(\Gamma)$. So

$$A_0(R)N = A_0(R)(I^1(P)\omega^* + M(K)) = (P\chi)A_0(R)H^1(\Gamma)\omega^* + M(K)$$
$$= (P\chi)\phi_0 H^1(\Gamma)\omega^* + M(K),$$

where ϕ_0 is an element in $A(R)$ with a simple zero at ζ_0 and non-vanishing elsewhere. Since $B \neq \{0\}$, $I^1(P)\omega^*$ is not weakly* dense in $M(\Gamma)$ so that there exists a non-zero function $f \in C(\Gamma)$ orthogonal to $I^1(P)\omega^*$. It follows from $I^1(P) = (P\chi)H^1(\Gamma)$ and Theorem 1 that $f(P\chi) \in H^\infty(\Gamma)$. Now we need the following special case of Bishop's theorem **(1)**:

LEMMA 4. *For any compact set $K \subseteq \Gamma$ of measure zero, there is a non-zero function $\in A(R)$ that vanishes identically on K.*

Since our set K satisfies the hypothesis of the lemma, there is such a function, which we denote by ϕ_1. We choose an integer $l > 0$ in such a way that

$$f\phi_1(P\chi)\phi_0^{-l}\omega^*$$

has a simple pole at ζ_0. Let $f_0 = f\phi_1\phi_0^{-l}$. Then it is immediate that f_0 is orthogonal to $A_0(R)I^1(P)\omega^*$ but not to $I^1(P)\omega^*$. As f_0 vanishes identically on K, it is orthogonal to $M(K)$. Since f_0 is a non-zero function, we have shown that $A_0(R)N$ is not weakly* dense in N, i.e. N is simply invariant.

Now we have $\phi_0^{-1} N > N$. Otherwise, $N = \phi_0 N$ and a fortiori $A_0(R)N = N$, which is a contradiction. Thus there exists a measure $\mu \in N$ such that $\phi_0^{-1}\mu \notin N$. So $\phi_0^{-1}\mu$ is orthogonal to $A_0(R)B$ but not to B. Hence $A_0(R)B$ is not uniformly dense in B. B is thus simply invariant.

(iii) Combining (i) and (ii), we see that (a) and (b) of Theorems 3 and 4 are true. Finally Theorem 3, (c) is also true because a non-trivial subspace B of the form $I^\infty(Q) \cap Z(K)$ cannot be equal to any $Z(K_1)$ with compact K_1. This completes the proof of the theorem.

6. Non-triviality and uniqueness of the expressions for the invariant subspaces of $C(\Gamma)$ and $M(\Gamma)$. In **(2)** we saw that, in the circle group case, $I^\infty(Q) \cap Z(K)$ can be trivial and we gave a necessary and sufficient condition for non-triviality. Now we wish to generalize the results in **(2)**. The following lemma is an immediate consequence of Lemma 4.

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LEMMA 5. *Let Q be an i-function and K a compact set $\subseteq \Gamma$ of measure zero. Then $I^\infty(Q) \cap Z(K)$ is trivial if and only if $I^\infty(Q) \cap C(\Gamma)$ is trivial.*

So, in what follows, we consider only the expression $I^\infty(Q) \cap C(\Gamma)$. We have the following

THEOREM 5. *Let Q be an i-function. Then $I^\infty(Q) \cap C(\Gamma)$ is non-trivial if and only if Q has the following factorization into i-functions:*

$$(1) \qquad Q = Q_1 Q_2 Q_3,$$

where Q_1 is conjugate inner (i.e. $\bar{Q}_1$ is an inner function in the sense of §2), *Q_2 is single-valued and continuous except on a compact set of measure zero, and*

$$Q_3(t, \alpha) = |h_0(t, \alpha)|/h_0(t, \alpha) \qquad (t \in \Gamma, \alpha \in \mathfrak{g})$$

for an outer function h_0 such that $|h_0|$ is continuous on Γ.

Proof. First suppose that $I^\infty(Q) \cap C(\Gamma)$ is non-trivial and let g be any non-zero function in it. By the definition of $I^\infty(Q)$, g/Q is induced from a function h in $\mathfrak{H}^\infty(R)$. Let $h = h_i h_0$ be a factorization of h into its inner and outer factors h_i and h_0. We may assume without loss of generality that

$$g(t) = Q(t, \alpha) h_i(t, \alpha) h_0(t, \alpha).$$

Since $|g(t)| = |h_0(t, \alpha)|$ and $|h_i(t, \alpha)| = 1$ a.e., we have

$$\begin{aligned} Q(t, \alpha) &= g(t)\overline{h_i(t, \alpha)} h_0(t, \alpha)^{-1} \\ &= \overline{h_i(t, \alpha)} \exp(i \arg g) |g(t)| h_0(t, \alpha)^{-1}. \end{aligned}$$

By putting

$$Q_1(t, \alpha) = \overline{h_i(t, \alpha)}, \qquad Q_2(t) = \exp(i \arg g(t)),$$

and

$$Q_3(t, \alpha) = |g(t)|/h_0(t, \alpha) = |h_0(t, \alpha)|/h_0(t, \alpha),$$

we get the desired factorization of Q into i-functions specified in the theorem.

Conversely, suppose that an i-function Q has a factorization of the form (1). Then there exists an inner function h_1 such that $Q_1 = \bar{h}_1$, Q_2 is single-valued and discontinuous only on a set K of measure zero, and $Q_3 = |h_2|/h_2$ for an outer function h_2 with continuous modulus. Let h_3 be a non-zero function in $A(R)$ that vanishes on K and let $g = |h_2| Q_2 h_3$. Since $|h_2|$ is continuous and h_3 vanishes at all discontinuities of Q_2, g is a continuous function on Γ. It is easy to see that

$$g/Q = g/(Q_1 Q_2 Q_3) = (|h_2| Q_2 h_3)/(\bar{h}_1 Q_2 |h_2| h_2^{-1}) = h_1 h_2 h_3 \in \mathfrak{H}^\infty(R).$$

Hence $g \in I^\infty(Q)$ and therefore $I^\infty(Q) \cap C(\Gamma)$ is non-trivial, as was to be proved.

Since $I^\infty(Q) \cap C(\Gamma)$ is the orthogonal complement of $I^1(\bar{Q})\omega^*$ in $C(\Gamma)$, we have the following

COROLLARY. *Let P be an i-function. Then $I^1(P)\omega^*$ is not weakly* dense in $M(\Gamma)$ if and only if P has the following factorization into i-functions:*

$$P = P_1 P_2 P_3,$$

where P_1 is an inner function, P_2 is single-valued and continuous except on a compact set of measure zero, and $P_3 = |h_0|/\overline{h_0}$ for an outer function h_0 such that $|h_0|$ is continuous on Γ.

Now we wish to discuss uniqueness of the expressions for the invariant subspaces of $C(\Gamma)$ and $M(\Gamma)$. First of all, it is obvious that the expression for the doubly invariant subspaces obtained in Theorem 3, (a) and Theorem 4, (a) is unique. It is also easy to see that, in the expression $N = I^1(P)\omega^* + M(K)$ obtained in Theorem 4, (*b*), K is unique and P is unique up to equivalence and a constant factor of modulus one. Finally we get the following, which is less trivial.

THEOREM 6. *In the expression $B = I^\infty(Q) \cap Z(K)$ of a simply invariant closed subspace B of $C(\Gamma)$ given by Theorem 3, (b), the i-function Q is determined uniquely by B up to equivalence and a constant factor of modulus one.*

Proof. The proof is nearly the same as that of the corresponding theorem in **(2)**. As our domain R is in general multiply connected, we need a little further consideration.

Let B be a closed simply invariant subspace of $C(\Gamma)$ and suppose that $B = I^\infty(Q) \cap Z(K)$ for an i-function Q and a compact set K of measure zero. Since $B^\perp$ is simply invariant in $M(\Gamma)$, Theroem 4 says that

$$B^\perp = I^1(P_0)\omega^* + M(K_0)$$

for some i-function P_0 and a compact set $K_0 \subseteq \Gamma$ of measure zero. So $B = I^\infty(Q_0) \cap Z(K_0)$ with $Q_0 = \bar{P}_0$. Since $B \subseteq I^\infty(Q)$, $B^\perp \supseteq I^1(\bar{Q})\omega^*$ and therefore $I^1(P_0) \supseteq I^1(\bar{Q})$. Thus $I^\infty(Q_0) \subseteq I^\infty(Q)$. Thus there exists an inner function W such that $Q_0(t, \alpha) = Q(t, \alpha)W(t, \alpha)$ on $\Gamma \times \mathfrak{g}$. We wish to show that W is a constant function.

LEMMA 6. *W has no zero in R.*

Proof. Suppose the statement is false. Let ζ_1 be a zero of W in R. Take a non-zero function $f \in B = I^\infty(Q) \cap Z(K)$. Then there exists a function $h \in \mathfrak{H}^\infty(R)$ such that $f/Q \equiv h$. Since there is a function in $A(R)$ that has a simple zero at ζ_1 and vanishes nowhere else on $\bar{R}$, we may assume, by modifying f if necessary, that $h(\zeta_1) \neq 0$. Now since $f \in I^\infty(Q_0) \cap Z(K_0)$, there is an $h_0 \in \mathfrak{H}^\infty(R)$ such that $f/Q_0 \equiv h_0$. So $h_0 \equiv f/Q_0 \equiv f/(QW) = h/W$. Thus $h \equiv h_0 W$, which is impossible because W vanishes at ζ_1 but h does not. Hence W has no zero in R, as was to be proved.

As shown in §2, there exists a positive singular measure μ on Γ such that

$$\log |W(\zeta)| = -\frac{1}{2\pi} \int_{\Gamma} \frac{\partial G(\zeta, t)}{\partial n_t} d\mu(t)$$

for $\zeta \in R$. Now we show

LEMMA 7. *W is a constant function.*

Proof. Suppose W is not constant. Then the measure μ defined above is non-trivial. Since μ is singular, there exists a compact set $K' \subseteq \Gamma$ of measure zero such that $\mu(K') > 0$. We define a singular inner function W' on R by

$$\log |W'(\zeta)| = -\frac{1}{2\pi} \int_{K'} \frac{\partial G(\zeta, t)}{\partial n_t} d\mu(t)$$

and also W'' by $W'W'' = W$. It is easy to see that W' is continuous on Γ except on K'.

Since $B = I^\infty(Q_0) \cap Z(K_0)$ is non-trivial, Theorem 5 says that $Q_0 = Q_1 Q_2 Q_3$, where Q_1 is conjugate inner, Q_2 is single-valued and continuous except on a compact set K'' of measure zero, and $Q_3 = |h_0|/h_0$ for an outer function h_0 such that $|h_0|$ is continuous on Γ. Let Q_b and Q_s be the Blaschke and the singular factors of the inner function $\bar{Q}_1$, respectively, as defined in §2. There is a positive singular measure ν on Γ such that

$$\log |Q_s(\zeta)| = -\frac{1}{2\pi} \int_{\Gamma} \frac{\partial G(\zeta, t)}{\partial n_t} d\nu(t).$$

We define, as before, inner functions Q' and Q'' by

$$\log |Q'(\zeta)| = -\frac{1}{2\pi} \int_{K'} \frac{\partial G(\zeta, t)}{\partial n_t} d\nu(t)$$

and $Q_s = Q'Q''$.

By Lemma 4, there exists a non-zero function $h_1 \in A(R)$ that vanishes on $K \cup K' \cup K''$. Let h_2 be the outer factor of h_1, which is determined uniquely up to a constant factor of modulus one. Since W', Q', and h_2 do not vanish anywhere on R, $\log W' + \log Q' - \log h_2$ is a well-defined additive analytic function on R. As we remarked in §2, there exists an analytic function u on R such that (i) u is analytic on $\bar{R}$, (ii) u never vanishes on $\bar{R}$, and (iii) u has the same period as $\log W' + \log Q' - \log h_2$. Put $h_3 = \exp u$. Since the period of $\log W' + \log Q' - \log h_2$ is pure imaginary, so are the periods of u. So h_3 is a multiplicative function that is continuous on $\bar{R}$.

Finally we define a multiplicative function h_4 by $h_4 = W''Q_b Q''h_0 h_2 h_3$. Clearly $h_4 \in \mathfrak{H}^\infty(R)$. We have

$$\begin{aligned} Qh_4 = \bar{W}Q_0 h_4 &= (\bar{W}'\bar{W}'')(\bar{Q}'\bar{Q}''\bar{Q}_b Q_2|h_0|h_0^{-1})(W''Q_b Q''h_0 h_2 h_3) \\ &= (Q_2|h_0|)(\bar{W}'\bar{Q}'h_2 h_3) = (Q_2|h_0|)(W'^{-1}Q'^{-1}h_2 h_3). \end{aligned}$$

We know that $Q_2|h_0|$ is single-valued. It is clear from our construction that

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$W'^{-1}Q'^{-1}h_2 h_3$ has no non-trivial period and so this is also single-valued. Since h_2 vanishes at all singularities of other factors, Qh_4 is single-valued and continuous. Since Qh_4 vanishes on $K \subseteq K_0$, it is in B. So $Qh_4 \in I^\infty(Q_0) \cap Z(K_0)$. There exists a function $h_5 \in \mathfrak{H}^\infty(R)$ such that $(Qh_4)/Q_0 \equiv h_5$. We have $Q_b Q'' h_6 h_2 h_3 \equiv W' h_5$. This implies that W' must divide $Q_b Q''$ and indeed W' must divide the singular part Q''. But this is impossible because the supports of measures corresponding to W' and Q'' are disjoint. Hence W must be a constant function. This proves Lemma 7 and thus Theorem 6 is established.

7. Some special cases. We have mentioned already that our results extend Voichick's and Sarason's theorems. Now we wish to indicate briefly the proof of these theorems.

(a) *Closed ideals of* $A(R)$. Let J be any non-trivial closed ideal of $A(R)$. Then J is a closed simply invariant subspace of $C(\Gamma)$. By Theorem 3,

$$J = I^\infty(Q) \cap Z(K)$$

with an i-function Q and a compact set $K \subseteq \Gamma$ of measure zero. Since

$$J^\perp = I^1(\bar{Q})\omega^* + M(K),$$

we see that the weak* closure $[J]_*$ of J in $L^\infty(\Gamma)$ is equal to $I^\infty(Q)$. Since $H^\infty(\Gamma)$ is weakly* closed in $L^\infty(\Gamma)$, we have $I^\infty(Q) \subseteq H^\infty(\Gamma)$. Hence $Q \in \mathfrak{H}^\infty(R)$, so Q is an inner function in the sense of Voichick **(8)**. This proves Theorem 1 of Voichick **(8)**.

(b) *Closed invariant subspaces of* $H^p(\Gamma)$. Let I be any closed (weakly* closed, if $p = +\infty$) invariant subspace of $H^p(\Gamma)$. Then I is either trivial or simply invariant. Suppose it is simply invariant. By Theorem 2, $I = I^p(Q)$ with an i-function Q. Since $I^p(Q)$ is now contained in $H^p(\Gamma)$, we again conclude that Q is an inner function. Theorem 2 of Voichick **(8)** is a special case ($p = 2$) of this fact.

(c) *Closed invariant subspaces of annulus operators.* Let R be an annulus $\{z: r_0 < |z| < 1\}$ $(r_0 > 0)$ and let $\mathfrak{M}$ be any closed (weakly* closed, if $p = +\infty$) invariant subspace of $L^p(\Gamma)$ with respect to the annulus operator, i.e. the multiplication by z restricted to the boundary Γ of the annulus. If $\mathfrak{M}$ is doubly invariant (in our sense), then $\mathfrak{M} = C_S L^p(\Gamma)$ for some measurable subset S of Γ. So $\mathfrak{M}$ consists of all L^p-functions that vanish at every point where C_S vanishes. Of course, C_S is a member of $\mathfrak{M}$.

Suppose now that $\mathfrak{M}$ is simply invariant. Then, by Theorem 2, $\mathfrak{M} = I^p(Q)$ for some i-function Q. We know that Q satisfies the relation

$$Q(t, \alpha + \beta) = c_\alpha c_\beta Q(t, 0)$$

for any $\alpha, \beta \in \mathfrak{g}$ and $t \in \Gamma$. Since R is an annulus, the integral homology group

$\mathfrak{g}$ of 1-cycles of R is an infinite cyclic group, i.e. $\mathfrak{g}$ is isomorphic to the additive group of integers. So our relation can be written in the form

$$Q(t, n) = \exp(-2\pi i\kappa n)Q(t, 0)$$

for any integer n and $t \in \Gamma$, where κ is a real number. We may assume that $0 \leqslant \kappa < 1$. This implies that $z^\kappa Q$ is single-valued and therefore $z^\kappa Q \in \mathfrak{M}$. Thus $z^\kappa QH^p(\Gamma) \subseteq \mathfrak{M}$. Take any $f \in \mathfrak{M} = I^p(Q)$. Then f/Q is in $\mathfrak{H}^p(R)$ so that $f/(z^\kappa Q) \in H^p(\Gamma)$. Hence $f \in z^\kappa QH^p(\Gamma)$. Consequently, $\mathfrak{M} = I^p(Q) = z^\kappa QH^p(\Gamma)$. If $p = 2$, then $z^\kappa H^2(\Gamma)$ is essentially the same as $H_\kappa^2(\Gamma)$ of Sarason **(4)**.

We shall determine the exponent κ. Take a non-zero $f \in \mathfrak{M}$. Then $f = z^\kappa Qh$ with an $h \in H^p(\Gamma)$. So we have

$$\log|f| = \begin{cases} \log|h| & \text{for } |z| = 1, \\ \log r_0 + \log|h| & \text{for } |z| = r_0. \end{cases}$$

We choose f in such a way that h is analytic on the closed unit disk and never vanishes there (e.g. $h = 1$). Then $\log|h|$ is harmonic on the unit disk and therefore we have

$$\begin{aligned} &\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta - \int_0^{2\pi} \log|f(r_0 e^{i\theta})|\,d\theta \\ &= \int_0^{2\pi} \log|h(e^{i\theta})|\,d\theta - \int_0^{2\pi} \log|h(r_0 e^{i\theta})|\,d\theta - 2\pi\kappa\log r_0 \\ &= -2\pi\kappa\log r_0, \end{aligned}$$

which is exactly the Sarason formula for the exponent. This proves Theorems 1 and 2 of Sarason **(4)**, where $p = 2$.

(d) *Cyclic vectors in* $H^p(\Gamma)$. An analytic function $h \in H^p(\Gamma)$ is called a *cyclic vector* if it generates the whole space $H^p(\Gamma)$. It is easy to see that h is cyclic if and only if h is outer in our sense, i.e.

$$\log|h(\zeta)| = -\frac{1}{2\pi}\int_\Gamma \frac{\partial G(\zeta, t)}{\partial n_t}\log|h(t)|\,ds_t. \tag{2}$$

If R is an annulus $\{z\colon r_0 < |z| < 1\}$, $r_0 > 0$, then it is known that

$$\begin{aligned} G(\zeta, t) &= -\delta\log r + \tfrac{1}{2}\log[r^2 - 2rr_0^{\delta}\cos(\theta - \vartheta) + r_0^{2\delta}] \\ &+ \sum_{\nu=1}^{\infty}\log[(1 - 2rr_0^{2\nu-\delta}\cos(\theta - \vartheta) + r^2r_0^{4\nu-2\delta})(1 - 2r^{-1}r_0^{2\nu+\delta}\cos(\theta - \vartheta) \\ &+ r^{-2}r_0^{4\nu+2\delta})] + \sum_{\nu=1}^{\infty}\log[(1 - 2rr_0^{2\nu-2+\delta}\cos(\theta - \vartheta) + r^2r_0^{4\nu-4+2\delta}) \\ &\times (1 - 2r^{-1}r_0^{2\nu-\delta}\cos(\theta - \vartheta) + r^{-2}r_0^{4\nu-2\delta})], \end{aligned}$$

where $t = re^{i\theta}$ $(r_0 < r < 1)$ and $\zeta = r_0^{\delta}e^{i\vartheta}$ $(0 < \delta < 1)$. Using this expression in (2) and then integrating both sides of (2) from 0 to 2π with respect to ϑ, we see that the Sarason formula for cyclic vectors (**4**, Theorem 4) is valid for all $p \geqslant 1$.

(e) *Maximality of the algebra $A(R)$ in $C(\Gamma)$.* Finally we shall show that $A(R)$ is a maximal closed subalgebra of $C(\Gamma)$. To see this, let B be any proper closed subalgebra of $C(\Gamma)$ containing A. Then B is an invariant subspace of $C(\Gamma)$ and indeed it is simply invariant. So $B^\perp = I^1(P)\omega^*$ for some i-function P. Since $B \supseteq A(R)$, we have, by Theorem 1, $B^\perp \subseteq A(R)^\perp = H^1(\Gamma)\omega^*$. Therefore $I^1(P) \subseteq H^1(\Gamma)$. On the other hand, B is an algebra so that $BB \subseteq B$. It follows that $BB^\perp \subseteq B^\perp$, i.e. $BI^1(P) \subseteq I^1(P)$. This immediately implies that

$$B \subseteq H^\infty(\Gamma).$$

Hence $B \subseteq H^\infty(\Gamma) \cap C(\Gamma) = A(R)$, as was to be proved. This extends a theorem of Wermer (**9**).

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