Compositio Mathematica **111:** 323–358, 1998.

© 1998 *Kluwer Academic Publishers. Printed in the Netherlands.*

Double complexes and Euler L-factors

CATERINA CONSANI
Department of Mathematics, M.I.T., Cambridge, MA. 02139 USA; e-mail:katia@math.mit.edu

Received: 2 July 1996; accepted in final form: 6 February 1997

Abstract. In this paper we take into consideration a conjecture of Bloch equating in a suitable range and under some standard conjectures, the rank of the motivic cohomology of the special fiber of a semistable degeneration over the ring of integers of a number field with the order of zero of the local Euler factor of the L-function over the semistable prime corresponding to the special fiber. We then develop, following the philosophy that the fiber 'at infinity' of an arithmetic variety should be considered as 'maximally degenerate', a construction that goes in parallel to the one we use for the non-archimedean fiber. Namely, a definition of a double complex and a weighted operator N on it that plays the role of the logarithm of the (local) monodromy map at infinity. We like to see this archimedean theory as the analogue of the limiting mixed Hodge structure theory of a degeneration of projective varieties over a disc. In particular, this yields to a description of the (motivic) Deligne cohomology as homology of the mapping cone of N. The main result arising from this construction is the proof of a conjecture of Deninger. Namely we show that the archimedean Γ-factor of the Zeta-function of the archimedean fiber can be seen as the characteristic polynomial of an archimedean Frobenius acting on the subgroup of the invariants of N into the hypercohomology of our double complex.

Key words: algebraic cycles, values of L-functions, motivic cohomology.

Introduction

In this paper we give an interpretation of the zeroes of the local L-factors related to degenerations of algebraic varieties. By degeneration we mean either a semistable degeneration of a family of proper and smooth varieties defined over a discrete valuation ring for which the special fiber Y is a reduced normal-crossings divisor over a finite field, or – following a suggestion of Manin (cf [15]) – the 'maximally degenerate' fiber at infinity X of an arithmetic variety. Let X be a proper, smooth variety over a number field K and let $\mathfrak{X}$ be a regular model of X defined over the ring of integers $\mathcal{O}_K$ of K (cf Section 1 for the notations). Let $\wp$ be a prime ideal of $\mathcal{O}_K$ and let $\mathfrak{X} \times \mathrm{Spec}(k(\wp)) = Y$ be the fiber over $\wp$. Assume that Y is a reduced, normal-crossings divisor on $\mathfrak{X}$ and that the residue field $k(\wp)$ is finite. Our main result in this non-archimedean case, is the proof of a conjecture by Bloch (cf Section 2) equating in a suitable range and under some standard conjectures, the rank of the motivic cohomology of Y with the order of zero of the local Euler factor of the L-function at $\wp$. The most interesting object we deal with is a double complex $(K^{\cdot,\cdot}, d', d'')$. $K^{\cdot,\cdot}$ is defined as a direct sum of l-adic cohomology groups of the different strata of the special fiber Y and the differentials d' and d'' are determined by means of corestrictions and the Gysin maps relating these groups. Furthermore,

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

$K^{\cdot,\cdot}$ is endowed with an operator N which plays a central role in this theory and behaves as the logarithm of the (local) monodromy map. To our knowledge, this complex was firstly introduced by Steenbrink (cf. [22]), with the purpose of making explicit the E_1-terms of the spectral sequence of the vanishing cycles for an algebraic degeneration over a disc and subsequently it was reexamined in the same context by Guillén and Navarro Aznar (cf. [11]) in their proof of the local invariant cycle theorem. Recently, Bloch, Gillet and Soulé (cf. [3]) have shown that a complex as above makes sense and can be studied in a more general set up i.e. for most cohomology theories and also for algebraic cycles modulo any adequate equivalence relation. If one for example uses the groups $CH^{\cdot}(Y^{(*)})$ of algebraic cycles of the strata modulo rational equivalence (cf. Section 1 for the notations), it is fairly easy to prove (cf. (3.14)) that the mapping cone of N is a complex quasi-isomorphic to

$$\begin{aligned} &\ldots CH^{\cdot-3}(Y^{(3)}) \xrightarrow{d''} CH^{\cdot-2}(Y^{(2)}) \to CH^{\cdot-1}(Y^{(1)}) \\ &\xrightarrow{i^*\cdot i_*} CH^{\cdot}(Y^{(1)}) \xrightarrow{d'} CH^{\cdot}(Y^{(2)}) \to CH^{\cdot}(Y^{(3)}) \ldots \end{aligned} \tag{0.1}$$

Here, the map in the middle $i^* \cdot i_*$ is the composite of a push-forward toward the group of algebraic cycles of the family, followed by a pullback to the groups of the components of Y. In [2], it has been shown that when resolution of singularities holds, both the kernel and the cokernel of $i^* \cdot i_*$ depend only on the generic fiber of the family. Furthermore, up to replacing the Chow groups by some appropriate cohomology theory $H^{\cdot}$, like $\mathbb{Q}_l$-étale or $\mathbb{Q}$-Betti, the cap product induces isomorphisms ($n = \dim Y$)

$$\frac{\mathrm{Ker}(d'\colon\ H^{\cdot}(Y^{(1)}) \to H^{\cdot}(Y^{(2)}))}{\mathrm{Image}(i^* \cdot i_*)} \simeq \frac{\mathrm{Ker}(i^* \cdot i_* : H_{2n-\cdot}(Y^{(1)}) \to H^{\cdot+2}(Y^{(1)}))}{\mathrm{Image}(d'')}.$$

We like to interpret these isomorphisms as the first step of a 'symmetry' between right and left hand side of two complexes analogous to (0.1) but built up using cohomology groups of the strata. In order to explain this symmetry for the $\mathbb{Q}_l$-étale cohomology theory, let us consider the spectral sequence of the vanishing cycles and recall some of its properties. The couple made by the E_1-term and the first differential of this spectral sequence is isomorphic to the complex $(K^{\cdot,\cdot}, d = d' + d'')$, further it is known that the spectral sequence degenerates from E_2 on, toward the cohomology of the geometric generic fiber of the family. The filtration (by the weights of the Frobenius on the special fiber i.e. the weight filtration) induced on the abutment coincides with the monodromy filtration when $\dim Y \leqslant 2$ (cf. [16]). In higher dimensions that coincidence is still an open conjecture. In the paper we refer to it as to the Monodromy Conjecture. The coincidence of the weight and the monodromy filtrations implies isomorphisms between the corresponding graded groups, by means of powers of N. A suitable decomposition of these maps

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

gives rise then, to the (symmetry-type) isomorphisms previously mentioned (cf. the proof of Theorem 3.5).

Our results can be also described in terms of the Beilinson conjecture relating the rank of the global motivic cohomology with the order of vanishing of a global L-function. Here are two possible examples.

1. The local Euler factor of an elliptic curve at a prime $p \in \mathbb{Z}$ ($K = \mathbb{Q}$) of multiplicative reduction is $(1 \pm |k(p)|^{-s})^{-1}$. The sign inside the parenteses is negative and the factor has a single pole at $s = 0$, if and only if the curve has split multiplicative reduction. This corresponds to the rationality of the 0-cycle which is the difference of the two points in the normalization of the fiber lying over the singular point p.

A much deeper example, suggested to us by K. Kato, is the following.

2. Let assume that the monodromy T around $\wp$ is unipotent. Then, one has for any integer $q \geqslant 0, N := \log(T) : H^q(X_{\bar{K}}, \mathbb{Q}_l) \to H^q(X_{\bar{K}}, \mathbb{Q}_l(-1))$. Hence, as an operator, N can be interpreted as a class in $H^{2d}((X \times X)_{\bar{K}}, \mathbb{Q}_l(d-1))$ ($d = \dim X_{\bar{K}}$) invariant under the decomposition group and therefore giving rise to (i.e. explaining the presence of) a pole of the local factor at $\wp$ of $L(H^{2d}((X \times X)_{\bar{K}}, \mathbb{Q}_l), s)$ at $s = d-1$. By our result N corresponds (assuming the conjectures mentioned above) to an algebraic cycle of codimension $d-1$ on the threefold intersections of components of the special fiber of a semistable model of $X \times X$. We hope to discuss this description of the monodromy in a subsequent paper.

Following this order of ideas, we were lead to think that for the reduced and irreducible fiber at infinity X of an arithmetic variety, a corresponding construction (i.e. a definition of a double complex $K^{\cdot,\cdot}$ and a weighted operator N) should yield to a description of the (motivic) Deligne cohomology as homology of the mapping cone of an endomorphism N. The complex we define in Section 4 is a direct sum of groups of real differential forms on X having certain weight and type and the differentials d' and d'' are described by means of the real differential operators d and d^c on X. Then, Proposition 4.1 shows what we expected. Furthermore, we prove that the hypercohomology of the simple complex $K^{\cdot,\cdot}$ associated to $K^{\cdot,\cdot}$ is a polarized bigraded Lefschetz module in the sense of [17]. Hence, N induces isomorphisms on the graded pieces $gr_*^W H^{\cdot}(\tilde{X}^*)$ of the hypercohomology groups of the complex $K^{\cdot}$ (cf. Proposition 4.8) and in turn these isomorphisms lead to (symmetry) isomorphisms (cf. Proposition 4.13) like in the case of a semistable degeneration.

This archimedean theory should be thought of as the analogue of the limiting mixed Hodge structure theory of a degeneration of projective varieties over a disc as developed in cf. [22]. This correspondence suggests a connection of our double complex $K^{\cdot,\cdot}$ with the E_0-terms of a spectral sequence of 'vanishing cycles' at infinity. Also we conjecture the existence of a real mixed Hodge structure, in the sense of Deligne (cf. [6]), on $K^{\cdot,\cdot}$ (cf. [5]).

Another application of these ideas is shown in Section 5. There, we define a linear operator Φ on the infinite dimensional hypercohomology vector spaces

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

$H^{\cdot}(\tilde{X}^*)(=\mathbb{H}^{\cdot}(K^{\cdot},d))$ studied in Section 4. Φ acts on the graded piece $gr_{2m}^W H^{\cdot}(\tilde{X}^*)$ of weight m as a multiplication by the weight. One might see Φ as a natural logarithm of a sort of geometric frobenius operator at infinity and try to deduce some analogies with the theory developed in Section 3 for a non archimedean semistable degenerate fiber. Then, our main result is Theorem 5.4 where we are able to recover, by means of the endomorphisms N and Φ on $H^{\cdot}(\tilde{X}^*)$, the Euler factor at infinity using the notion of infinite determinant introduced by Deninger in [7]. In *op. cit*. the Euler factors are reproduced as infinite determinants defined over the infinite dimensional vector spaces $H^{\cdot}_{\text{ar}}(X)$ (the 'archimedean cohomology') and by means of a weight graded linear endomorphism Θ. Proposition 5.3 proves that $(H^{\cdot}_{\text{ar}}(X),\Theta)$ coincides with our couple $(H^{\cdot}(\tilde{X}^*)^{N=0},\Phi)$. This result shows what was expected in [7], i.e. a natural geometric definition of $(H^{\cdot}_{\text{ar}}(X),\Theta)$ and also it seems to agree with the point of view expressed in [9] that the 'archimedean cohomology' should be thought as 'a fixed module under inertia' of a sort of universal cohomology theory over a (still unknown!) arithmetic site.

The paper is organized as follows. Section 1 contains the definitions of the main notations we have used in this paper. In Section 2 we state Bloch's conjecture. In Section 3 we introduce a double complex $K^{\cdot,\cdot}$, together with two differentials d' and d'' and an operator N. After recalling the main properties of these objects, we show how to prove Bloch's conjecture under suitable conditions (cf. Theorem 3.5). In Section 4 we define a second bigraded group $(K^{\cdot,\cdot},d',d'')$ and a corresponding map N, in order to study the closed fiber at infinity. The first part of this paragraph deals with the main properties of $K^{\cdot,\cdot}$. In particular, we show that the hypercohomology of the associated simple complex is a polarized bigraded Lefschetz module. As an application of this result, we prove (symmetry) isomorphisms between some Deligne cohomology groups (cf. Proposition 4.13). In Section 5 we use the theory developed in the previous paragraph to construct an infinite dimensional vector space $H^{\cdot}(\tilde{X}^*)^{N=0}$ together with an endomorphism Φ. Then, borrowing a notion of infinite determinant introduced in [7], we recover the Euler factor using the couple $(H^{\cdot}(\tilde{X}^*)^{N=0},\Phi)$ (cf. Theorem 5.4). Finally, we compare our construction with Deninger's one (cf. [7]): Proposition 5.3 shows a compatibility of the objects involved.

1. Notations

We denote by S the spectrum of a Henselian discrete valuation ring Λ. Let η and v be respectively the generic and closed points of S; $\bar{\eta}$ and $\bar{v}$ are the corresponding geometric points. By $k(\eta)$ and $k(v)$ we mean the residue fields of Λ at η and v. We write $k(\bar{\eta})$ for a separable closure of $k(\eta)$ and $k(\bar{v})$ for the residue field of the integral closure of Λ in $k(\bar{\eta})$, defining the geometric point $\bar{v}$. Let $\text{Gal}(\bar{\eta}/\eta)$ resp. $\text{Gal}(\bar{v}/v)$ be the Galois groups $\text{Gal}(k(\bar{\eta})/k(\eta))$ resp. $\text{Gal}(k(\bar{v})/k(v))$. Let $I\subset\text{Gal}(\bar{\eta}/\eta)$ be the inertia group, defined as the kernel of the map $\text{Gal}(\bar{\eta}/\eta)\to\text{Gal}(\bar{v}/v)$ (cf. [19] Chapt. 1 Section 7).

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

For Z a scheme of finite type over a field k, and l a prime number such that $(\mathrm{char}\, k, l) = 1$, we denote by $H^m(\bar{Z}, \mathbb{Q}_l)$ the l-adic geometric cohomology of $\bar{Z} = Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$, being $\bar{k}$ an algebraic closure of k.

Let X be a proper, smooth variety over a field. By a *model* of X we mean a flat, proper scheme over $\Lambda : \pi : \mathfrak{X} \to \mathrm{Spec}(\Lambda)$, together with an isomorphism (i.e. identification) of the generic fiber $\mathfrak{X}_\eta \simeq X(\mathfrak{X}_\eta := \mathfrak{X} \times \mathrm{Spec}(k(\eta)))$. Given a model $\mathfrak{X}$, we write Y for the *special fiber* $\mathfrak{X} \times \mathrm{Spec}(k(v))$.

In this paper we always assume that $\mathfrak{X}$ is a *semistable model*, by which we mean that $\mathfrak{X}$ is a regular model and the special fiber Y is a reduced divisor with normal crossings in $\mathfrak{X}$. We will often refer to these conditions by saying that π is a *semistable fibration*. In that case, each irreducible component Y_i of $Y = \cup_{i=1}^t Y_i$ is a regular scheme. For any subset $I \subseteq \{1, \dots, t\} I \neq \emptyset$, we set $Y_I = \cap_{i \in I} Y_i$. It follows from our assumptions that Y_I is a regular scheme. We define $Y_\emptyset = \mathfrak{X}$. Let $r = |I|$ be the cardinality of I and let $n = \dim Y$. Define

$$Y^{(r)} = \begin{cases} Y^{(0)} = \mathfrak{X} & \text{if } r = 0 \\ \coprod_{|I|=r} Y_I & \text{if } 1 \leqslant r \leqslant n \\ \emptyset & \text{if } r > n. \end{cases}$$

We write $CH_m(Y_I)$ (resp. $CH^m(Y_I)$) for the Chow homology group (resp. cohomology group) (in the sense of [10]) of dimension (resp. codimension) m algebraic cycles modulo rational equivalence on the stratum Y_I. We set $CH_m(Y^{(r)}) = \oplus_{|I|=r} CH_m(Y_I)$ (resp. $CH^m(Y^{(r)}) = \oplus_{|I|=r} CH^m(Y_I)$).

For m, t and u non negative integers such that $1 \leqslant u \leqslant t \leqslant r - 1 (r = |I|)$, the homomorphisms

$$\delta_{u*} : CH_m(Y^{(t+1)}) \to CH_m(Y^{(t)})$$

and

$$\delta_u^* : CH^m(Y^{(t)}) \to CH^m(Y^{(t+1)}),$$

are defined as follows. Let $I = \{i_1, \dots, i_{t+1}\}$, with $i_1 < i_2 < \cdots < i_{t+1}$ and let $J = I - \{i_u\}$. Then, the restriction of δ_{u*} to $CH_m(Y_I)$ is the push forward map on the Chow homology groups induced by the embedding $Y_I \to Y_J$. The component of δ_u^* in $CH^m(Y_J)$, is the pullback map on the Chow cohomology groups induced again by the above embedding of strata. We define Gysin morphisms γ (resp. restriction morphisms ρ)

$$\gamma : CH_m(Y^{(r+1)}) \to CH_m(Y^{(r)})$$

$$(\rho : CH^m(Y^{(r)}) \to CH^m(Y^{(r+1)})),$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

by the formulae

$$\gamma = \sum_{u=1}^{r} (-1)^{u-1} \delta_{u*}$$

$$\left(\rho = \sum_{u=1}^{r} (-1)^{u-1} \delta_u^* \right).$$

Similar definitions hold when we replace Chow cohomology groups by l-adic cohomology groups of the strata and we interpret the Gysin maps as covariant morphisms between l-adic cohomology groups shifting degrees.

One can show (cf. [3] Lemma 1 (i) and [11] Proposition (2.9) (iii)) that $\gamma^2 = 0 = \rho^2$ and $\gamma \cdot \rho + \rho \cdot \gamma = 0$.

2. Higher chow groups of the special fiber of a semistable fibration and L-functions

In this section we formulate a conjecture due to S. Bloch, on the rank of the higher Chow groups of the special fiber Y of a semistable fibration. In the next paragraph we will prove it under suitable conditions.

We assume all the notations introduced in Section 1; in particular our starting geometric setting consists of the following diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\ i\ } & \mathfrak{X} & \xleftarrow{\ j\ } & \bar{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k(v)) & \longrightarrow & \mathrm{Spec}(\Lambda) & \longleftarrow & \mathrm{Spec}(\bar{\eta}) \end{array}$$

where π a semistable fibration.

In the following we will deal with the *motivic cohomology* theory defined by Bloch in [1]. For a proper, equidimensional scheme V of finite type over a field k and for any couple of non negative integers q, r, the groups of integral motivic cohomology are defined as

$$H^q_{\mathcal{M}}(V, \mathbb{Z}(r)) := CH^r(V, 2r - q).$$

$CH^*(V, \cdot)$ are the *higher Chow groups* of algebraic cycles modulo rational equivalence defined as follows. For m any non negative integer, let

$$\Delta^m = \mathrm{Spec} \left(k[t_0, \ldots, t_m] \Big/ \left(\sum_i t_i - 1 \right) \right) \simeq A_k^m.$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Given an increasing map $\rho\colon \{0,\dots,t\} \to \{0,\dots,m\}$, define $\tilde{\rho}\colon \Delta^t \to \Delta^m$ as $\tilde{\rho}^*(t_i) = \Sigma_{\rho(j)=i} t_j$ and $\tilde{\rho}^*(t_i) = 0$ if $\rho^{-1}(\{i\}) = \emptyset$. If ρ is injective, the image $\tilde{\rho}(\Delta^t) \subset \Delta^m$ is called a *face*; if ρ is surjective, $\tilde{\rho}$ is a *degeneracy*.

Let $Z^*(V,m) \subset Z^*(V\times\Delta^m)$ be the free abelian group generated by irreducible subvarieties meeting all faces $V\times\Delta^t \subset V\times\Delta^m$ properly. Let ∂_i (resp. S_i) be the pullback along the face map

$$(t_0,\dots,t_{m-1}) \to (t_0,\dots,t_{i-1},0,t_{i+1},\dots,t_{m-1})$$

(resp. degeneracy)

$$(t_0,\dots,t_m) \to (t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_m).$$

Then, following [1], the groups $CH^*(V,m)$ are defined to be the homotopy of the complex of simplicial abelian groups

$$\mathcal{Z}^*(V,\cdot) := (Z^*(V,m),\{\partial_i\},\{S_i\})_{m\geqslant 0}.$$

Because the homotopy of this complex in fact coincides with its homology, one gets

$$\begin{aligned} CH^*(V,m) &= \frac{\cap_{i=0}^m \operatorname{Ker}(\partial_i\colon\ Z^*(V,m) \to Z^*(V,m-1))}{\partial_{m+1}(\cap_{i=0}^m \operatorname{Ker}(\partial_i\colon\ Z^*(V,m+1) \to Z^*(V,m)))} \\ &\simeq \frac{\operatorname{Ker}(\Sigma_i(-1)^i\partial_i\colon\ Z^*(V,m) \to Z^*(V,m-1))}{\operatorname{Image}(\Sigma_i(-1)^i\partial_i\colon\ Z^*(V,m+1) \to Z^*(V,m))}. \end{aligned}$$

It follows from the definition of the complexes $\mathcal{Z}^*(Y^{(j)},\cdot)$, (note that each stratum $Y_I, |I| = j$, is a regular scheme) that one has the following exact sequence of complexes calculating higher Chow groups, where the horizontal maps are Gysin homomorphisms γ on the level of cycles that we have defined in the last paragraph

$$\begin{aligned} &0 \to \mathcal{Z}^{r-n-1}(Y^{(n+1)},\cdot) \to \cdots \\ &\quad \to \mathcal{Z}^{r-2}(Y^{(2)},\cdot) \to \mathcal{Z}^{r-1}(Y^{(1)},\cdot) \to \mathcal{Z}^{r-1}(Y,\cdot) \to 0 \end{aligned} \tag{2.1}$$

In particular, one has

$$CH^*(Y) \simeq \operatorname{Coker}(\gamma\colon\ CH_{n-*}(Y^{(2)}) \to CH_{n-*}(Y^{(1)})). \tag{2.2}$$

We assume from now on that $k(v)$ is a finite field: let $N(v)$ be its number of elements. We also assume for the rest of this section that $2r-q \geqslant 1$.

Since the strata Y_I are proper and smooth over a finite field, the results obtained in [21] may suggest the following

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

CONJECTURE 2.1. *$K_m(Y_I) \otimes \mathbb{Q} = 0$ for $m > 0$.*

The Riemann–Roch isomorphism (9.1) of [1] would then imply

$$CH^*(Y_I, m) \otimes \mathbb{Q} = (0) \qquad \text{for } m \geqslant 1.$$

Hence, when $m \geqslant 1$, this vanishing together with (2.1) would yield, by a simple diagram chase

$$CH^*(Y, m) \otimes \mathbb{Q} \simeq \frac{\mathrm{Ker}(\gamma : CH_{n-*}(Y^{(m+1)}) \to CH_{n-*}(Y^{(m)}))}{\mathrm{Image}(\gamma : CH_{n-*}(Y^{(m+2)}) \to CH_{n-*}(Y^{(m+1)}))} \otimes \mathbb{Q}. \tag{2.3}$$

Therefore, (2.2) and (2.3) would imply

$$\begin{aligned} &\mathrm{rank}\, CH^{r-1}(Y, 2r-q-1) \otimes \mathrm{Q} \\ &= \begin{cases} \mathrm{rank}\, \frac{\mathrm{Ker}(\gamma: CH_{n-r+1}(Y^{(2r-q)}) \to CH_{n-r+1}(Y^{(2r-q-1)}))}{\mathrm{Image}(\gamma: CH_{n-r+1}(Y^{(2r-q+1)}) \to CH_{n-r+1}(Y^{(2r-q)}))} \otimes \mathbb{Q} & \text{if } 2r-q \geqslant 2 \\ \mathrm{rank}\, \mathrm{Coker}(\gamma\colon\ CH_{n-r+1}(Y^{(2)}) \to CH_{n-r+1}(Y^{(1)})) \otimes \mathbb{Q} & \text{if } 2r-q = 1. \end{cases} \end{aligned} \tag{2.4}$$

Let denote by F^* the (geometric) Frobenius automorphism acting on the groups of geometric l-adic cohomology of $\bar{X}$. Under these conditions, Bloch has formulated the following

CONJECTURE 2.2 (Bloch). *$\mathrm{ord}_{s=q-r} \det(I - F^* N(v)^{-s} \mid H^{q-1}(\bar{X}, \mathbb{Q}_l)^I) =$*

$$= \begin{cases} \mathrm{rank}\, CH^{r-1}(Y, 2r-q-1) \otimes \mathbb{Q}_l & \text{if } 2r-q \geqslant 2 \\ \mathrm{rank} \frac{\mathrm{Ker}(i^* \cdot i_*\colon CH_{n-r+1}(Y^{(1)}) \to CH^r(Y^{(1)}))}{\mathrm{Image}(\gamma\colon CH_{n-r+1}(Y^{(2)}) \to CH_{n-r+1}(Y^{(1)}))} \otimes \mathbb{Q}_l & \text{if } 2r-q = 1. \end{cases}$$

The map $i^* \cdot i_*$ is the composite of the push forward map i_* toward the Chow groups of the model $\mathfrak{X}$, followed by the pullback i^*.

A nice consequence of this conjecture is the independence of the rank of the groups $H^{q-1}_{\mathcal{M}}(Y, \mathbb{Q}_l(r-1)) = CH^{r-1}(Y, 2r-q-1) \otimes \mathbb{Q}_l$ on the special fiber Y, when $2r - q \geqslant 2$. Results in this direction have been recently obtained by Gillet and Soulé (cf. [13]) and by Hanamura (cf. [14]).

In the next section we will show how Conjecture 2.2 follows from Monodromy and Tate Conjectures, together with the Conjecture 2.1 and the assumption of the semisimplicity of the frobenius. Finally, we anticipate that the homology groups involved in the Conjecture 2.2 can be seen as homologies of a mapping cone of an endomorphism of a suitable bigraded complex $K^{\cdot,\cdot}$ (cf. (3.14).

3. A bigraded complex with monodromy on the special fiber

In this section we introduce a bigraded complex endowed with a monodromy-type map N and we explain some properties of it which we use in the proof of the Conjecture 2.2.

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

We assume all of the notations introduced in Seciton 1; in particular, we denote by n the dimension of the special fiber Y.

Let $i, j, k \in \mathbb{Z}$. We define, following [22], [11]:

$$K^{i,j,k} := \begin{cases} H^{i+j-2k+n}(\bar{Y}^{(2k-i+1)}, \mathbb{Q}_l(i-k)) & \text{if } k \geqslant \max(0, i) \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Set: $K^{i,j} = \oplus_k K^{i,j,k}$ and $K^* = \oplus_{i+j=*} K^{i,j}$.

The corestriction map ρ and the Gysin homomorphism γ on l-adic cohomology groups, define respectively maps:

$$d'\colon\ K^{i,j,k} \to K^{i+1,j+1,k+1} \qquad d'(a) = \rho(a)$$

$$d''\colon\ K^{i,j,k} \to K^{i+1,j+1,k} \qquad d''(a) = -\gamma(a)$$

and

$$N : K^{i,j,k} \to K^{i+2,j,k+1}(-1) \qquad N(a) = a$$

for any $a \in K^{i,j,k}$.

On $K^{i,j}$, let $d = d' + d''$. It follows from the definitions that $[d, N] = 0$ and $d^2 = 0$.

We denote by Cone $(N\colon\ K^* \to K^*)$ the complex $K^* \oplus K^*[-1]$ endowed with the differential $D(a, b) = (\mathrm{d}a, N(a) - \mathrm{d}b)$.

In [3], a similar complex has been defined using the groups of algebraic cycles of the strata Y_I modulo rational, algebraic, homological or numerical equivalence. More precisely, we would like to point out here that it was the axiomatic description given in *op. cit.* and in [11] that suggested to us the definition of (3.1). In (3.13) we will denote by the same name $K^{i,j,k}$ the complex of algebraic cycles modulo rational equivalence.

The following technical lemma will be used in the proof of Proposition 3.2.

LEMMA 3.1. *Let $N\colon\ C^* \to D^*$ be a morphism of complexes. Suppose there exists an integer m such that*

$$N_i\colon\ C^i \to D^i \quad \textit{is} \begin{cases} \textit{injective} & \textit{if } i \leqslant m-1 \\ \textit{bijective} & \textit{if } i = m \\ \textit{surjective} & \textit{if } i \geqslant m+1 \end{cases}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Let us define a complex L^* via the diagram with exact columns

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
0 & & 0 & & & & L^{m+1} & \longrightarrow & L^{m+2}\cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^{m-2} & \longrightarrow & C^{m-1} & \xrightarrow{d_C} & C^{m} & \longrightarrow & C^{m+1} & \longrightarrow & C^{m+2}\cdots \\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \simeq\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\
D^{m-2} & \longrightarrow & D^{m-1} & \xrightarrow{d_D} & D^{m} & \longrightarrow & D^{m+1} & \longrightarrow & D^{m+2}\cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
L^{m-1} & \longrightarrow & L^{m} & & & & 0 & & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}
$$

where the differential $L^m \to L^{m+1}$ is defined by a diagram chase.

Then, L^* is quasi-isomorphic to Cone$(N\colon\ C^* \to D^*)$.

Proof. Let think Cone$(N\colon\ C^* \to D^*)$ as the double complex

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{i} & \longrightarrow & C^{i+1} & \longrightarrow & C^{i+2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & D^{i} & \longrightarrow & D^{i+1} & \longrightarrow & D^{i+2} & \longrightarrow & \cdots
\end{array}
$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

It contains the following acyclic sub-double complex

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C^{m-2} & \longrightarrow & C^{m-1} & \longrightarrow & \mathrm{Image}(d_C) & \longrightarrow & 0 & \longrightarrow & 0\cdots\\
 & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle =} & & & & \\
\cdots & \longrightarrow & C^{m-2} & \longrightarrow & C^{m-1} & \longrightarrow & \mathrm{Image}(N\cdot d_C) & \longrightarrow & 0 & \longrightarrow & 0\cdots
\end{array}$$

Taking the quotient, we get that $\mathrm{Cone}(N\colon\, C^* \to D^*)$ is quasi-isomorphic to the complex

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \bar{C}^m & \longrightarrow & C^{m+1} & \longrightarrow & C^{m+2} & \longrightarrow & \cdots\\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle N=1} & & \Big\downarrow{\scriptstyle \mathrm{epi}} & & \Big\downarrow{\scriptstyle \mathrm{epi}} & & \\
 & & L^{m-1} & \longrightarrow & L^m & \xrightarrow{d_D} & \overline{D^m} & \longrightarrow & D^{m+1} & \longrightarrow & D^{m+2} & \longrightarrow & \cdots
\end{array}$$

where $\overline{C^m} = \frac{C^m}{\mathrm{Image}(d_C)}$ and $\overline{D^m} = \frac{D^m}{\mathrm{Image}(N\cdot d_C)}$. The acyclic complex

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \overline{C^m}/N^{-1}(\mathrm{Image}(d_D)) & \longrightarrow & D^{m+1} & \longrightarrow & D^{m+2} & \longrightarrow & \cdots\\
 & & \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle =} & & \\
 & & \overline{D^m}/\mathrm{Image}(d_D) & \longrightarrow & D^{m+1} & \longrightarrow & D^{m+2} & \longrightarrow & \cdots
\end{array}$$

is a quotient of the previous one. The kernel, which is quasi-isomorphic to $\mathrm{Cone}(N\colon C^* \to D^*)$, is the complex

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & N^{-1}(\mathrm{Image}(d_D)) & \longrightarrow & L^{m+1} & \longrightarrow & L^{m+2} & \longrightarrow & \cdots\\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow & & \Big\downarrow & & \\
L^{m-1} & \longrightarrow & L^m & \xrightarrow{d_D} & \mathrm{Image}(d_D) & \longrightarrow & 0 & & 0 & &
\end{array}$$

It is easy to check that this complex is quasi-isomorphic to L^*.

PROPOSITION 3.2. *Let $*$ be a fixed integer. Then, the complex* $(q \in \mathbb{Z})$

$$\mathrm{Cone}(N\colon\, K^{q-*,q-n} \to K^{q-*+2,q-n}(-1))$$

is quasi-isomorphic to the following complex

$$\mathcal{C}^q(*) := \begin{cases} H^{2q-*}(\bar{Y}^{(*-q)}, \mathbb{Q}_l(q-*)) & \text{if } q \leqslant * - 1\\ H^*(\bar{Y}^{(q-*+1)}, \mathbb{Q}_l) & \text{if } q \geqslant * \end{cases}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

for $a \in \mathcal{C}^q(*)$, the differential $d_{\mathcal{C}}$ is given by

$$d_{\mathcal{C}}(a) = \begin{cases} d''(a) & \text{if } q < * - 1 \\ -i^{\star} i_{\star}(a) & \text{if } q = * - 1 \\ d'(a) & \text{if } q \geqslant * \end{cases}$$

i.e.

$$\ldots H^{2q-*}(\bar{Y}^{(*-q)}, \mathbb{Q}_l(q-*)) \xrightarrow{d''} \cdots \to H^{*-4}(\bar{Y}^{(2)}, \mathbb{Q}_l(-2))$$

$$\to H^{*-2}(\bar{Y}^{(1)}, \mathbb{Q}_l(-1)) \xrightarrow{-i^{\star} \cdot i_{\star}} H^{*}(\bar{Y}^{(1)}, \mathbb{Q}_l) \xrightarrow{d'} H^{*}(\bar{Y}^{(2)}, \mathbb{Q}_l)$$

$$\to H^{*}(\bar{Y}^{(3)}, \mathbb{Q}_l) \to \cdots \to H^{*}(\bar{Y}^{(q-*+1)}, \mathbb{Q}_l) \xrightarrow{d'} \cdots$$

In particular, $\mathcal{C}^*(*) = H^*(\bar{Y}^{(1)}, \mathbb{Q}_l)$ and $\mathcal{C}^{*-1}(*) = H^{*-2}(\bar{Y}^{(1)}, \mathbb{Q}_l(-1))$. The composite $i^{\star} \cdot i_{\star}$ is the push forward map $i_{\star}\colon H^{*-2}(\bar{Y}^{(1)}, \mathbb{Q}_l(-1)) \to H^*(\mathfrak{X}, \mathbb{Q}_l)$ followed by (the identity N and) the pullback $i^{\star} : H^*(\mathfrak{X}, \mathbb{Q}_l) \to H^*(\bar{Y}^{(1)}, \mathbb{Q}_l)$.

Proof. Let fix an integer $*$ and define $C^q = K^{q-*,q-n}, D^q = K^{q-*+2,q-n}$. Then, the map $N\colon C^q \to D^q$ looks like:

$$\begin{array}{lll} q - * \geqslant 0\colon & N\colon \oplus_{k \geqslant q-*} K^{q-*,q-n,k} & \operatorname{Ker} N = K^{q-*,q-n,q-*} \\ & \quad \twoheadrightarrow \oplus_{k \geqslant q-*+2} K^{q-*+2,q-n,k}, & \\ q - * = -1\colon & N\colon \oplus_{k \geqslant 0} K^{-1,q-n,k} & \\ & \quad \xrightarrow{\simeq} \oplus_{k \geqslant 1} K^{1,q-n,k}, & \\ q - * \leqslant -2\colon & N\colon \oplus_{k \geqslant 0} K^{q-*,q-n,k} & \operatorname{Coker} N = K^{q-*+2,q-n,0} \\ & \quad \hookrightarrow \oplus_{k \geqslant 0} K^{q-*+2,q-n,k}, & \end{array}$$

We are now in the situation of Lemma 3.1, with $m = * - 1$ and

$$L^q = \begin{cases} K^{q-*+1,q-n-1,0}; & \text{if } q \leqslant * - 1 \\ K^{q-*,q-n,q-*}; & \text{if } q \geqslant * \end{cases}$$

Therefore the claim follows. □

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Since $d^2 = 0$ and d commutes with N, d defines a differential on the bigraded groups:

$$\mathrm{Ker}(N)^{\cdot,\cdot} = \mathrm{Ker}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot}(-1)),$$

$$\mathrm{Coker}(N)^{\cdot,\cdot} = \mathrm{Coker}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot}(-1))$$

as well on the mapping cone of N

$$\mathrm{Cone}(N)^{\cdot,\cdot} = \mathrm{Cone}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot}(-1)).$$

We introduce following [3], the groups ($q \geqslant 0, r \in \mathbb{Z}$):

$$gr^W_{q+r} H^q(\tilde{X}^*) := \frac{\mathrm{Ker}(d\colon\ K^{-r,q-n} \to K^{-r+1,q-n+1})}{\mathrm{Image}(d\colon\ K^{-r-1,q-n-1} \to K^{-r,q-n})} \tag{3.2}$$

$$gr^W_{q+r} H^q(X^*) := \frac{\mathrm{Ker}(d\colon\ \mathrm{Cone}(N)^{-r+1,q-n-1} \to \mathrm{Cone}(N)^{-r+2,q-n})}{\mathrm{Image}(d\colon\ \mathrm{Cone}(N)^{-r,q-n-2} \to \mathrm{Cone}(N)^{-r+1,q-n-1})} \tag{3.3}$$

$$gr^W_{q+r} H^q(Y) := \frac{\mathrm{Ker}(d\colon\ \mathrm{Ker}(N)^{-r,q-n} \to \mathrm{Ker}(N)^{-r+1,q-n+1})}{\mathrm{Image}(d\colon\ \mathrm{Ker}(N)^{-r-1,q-n-1} \to \mathrm{Ker}(N)^{-r,q-n})} \tag{3.4}$$

$$gr^W_{q+r} H^q_Y(X) := \frac{\mathrm{Ker}(d\colon\ \mathrm{Coker}(N)^{-r,q-n-2} \to \mathrm{Coker}(N)^{-r+1,q-n-1})}{\mathrm{Image}(d\colon\ \mathrm{Coker}(N)^{-r-1,q-n-3} \to \mathrm{Coker}(N)^{-r,q-n-2})} \tag{3.5}$$

These formal definitions (inspired by the theory of variations of Hodge structures) imply analogs of the Wang exact sequence and the standard exact sequence for cohomology with supports. In this abstract setting that means the exactness of ($m \in \mathbb{Z}$):

$$\ldots gr^W_m H^q(X^*) \to gr^W_m H^q(\tilde{X}^*) \xrightarrow{N} gr^W_{m-2} H^q(\tilde{X}^*)(-1) \to gr^W_m H^{q+1}(X^*) \ldots \tag{3.6}$$

$$\ldots gr^W_m H^q(Y) \to gr^W_m H^q(X^*) \to gr^W_m H^{q+1}_Y(X) \to gr^W_m H^{q+1}(Y) \ldots \tag{3.7}$$

(cf. *op. cit.* Lemma 3 for a proof).

As in *op. cit.* Lemma 4 we have the following formulae

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

LEMMA 3.3.
(i)

$$gr^W_{q+r}H^q(Y) = \begin{cases} \frac{\mathrm{Ker}(d''\colon H^{q+r}(\bar{Y}^{(-r+1)},\mathbb{Q}_l)\to H^{q+r}(\bar{Y}^{(-r+2)},\mathbb{Q}_l))}{\mathrm{Image}(d'\colon H^{q+r}(\bar{Y}^{(-r)},\mathbb{Q}_l)\to H^{q+r}(\bar{Y}^{(-r+1)},\mathbb{Q}_l))} & \textit{if } r \leqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

In particular,

$$gr^W_m H^m(Y) = \mathrm{Ker}(d'\colon\ H^m(\bar{Y}^{(1)},\mathbb{Q}_l) \to H^m(\bar{Y}^{(2)},\mathbb{Q}_l))$$

(ii)

$$gr^W_{q+r}H^q_Y(X)$$

$$= \begin{cases} \frac{\mathrm{Ker}(d''\colon H^{q-r-2}(\bar{Y}^{(r+1)},\mathbb{Q}_l(-r-1))\to H^{q-r}(\bar{Y}^{(r)},\mathbb{Q}_l(-r)))}{\mathrm{Image}(d''\colon H^{q-r-4}(\bar{Y}^{(r+2)},\mathbb{Q}_l(-r-2))\to H^{q-r-2}(\bar{Y}^{(r+1)},\mathbb{Q}_l(-r-1)))} & \textit{if } r \geqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

In particular,

$$gr^W_m H^m_Y(X) = \mathrm{Coker}(d''\colon\ H^{m-4}(\bar{Y}^{(2)},\mathbb{Q}_l(-2)) \to H^{m-2}(\bar{Y}^{(1)},\mathbb{Q}_l(-1)))$$

(iii) *The map* $gr^W_m H^q_Y(X) \to gr^W_m H^q(Y)$ *in the exact sequence* (3.7) *is zero unless* $m = q$. *On* $gr^W_m H^m_Y(X)$ *this map coincides with the morphism:*

$$-i^* \cdot i_* : H^{m-2}(\bar{Y}^{(1)},\mathbb{Q}_l(-1)) \to H^m(\bar{Y}^{(1)},\mathbb{Q}_l).$$

(iv)

$$gr^W_{q+r}H^q(X^*) = \begin{cases} gr^W_{q+r}H^{q+1}_Y(X) & \textit{if } r > 1, \\ \frac{\mathrm{Ker}(-i^*\cdot i_*\colon H^{q-1}(\bar{Y}^{(1)},\mathbb{Q}_l(-1))\to H^{q+1}(\bar{Y}^{(1)},\mathbb{Q}_l))}{\mathrm{Image}(d'')} & \textit{if } r = 1, \\ \frac{\mathrm{Ker}(d'\colon H^q(\bar{Y}^{(1)},\mathbb{Q}_l)\to H^q(\bar{Y}^{(2)},\mathbb{Q}_l))}{\mathrm{Image}(-i^*\cdot i_*)} & \textit{if } r = 0, \\ gr^W_{q+r}H^q(Y) & \textit{if } r \leqslant -1 \end{cases}$$

COROLLARY 3.4.

$$N\colon\ gr^W_{q+r}H^q(\tilde{X}^*) \to gr^W_{q+r-2}H^q(\tilde{X}^*)(-1)\ \ \textit{is}\ \begin{cases} \textit{injective} & \textit{if } r > 1 \\ \textit{bijective} & \textit{if } r = 1 \\ \textit{surjective} & \textit{if } r \leqslant 0 \end{cases}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Furthermore:

$$\textit{if } r>1\colon\quad \mathit{Coker}(N\colon gr^W_{q+r}H^q(\tilde{X}^*)\to gr^W_{q+r-2}H^q(\tilde{X}^*)(-1))$$

$$= gr^W_{q+r}H^{q+1}(X^*)$$

$$\textit{if } r\leqslant 0\colon\quad \mathit{Ker}(N\colon gr^W_{q+r}H^q(\tilde{X}^*)\to gr^W_{q+r-2}H^q(\tilde{X}^*)(-1))$$

$$= gr^W_{q+r}H^q(X^*)$$

Proof. Follows from the Wang exact sequence (3.6) together with Lemma 3.3 (iv). □

Let us consider, following [16] (cf. Theorem 2.10), the $\mathrm{Gal}(\bar{\eta}/\eta)$-equivariant spectral sequence of the vanishing cycles $(r,q\in\mathbb{Z},q\geqslant 0)$:

$$\begin{aligned} &E_1^{-r,q+r}\\ &\quad=\bigoplus_{k\geqslant\max(0,-r)} H^{q-r-2k}(\bar{Y}^{(r+2k+1)},\mathbb{Q}_l(-r-k))\Rightarrow H^q(\bar{X},\mathbb{Q}_l);\\ &E_\infty^{-r,q+r}=gr^W_{q+r}H^q(\bar{X},\mathbb{Q}_l). \end{aligned} \tag{3.8}$$

The differential on the E_1-term is defined as

$$d_1=\sum_k((-1)^{r+k}d_1'+(-1)^{k-r}d_1'')$$

for $d_1'=d'=\rho$ and $d_1''=d''=-\gamma$, where γ and ρ are respectively the Gysin and the corestriction homomorphisms as defined in Section 1. Furthermore, since $\gamma\cdot\rho+\rho\cdot\gamma=0$, one has $d_1''\cdot d_1'+d_1'\cdot d_1''=0$.

It follows from Weil conjectures that $E_1^{-r,q+r}$ is a pure Galois module of weight $q+r$. Therefore, the spectral sequence degenerates at the E_2-term and $E_\infty^{-r,q+r}$ has pure weight $q+r$. From the definitions (3.8) and (3.1) we get that

$$E_1^{-r,q+r}=K^{-r,q-n}$$

Hence,

$$(E_1^{-r,q+r};d_1=d_1'+d_1'')\simeq(K^{-r,q-n};d=d'+d'')$$

and that is enough to conclude

$$gr^W_{q+r}H^q(\bar{X},\mathbb{Q}_l)\simeq gr^W_{q+r}H^q(\tilde{X}^*).$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

If n $(= \dim\ Y) \leqslant 2$, Rapoport and Zink have proved (cf. *op. cit.* Theorem 2.13) that the filtration induced on the abutment $H^q(\bar{X}, \mathbb{Q}_l)$ by the weight filtration on the special fiber Y, coincides with the monodromy filtration. This means that the action of $(T-1)$ on E_1, where T is a topological generator of the maximal pro-l-quotient of $I \subset \mathrm{Gal}(\bar{\eta}/\eta)$, induces the monodromy transformation on $gr^W_{q+r} H^q(\bar{X}, \mathbb{Q}_l)$. Hence, the corresponding nilpotent map $N = \log\ T$ defines the monodromy filtration on $H^q(\bar{X}, \mathbb{Q}_l)$. Therefore, the operator N on the associated graded group determines isomorphisms of weighted pure Galois structures $(q,\ r \geqslant 0)$

$$N^r\colon\ gr^W_{q+r} H^q(\bar{X}, \mathbb{Q}_l) \xrightarrow{\simeq} gr^W_{q-r} H^q(\bar{X}, \mathbb{Q}_l)(-r). \tag{3.9}$$

When $n > 2$, this result is still unproved in the full generality (although some particular cases have been studied, e.g. abelian varieties over number fields: cf. [16]). We will refer to it as the Monodromy Conjecture.

In the rest of this section we will assume the Monodromy Conjecture.

The isomorphisms (3.9) together with the exact sequences (3.6) and (3.7) imply the exactness of the following weighted sequence $m \in \mathbb{Z}$ (Clemens–Schmidt type)

$$\begin{aligned} &\dots gr^W_m H^q_Y(X) \to gr^W_m H^q(Y) \to gr^W_m H^q(\tilde{X}^*) \\ &\xrightarrow{N} gr^W_{m-2} H^q(\tilde{X}^*)(-1) \to gr^W_m H^{q+2}_Y(X) \dots \end{aligned} \tag{3.10}$$

We refer to [22] (5.12), [18] and [12] (IV, 7.14) for a proof of this claim.

We are now ready to state our main result

THEOREM 3.5. *Let suppose that the field $k(v)$ is finite and let $N(v)$ be its number of elements. Let F^* be the (geometric) frobenius automorphism acting on the geometric l-adic cohomology groups of $\bar{X}$. Assume the Monodromy Conjecture and suppose F^* acts semisimply on $H^*(\bar{X}, \mathbb{Q}_l)^I$. Let $m = 2r - q \geqslant 1$ and $a = q - r$, then*

$$\operatorname*{ord}_{s=a} \det(I - F^* N(v)^{-s} | H^{q-1}(\bar{X}, \mathbb{Q}_l)^I)$$

$$= \begin{cases} rk \left(\dfrac{\mathrm{Ker}(d''\colon H^{2a}(\bar{Y}^{(m)}, \mathbb{Q}_l(a)) \to H^{2(a+1)}(\bar{Y}^{(m-1)}, \mathbb{Q}_l(a+1)))}{\mathrm{Image}(d'')} \right)^{F=id} & m \geqslant 2 \\ rk \left(\dfrac{\mathrm{Ker}(i^* \cdot i_*\colon H^{2a}(\bar{Y}^{(1)}, \mathbb{Q}_l(a)) \to H^{2(a+1)}(\bar{Y}^{(1)}, \mathbb{Q}_l(a+1)))}{\mathrm{Image}(d'')} \right)^{F=id} & m = 1 \end{cases}$$

Proof. Since F^* acts semisimply on the inertia invariants in $H^{q-1}(\bar{X}, \mathbb{Q}_l)$, we have

$$\operatorname*{ord}_{s=a} \det(I - F^* N(v)^{-s} | H^{q-1}(\bar{X}, \mathbb{Q}_l)^I) = \mathrm{rank}\, H^{q-1}(\bar{X}, \mathbb{Q}_l)^{\langle I, F = N(v)^a \rangle}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Since the wild inertia acts trivially (we deal with a semistable fibration)

$$\begin{aligned}&\operatorname{rank}(H^{q-1}(\bar{X},\mathbb{Q}_l)^I)^{F=N(v)^a}\\&\quad=\operatorname{rank}\Big(\operatorname{Ker}(N\colon gr^W_{2a}H^{q-1}(\tilde{X}^*)(a)\\&\qquad\to gr^W_{2(a-1)}H^{q-1}(\tilde{X}^*)(a-1))\Big)^{\langle F=id\rangle}.\end{aligned}$$

It follows from the Monodromy Conjecture together with Corollary 3.4 and Lemma 3.3 that

$$\operatorname*{ord}_{s=a}\det(I-F^*N(v)^{-s}|H^{q-1}(\bar{X},\mathbb{Q}_l)^I)=\operatorname{rank}(gr^W_{2a}H^{q-1}(X^*)(a))^{\langle F=id\rangle}$$

$$=\begin{cases}\operatorname{rank}(gr^W_{2a}H^{q-1}(Y)(a))^{\langle F=id\rangle} & \text{if } m\geqslant 2\\ \operatorname{rank}\left(\frac{gr^W_{2a}H^{2a}(Y)(a)}{\operatorname{Image}(i^*\cdot i_*)}\right)^{\langle F=id\rangle} & \text{if } m=1\end{cases}$$

$$=\begin{cases}rk\left(\frac{\operatorname{Ker}(d'\colon H^{2a}(\bar{Y}^{(m)},\mathbb{Q}_l(a))\to H^{2a}(\bar{Y}^{(m+1)},\mathbb{Q}_l(a)))}{\operatorname{Image}(d')}\right)^{\langle F=id\rangle} & m\geqslant 2\\ rk\left(\frac{\operatorname{Ker}(d'\colon H^{2a}(\bar{Y}^{(1)},\mathbb{Q}_l(a))\to H^{2a}(\bar{Y}^{(2)},\mathbb{Q}_l(a)))}{\operatorname{Image}(i^*\cdot i_*)}\right)^{\langle F=id\rangle} & m=1\end{cases}$$

From the Monodromy Conjecture we also have an isomorphism like (3.9)

$$gr^W_{2r}H^{q-1}(\tilde{X}^*)(r)\xrightarrow{N^{2r-q+1}}gr^W_{2(a-1)}H^{q-1}(\tilde{X}^*)(a-1)$$

$(2r-q+1\geqslant 2$, under our hypotheses). Let decompose this isomorphism as

$$\begin{aligned}&0\to gr^W_{2r}H^{q-1}(\tilde{X}^*)(r)\xrightarrow{N}gr^W_{2(r-1)}H^{q-1}(\tilde{X}^*)(r-1)\xrightarrow{N}\cdots\\&\quad\cdots\xrightarrow{N}gr^W_{2a}H^{q-1}(\tilde{X}^*)(a)\xrightarrow{N}gr^W_{2(a-1)}H^{q-1}(\tilde{X}^*)(a-1)\to 0\end{aligned}$$

where the first N on the left is injective and the last N on the right is surjective (cf. Corollary 3.4). The composite

$$gr^W_{2(r-1)}H^{q-1}(\tilde{X}^*)(r-1)\xrightarrow{N^{2r-q-1}}gr^W_{2a}H^{q-1}(\tilde{X}^*)(a),$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

is also an isomorphism for $2r - q \geqslant 2$, therefore in this range we get

$$\begin{aligned} & gr^W_{2a} H^{q-1}(Y)(a) \\ & \quad = \mathrm{Ker}(N\colon\ gr^W_{2a} H^{q-1}(\tilde{X}^*)(a) \to gr^W_{2(a-1)} H^{q-1}(\tilde{X}^*)(a-1)) \\ & \quad \simeq \mathrm{Coker}(N\colon\ gr^W_{2r} H^{q-1}(\tilde{X}^*)(r) \to gr^W_{2(r-1)} H^{q-1}(\tilde{X}^*)(r-1)) \\ & \quad = gr^W_{2r} H^{q+1}_Y(X)(r) \\ & \quad = \frac{\mathrm{Ker}(d''\colon\ H^{2a}(\bar{Y}^{(2r-q)}, \mathbb{Q}_l(a)) \to H^{2(a+1)}(\bar{Y}^{(2r-q-1)}, \mathbb{Q}_l(a+1)))}{\mathrm{Image}(d'')} \end{aligned} \tag{3.11}$$

For $2r - q = 1$:

$$\begin{aligned} & \frac{gr^W_{2a} H^{2a}(Y)(a)}{\mathrm{Image}(i^* \cdot i_*)} \\ & \quad = \mathrm{Ker}(N : gr^W_{2a} H^{2a}(\tilde{X}^*)(a) \to gr^W_{2(a-1)} H^{2a}(\tilde{X}^*)(a-1)) \\ & \quad \simeq \mathrm{Coker}(N\colon\ gr^W_{2(a+1)} H^{2a}(\tilde{X}^*)(a+1) \to gr^W_{2a} H^{2a}(\tilde{X}^*)(a)) \\ & \quad = \frac{\mathrm{Ker}(i^* \cdot i_*\colon\ H^{2a}(\bar{Y}^{(1)}, \mathbb{Q}_l(a)) \to H^{2(a+1)}(\bar{Y}^{(1)}, \mathbb{Q}_l(a+1)))}{\mathrm{Image}(d'')} \end{aligned} \tag{3.12}$$

□

We would like to remark explicitly here that the isomorphisms (3.11) and (3.12) established in the proof of the previous Theorem should be interpreted as higher analogs of the isomorphism

$$\tau\colon\ H^{a,a}_{II}(X)_{\text{ét}} \to H^{a,a}_{I}(X)_{\text{ét}},$$

described in [2] Theorem (6.3.1). Therefore, the assumption of the Monodromy Conjecture in the Theorem 3.5 allows us to prove isomorphisms between I-cohomological invariants and coinvariants (I being the inertia group), or equivalently between right and left hand side with respect to the map $i^* \cdot i_*$, of the complexes Cone(N) described in Proposition 3.2 and in (3.13) below.

In analogy to (3.1) one can define for $i, j, k \in \mathbb{Z}$:

$$K^{i,j,k} := \begin{cases} CH^{\frac{i+j-2k+n}{2}}(Y^{(2k-i+1)}) & \text{if } k \geqslant \max(0,i) \text{ and } \\ & \quad i+j+n \equiv 0\ (2) \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Gysin homomorphisms and restriction maps define differentials d' and d'' as we have already described for the complex (3.1). Let $d = d' + d''$.
The operator $N\colon K^{i,j,k} \to K^{i+2,j,k+1}$ is set to be the identity: i.e. $N(a) = a$, for $a \in K^{i,j,k}$.

The analogue of Proposition 3.2 for this complex states that for a fixed integer $*$, Cone$(N\colon K^{q-2*,q-n} \to K^{q-2*+2,q-n})$ $(q \in \mathbb{Z})$ is quasi-isomorphic to

$$\ldots CH^{q-*}(Y^{(2*-q)}) \xrightarrow{d''} \cdots \to CH^{*-2}(Y^{(2)}) \to CH^{*-1}(Y^{(1)})$$

$$\xrightarrow{i^{\star}\cdot i_{\star}} CH^{*}(Y^{(1)}) \xrightarrow{d'} CH^{*}(Y^{(2)}) \to \cdots \to CH^{*}(Y^{(q-2*)}) \to \cdots$$

The complex (3.13) has been firstly studied in [3]. For this geometric theory the Monodromy Conjecture is achieved if one assumes that the strata of Y satisfy both the hard Lefschetz theorem and the Hodge index theorem.

We are now able to complete a proof of the Conjecture 2.2.

COROLLARY 3.6. *Under the same hypotheses and notations of Theorem 3.5, assuming Tate conjectures and the injectivity of the cycle class map on each stratum Y_I and Conjecture 2.1, we have*

$$\operatorname*{ord}_{s=q-r} \det\,(I - F^*N(v)^{-s}|H^{q-1}(\bar{X}, \mathbb{Q}_l)^I)$$

$$= \begin{cases} \text{rank } CH^{r-1}(Y, 2r-q-1) \otimes \mathbb{Q}_l & \text{if } 2r-q \geqslant 2 \\ \text{rank} \dfrac{\text{Ker}(i^*\cdot i_*\colon CH_{n-r+1}(Y^{(1)}) \to CH^r(Y^{(1)}))}{\text{Image}(\gamma\colon CH_{n-r+1}(Y^{(2)}) \to CH_{n-r+1}(Y^{(1)}))} \otimes \mathbb{Q}_l & \text{if } 2r-q = 1. \end{cases}$$

Proof. Since Tate conjectures hold and the spectral sequence 3.8 is Galois equivariant, it follows from Theorem 3.5 that ($G = \text{Gal}(\bar{v}/v)$)

$$\operatorname*{ord}_{s=q-r} \det(I - F^*N(v)^{-s}|H^{q-1}(\bar{X}, \mathbb{Q}_l)^I)$$

$$= \begin{cases} rk \left(\dfrac{\text{Ker}(d''\colon H^{2(q-r)}(\bar{Y}^{(2r-q)}, \mathbb{Q}_l(q-r)) \to H^{2(q-r+1)}(\bar{Y}^{(2r-q-1)}, \mathbb{Q}_l(q-r+1)))}{\text{Image}(d'')}\right)^{G} & 2r-q \geqslant 2 \\ rk \left(\dfrac{\text{Ker}(i^*\cdot i_*\colon H^{2(r-1)}(\bar{Y}^{(1)}, \mathbb{Q}_l(r-1)) \to H^{2r}(\bar{Y}^{(1)}, \mathbb{Q}_l(r)))}{\text{Image}(d'')}\right)^{G} & 2r-q = 1 \end{cases}$$

$$= \begin{cases} rk \dfrac{\text{Ker}(\gamma\colon CH_{n-r+1}(Y^{(2r-q)}) \to CH_{n-r+1}(Y^{(2r-q-1)}))}{\text{Image}(\gamma\colon CH_{n-r+1}(Y^{(2r-q+1)}) \to CH_{n-r+1}(Y^{(2r-q)}))} \otimes \mathbb{Q}_l & 2r-q \geqslant 2 \\ rk \dfrac{\text{Ker}(i^*\cdot i_*\colon CH_{n-r+1}(Y^{(1)}) \to CH^r(Y^{(1)}))}{\text{Image}(\gamma\colon CH_{n-r+1}(Y^{(2)}) \to CH_{n-r+1}(Y^{(1)}))} \otimes \mathbb{Q}_l & 2r-q = 1. \end{cases}$$

Then, the claim follows from Conjecture 2.1 as we showed in Section 2.

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

4. A bigraded complex with monodromy for the closed fiber at infinity

Let X be a smooth projective variety of dimension n over $\mathbb{C}$ or $\mathbb{R}$. In this paragraph we define a bigraded complex, endowed with a monodromy-type map N, which should be interpreted as the archimedean analogue of the complex $K^{\cdot,\cdot}$ studied in the last section. In particular, we show that real Deligne cohomology can be described as the homology of the mapping cone of this bigraded complex. Our construction is intended to stress some similarities occurring at the semistable and the archimedean places. In particular, the symmetry between cohomological I-invariants and coinvariants at the semistable places described in Section 3, corresponds here to the (already known!) isomorphisms

$$H^q_{\mathcal{D}}(X, \mathbb{R}(p)) \simeq H^{q+1}_{\mathcal{D}}(X, \mathbb{R}(q+1-p)), \qquad q \geqslant 2p \geqslant 0.$$

In this section, we will prove that $H^q_{\mathcal{D}}(X, \mathbb{R}(p))$ is in fact symmetric to $H^{q+1}_{\mathcal{D}}(X, \mathbb{R}(q+1-p))$, with respect to the map $(-2\pi\sqrt{-1})dd^c$. This result will be taken up again at the end of the next paragraph where we show that, by means of a non degenerate pairing, these two groups are dual.

We write $(A^{a,b}+A^{b,a})_{\mathbb{R}}$ for the abelian group of real differential forms of type $(a,b)+(b,a)$ on X. By the expression $(A^{a,b}+A^{b,a})_{\mathbb{R}}(p)(p \in \mathbb{Z})$, we mean the pth-Tate twist of $(A^{a,b}+A^{b,a})_{\mathbb{R}}$ i.e. $(A^{a,b}+A^{b,a})_{\mathbb{R}}(p) = (2\pi\sqrt{-1})^p(A^{a,b}+A^{b,a})_{\mathbb{R}}$.

Let $i, j, k \in \mathbb{Z}$. We define the following complex:

$$K^{i,j,k} := \begin{cases} \left(\bigoplus_{\substack{a+b=j+n \\ |a-b|\leqslant 2k-i}} A^{a,b}\right)_{\mathbb{R}} \left(\dfrac{n+j-i}{2}\right) & \text{if } n+j-i \equiv 0(2) \\ & \quad \text{and } k \geqslant \max(0,i) \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Set $K^{i,j} = \oplus_k K^{i,j,k}$ and $K^* = \oplus_{i+j=*} K^{i,j}$.

Consider the real differential operators $d = (\partial + \bar{\partial})$ and $d^c = -\sqrt{-1}(\partial - \bar{\partial})$. One defines the following maps:

$$d'\colon K^{i,j,k} \to K^{i+1,j+1,k+1}, \quad d'(a) = d(a) \tag{4.2}$$

$$\begin{aligned} &d''\colon K^{i,j,k} \to K^{i+1,j+1,k} \\ &d''(a) = -d^c(a) \quad (\text{projected onto } K^{i+1,j+1,k}) \end{aligned} \tag{4.3}$$

and

$$N\colon K^{i,j,k} \to K^{i+2,j,k+1} \qquad N(a) = (2\pi\sqrt{-1})^{-1}a \tag{4.4}$$

for any $a \in K^{i,j,k}$.

On $K^{i,j}$, let $d = d' + d''$. The complex Cone(N) = Cone$(N\colon\ K^* \to K^*)$ is defined to be $K^* \oplus K^*[-1]$ endowed with the differential $D(a,b) = (\mathrm{d}a, N(a) - \mathrm{d}b)$.

Using the same arguments presented in the proof of Proposition 3.2, one gets

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

PROPOSITION 4.1. *Let p be a fixed non negative integer. Then, the complex* $\mathrm{Cone}(N\colon K^{q-2p,q-n} \to K^{q-2p+2,q-n})(q \in \mathbb{Z})$ *is quasi-isomorphic to the 'Deligne-complex'* $(C^q_{\mathcal{D}}(p), \mathrm{d}_{\mathcal{D}})$

$$C^q_{\mathcal{D}}(p) := \begin{cases} \left(\bigoplus_{\substack{a+b=j-1 \\ |a-b| \leqslant 2p-q-1}} A^{a,b} \right)_{\mathbb{R}} (p-1) & \text{if } q \leqslant 2p-1 \\ \left(\bigoplus_{\substack{a+b=q \\ |a-b| \leqslant q-2p}} A^{a,b} \right)_{\mathbb{R}} (p) & \text{if } q \geqslant 2p \end{cases}$$

for $a \in C^q_{\mathcal{D}}(p)$, the differential $\mathrm{d}_{\mathcal{D}}$ is given by

$$\mathrm{d}_{\mathcal{D}}(a) = \begin{cases} d''(a) & \text{if } q < 2p-1 \\ 2\pi\sqrt{-1}d'd''(a) & \text{if } q = 2p-1 \\ d'(a) & \text{if } q \geqslant 2p \end{cases}$$

i.e.

$$\left(\bigoplus_{\substack{a+b=q-1 \\ |a-b| \leqslant 2p-q-1}} A^{a,b} \right)_{\mathbb{R}} (p-1) \xrightarrow{d''} \cdots \to \left(\bigoplus_{\substack{a+b=2p-3 \\ |a-b| \leqslant 1}} A^{a,b} \right)_{\mathbb{R}} (p-1)$$

$$\to (A^{p-1,p-1})_{\mathbb{R}}(p-1) \xrightarrow{2\pi\sqrt{-1}d'd''}$$

$$\xrightarrow{2\pi\sqrt{-1}d'd''} (A^{p,p})_{\mathbb{R}}(p) \xrightarrow{d'} \left(\bigoplus_{\substack{a+b=2p+1 \\ |a-b| \leqslant 1}} A^{a,b} \right)_{\mathbb{R}} (p)$$

$$\to \cdots \to \left(\bigoplus_{\substack{a+b=q \\ |a-b| \leqslant q-2p}} A^{a,b} \right)_{\mathbb{R}} (p) \to \cdots$$

In particular, $C^{2p-1}_{\mathcal{D}}(p) = (A^{p-1,p-1})_{\mathbb{R}}(p-1), C^{2p}_{\mathcal{D}}(p) = (A^{p,p})_{\mathbb{R}}(p)$. *The map* $2\pi\sqrt{-1}d'd''$ *is the composite of d'' followed by (the identity N and) the map d'.*

When X is defined over $\mathbb{C}$, the homology of this complex computes the real Deligne cohomology of X (cf. [4] Theorem 1.10). We recall that the real Deligne cohomology of $X_{\mathbb{R}}$ (i.e. a variety defined over $\mathbb{R}$) is defined as

$$H^*_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) := H^*_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(p))^{\bar{F}_\infty = id}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

$\bar{F}_\infty$ being the De-Rham conjugation. Therefore, taking the $\bar{F}_\infty$-invariants of the homology of the above complex yields a description of $H^*_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p))$.

In analogy with the properties stated in [2] Lemma 1, we have

LEMMA 4.2.

(i) $d^2 = 0 = d'd'' + d''d'$.

(ii) *N commutes with d' and d'', hence* $[N, d] = 0$.

(iii) *For any* $i, j \in \mathbb{Z}, i \geqslant 0$ *the map*

$$N^i\colon K^{-i,j} \to K^{i,j}$$

is an isomorphism.

(iv) *For any* $i, j \in \mathbb{Z}, i \geqslant 0$

$$\mathrm{Ker}(N^{i+1}) \cap K^{-i,j} = K^{-i,j,0} = \left(\bigoplus_{\substack{a+b=j+n \\ |a-b| \leqslant i}} A^{a,b} \right)_{\mathbb{R}} \left(\frac{n+j+i}{2} \right).$$

Proof. For (i): let f be a differential form of pure type (a, b) with $a < b$. Let $g = f + \bar{f}$ be an element of $K^{i,j,k}$. Then, $a + b = j + n$. If $|a - b| < 2k - i$, then the statement follows from the well known equality $dd^c + d^c d = 0$. If $|a - b| = 2k - i$, then

$$d'd''(g) = -\sqrt{-1}d'(\partial f - \bar{\partial}\bar{f}) = -\sqrt{-1}(-\partial\bar{\partial}\bar{f} + \bar{\partial}\partial f).$$

On the other hand, since $d''(\partial f + \bar{\partial}\bar{f}) = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$

$$d''d'(g) = d''(\partial\bar{f} + \bar{\partial}f) = -\sqrt{-1}(\partial\bar{\partial}f - \bar{\partial}\partial\bar{f}) = \sqrt{-1}(-\partial\bar{\partial}\bar{f} + \bar{\partial}\partial f).$$

(ii), (iii) and (iv) are direct consequences of the definitions by checking degrees. □

Since $d^2 = 0 = [N, d]$, d defines a differential on the bigraded groups:

$$\mathrm{Ker}(N)^{\cdot,\cdot} = \mathrm{Ker}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot}), \quad \mathrm{Coker}(N)^{\cdot,\cdot} = \mathrm{Coker}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot})$$

as well as on the mapping cone of N

$$\mathrm{Cone}(N)^{\cdot,\cdot} = \mathrm{Cone}(K^{\cdot,\cdot} \xrightarrow{N} K^{\cdot+2,\cdot}).$$

We define as in (3.2)–(3.5) the groups $gr^W_{q+r} H^q(\tilde{X}^*), gr^W_{q+r} H^q(X^*), gr^W_{q+r} H^q(Y)$ and $gr^W_{q+r} H^q_Y(X)$. These groups fit into exact sequences like (3.6) and (3.7). In analogy with Lemma 3.3 of Section 3 we have ($p \in \mathbb{Z}$).

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

LEMMA 4.3.

(i)

$$gr^W_{q+r} H^q(Y)$$

$$= \begin{cases} \dfrac{\mathrm{Ker}\left(d' \colon \left(\bigoplus_{\substack{a+b=q \\ |a-b|\leqslant -r}} A^{a,b}\right)_{\mathbb{R}}(p) \to \left(\bigoplus_{\substack{a+b=q+1 \\ |a-b|\leqslant -r+1}} A^{a,b}\right)_{\mathbb{R}}(p)\right)}{\mathrm{Image}(d')} & r \leqslant 0, q+r = 2p \\ 0 & \textit{otherwise} \end{cases}$$

In particular,

$$gr^W_{2p} H^{2p}(Y) = \mathrm{Ker}\left(d' \colon (A^{p,p})_{\mathbb{R}}(p) \to \left(\bigoplus_{\substack{a+b=2p+1 \\ |a-b|\leqslant 1}} A^{a,b}\right)_{\mathbb{R}}(p)\right)$$

(ii)

$$gr^W_{q+r} H^q_Y(X)$$

$$= \begin{cases} \dfrac{\mathrm{Ker}\left(d'' \colon \left(\bigoplus_{\substack{a+b=q-2 \\ |a-b|\leqslant -r}} A^{a,b}\right)_{\mathbb{R}}(p-1) \to \left(\bigoplus_{\substack{a+b=q-1 \\ |a-b|\leqslant r-1}} A^{a,b}\right)_{\mathbb{R}}(p-1)\right)}{\mathrm{Image}(d'')} & r \geqslant 0, q+r = 2p \\ 0 & \textit{otherwise} \end{cases}$$

In particular,

$$gr^W_{2p} H^{2p}_Y(X)$$

$$= \mathrm{Coker}\left(d'' \colon \left(\bigoplus_{\substack{a+b=2p-3 \\ |a-b|\leqslant 1}} A^{a,b}\right)_{\mathbb{R}}(p-1) \to (A^{p-1,p-1})_{\mathbb{R}}(p-1)\right)$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

(iii) *The map* $gr_m^W H_Y^q(X) \to gr_m^W H^q(Y)$ *in the exact sequence* (3.7) *is zero unless* $m = q = 2p$. *On* $gr_{2p}^W H_Y^{2p}(X)$ *this map coincides with the morphism:*

$$2\pi\sqrt{-1}d'd'' : \frac{(A^{p-1,p-1})_{\mathbb{R}}(p-1)}{\mathrm{Image}(d'')} \\ \to \mathrm{Ker}\left(d' : (A^{p,p})_{\mathbb{R}}(p) \to \left(\bigoplus_{\substack{a+b=2p+1 \\ |a-b|\leqslant 1}} A^{a,b}\right)_{\mathbb{R}}(p)\right)$$

(iv) $gr_{q+r}^W H^q(X^*) = 0$ *unless* $q + r = 2p$, *in which case*

$$gr_{2p}^W H^q(X^*)$$

$$= \begin{cases} gr_{2p}^W H_Y^{q+1}(X) & \text{if } q < 2p-1, \\ \frac{\mathrm{Ker}(2\pi\sqrt{-1}d'd'' : (A^{p-1,p-1})_{\mathbb{R}}(p-1) \to (A^{p,p})_{\mathbb{R}}(p))}{\mathrm{Image}(d'')} & \text{if } q = 2p-1, \\ \frac{\mathrm{Ker}\left(d' : (A^{p,p})_{\mathbb{R}}(p) \to \left(\bigoplus_{\substack{a+b=2p+1 \\ |a-b|\leqslant 1}} A^{a,b}\right)_{\mathbb{R}}(p)\right)}{\mathrm{Image}(2\pi\sqrt{-1}d'd'')} & \text{if } q = 2p, \\ gr_{2p}^W H^q(Y) & \text{if } q \geqslant 2p+1 \end{cases}$$

Note that for $p \geqslant 0$ one gets isomorphisms

$$H_{\mathcal{D}}^q(X_{\mathbb{C}}, \mathbb{R}(p)) \simeq gr_{2p}^W H^q(X^*)$$

$$(\text{resp. } H_{\mathcal{D}}^q(X_{\mathbb{R}}, \mathbb{R}(p)) \simeq gr_{2p}^W H^q(X^*)^{\bar{F}_\infty = id})$$

The statement which corresponds to Corollary 3.4 in this context is.

COROLLARY 4.4. $gr_{q+r}^W H^q(\tilde{X}^*) = 0$ *unless* $q + r = 2p$, *in which case*

$$N : gr_{2p}^W H^q(\tilde{X}^*) \to gr_{2p-2}^W H^q(\tilde{X}^*)$$

is

$$\begin{cases} \textit{injective} & \text{if } q < 2p-1 \\ \textit{bijective} & \text{if } q = 2p-1 \\ \textit{surjective} & \text{if } q \geqslant 2p \end{cases}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Furthermore,

if $q < 2p-1$,

$$\mathrm{Coker}N = \frac{H^q(X,\mathcal{C})}{F^pH^q(X,\mathcal{C}) + H_B^q(X,\mathbb{R}(p))} \simeq H_{\mathcal{D}}^{q+1}(X_{\mathbb{C}},\mathbb{R}(p))$$

if $q \geqslant 2p$, $\quad \mathrm{Ker}\, N = gr_{2p}^W H^q(X^*)$.

If $q = 2p$:

$$gr_{2p}^W H^{2p}(X^*) = F^pH^{2p}(X,\mathbb{C}) \cap H_B^{2p}(X,\mathbb{R}(p)) \simeq H_{\mathcal{D}}^{2p}(X_{\mathbb{C}},\mathbb{R}(p)).$$

Finally, for $X_{\mathbb{R}}$*, we have the corresponding statements by taking the* $\bar{F}_\infty$*-invariants. In particular*

$$gr_{2p}^W H^{2p}(X^*)^{\bar{F}_\infty = id} \simeq H_{\mathcal{D}}^{2p}(X_{\mathbb{R}},\mathbb{R}(p)).$$

Since X is projective over $\mathbb{C}$ (or $\mathbb{R}$), the fundamental real $(1,1)$-form ω on it defines the Lefschetz operator

$$l\colon\ K^{i,j,k} \to K^{i,j+2,k} \qquad l(a) = (2\pi\sqrt{-1})a \wedge \omega \tag{4.5}$$

This operator satisfies the following properties.

LEMMA 4.5.
(i) $[l,N] = [l,d'] = [l,d''] = 0$ *and hence* $[l,d] = 0$.
(ii) *For any integer* i *and* $\forall j \geqslant 0, l$ *induces isomorphisms*

$$l^j : K^{i,-j} \to K^{i,j}$$

(iii) *For any non negative integers* i,j

$$\begin{aligned}
&(K^{-i,-j})_0 \\
&:= K^{-i,-j} \cap (\mathrm{Ker}\ l^{j+1}) \cap\ (\mathrm{Ker}\ N^{i+1}) \\
&= K^{-i,-j,0} \cap\ (\mathrm{Ker}\ l^{j+1}) \\
&= \mathrm{Ker}\left(l^{j+1}\colon \left(\bigoplus_{\substack{a+b=-j+n\\|a-b|\leqslant i}} A^{a,b}\right)_{\mathbb{R}} \left(\frac{n+i-j}{2}\right)\right. \\
&\qquad \left.\to \left(\bigoplus_{\substack{a+b=j+n+2\\|a-b|\leqslant i}} A^{a,b}\right)_{\mathbb{R}} \left(\frac{n+i+j}{2}+1\right)\right).
\end{aligned}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Proof. It is immediate that $[l, N] = 0.[l, d'] = 0 = [l, d'']$ follow from the definitions and the property that ω is d'-closed. (ii) is the hard Lefschetz theorem on the level of differential forms (cf. [23] Chapt. V, Sect. 2 and Theorem 3.12(c)). Finally, (iii) follows from Lemma 4.2(iv). □

We set, for $m \in \mathbb{Z}$

$$\epsilon(m) = (-1)^{\frac{m(m+1)}{2}}.$$

Let denote by C the Weil operator: on a differential form f of type (a, b)

$$C(f) := (\sqrt{-1})^{a-b}(f).$$

(cf. *op. cit.* Chapt. V, Sect. 1).

Define pairings

$$\begin{aligned} &\psi\colon\ K^{-i,-j,k} \otimes K^{i,j,k+i} \to \mathbb{R}(n) \\ &\psi(x, y) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \epsilon(n-j)(-1)^k \int_X x \wedge Cy \end{aligned} \tag{4.6}$$

$$\begin{aligned} x &\in K^{-i,-j,k} \\ &= \left(\bigoplus_{\substack{a+b=-j+n \\ |a-b|\leqslant 2k+i}} A^{a,b}\right)_{\mathbb{R}} \left(\frac{n-j+i}{2}\right), y \in K^{i,j,k+i} \\ &= \left(\bigoplus_{\substack{a+b=j+n \\ |a-b|\leqslant 2k+i}} A^{a,b}\right)_{\mathbb{R}} \left(\frac{n+j-i}{2}\right). \end{aligned}$$

One extends ψ by zero: i.e. ψ vanishes on $K^{i,j,k} \otimes K^{u,v,w}$ unless $i + u = j + v = w + i - k = 0$. Hence, we get a pairing

$$\psi\colon\ K^{\cdot,\cdot} \otimes K^{\cdot,\cdot} \to \mathbb{R}(n).$$

LEMMA 4.6. *The following identities hold*

(i) $\psi(x, y) = (-1)^n \psi(y, x)$.
(ii) $\psi(Nx, y) + \psi(x, Ny) = 0$.
(iii) $\psi(lx, y) + \psi(x, ly) = 0$.
(iv) $\psi(d'x, y) = \psi(x, d''y)$.
(v) $\psi(d''x, y) = \psi(x, d'y)$.

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Proof. (i) is a direct consequence of the equality $(-1)^k\epsilon(n-j) = (-1)^{k+i}\epsilon(n+j)(-1)^j$. Note that it follows from the definition of the complex $K^{i,j,k}$ given in (4.1) that $\psi(x,y) = (-1)^{i+j}\psi(y,x) = (-1)^n\psi(y,x)$. (ii) is immediate from the definitions. (iii) is a consequence of $\epsilon(n-j) = -\epsilon(n-j-2)$. For (iv): let $x \in K^{-i-1,-j-1,k-1}$ and $y \in K^{i,j,k+i}$. Since $d^c = C^{-1}d'C$, it follows from Stokes's theorem and the definitions of d' and d''

$$\begin{aligned}\psi(d'x,y) &= \left(\tfrac{1}{2\pi\sqrt{-1}}\right)^n \epsilon(n-j)(-1)^k \textstyle\int_X d'x \wedge Cy\\ &= \left(\tfrac{1}{2\pi\sqrt{-1}}\right)^n \epsilon(n-j)(-1)^{n-j+k} \textstyle\int_X x \wedge C(C^{-1}d'C)y\\ &= \left(\tfrac{1}{2\pi\sqrt{-1}}\right)^n (\epsilon(n-j)(-1)^{n-j})(-1)^{k-1} \textstyle\int_X x \wedge Cd''y\\ &= \psi(x,d''y).\end{aligned}$$

(v) is a direct consequence of (iv) and (i). □

PROPOSITION 4.7. *The bilinear form $\psi(\cdot, l^jN^i\cdot)$ induces on $K_0^{-i,-j}$ a polarization i.e. the bilinear form*

$$Q\colon\ K_0^{-i,-j} \otimes K_0^{-i,-j} \to \mathbb{R},$$

defined as

$$Q(x,y) = \psi(x, l^jN^iy),$$

is symmetric and positive definite.

Proof. Let

$$\begin{aligned}&x,y \in (K^{-i,-j})_0\\ &\quad = (K^{-i,-j,0})_0 = \Bigg(\bigoplus_{\substack{a+b=n-j\\ |a-b|\leqslant i}} A^{a,b}\Bigg)_{\mathbb{R}} \left(\frac{n-j+i}{2}\right),\end{aligned}$$

then

$$(2\pi\sqrt{-1})^{\frac{-n+j-i}{2}}x,\quad (2\pi\sqrt{-1})^{\frac{-n+j-i}{2}}y \in \Bigg(\bigoplus_{\substack{a+b=n-j\\ |a-b|\leqslant i}} A^{a,b}\Bigg)_{\mathbb{R}}.$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

Since $N(a) = (2\pi\sqrt{-1})^{-1}a$, and $l(a) = (2\pi\sqrt{-1})a \wedge \omega$, one has

$$Q(x,y) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \epsilon(n-j) \int_X x \wedge l^j N^i C y$$
$$= \epsilon(n-j) \int_X (2\pi\sqrt{-1})^{\frac{-n+j-i}{2}} x \wedge C (2\pi\sqrt{-1})^{\frac{-n+j-i}{2}} y.$$

Hence, the claim follows from classical Hodge theory: cf. [23] Chapt. V, Sect. 6, Theorem 6.1(d). □

From Lemmas 4.2, 4.5, 4.6 and Proposition 4.7 we deduce that $(K^{\cdot,\cdot}, N, l, \psi)$ is a polarized bigraded Lefschetz module in the sense of Saito (cf. [17]). Hence, via the one-to-one correspondence between bigraded Lefschetz modules and representations of the Lie group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, we associate to $(K^{\cdot,\cdot}, N, l, \psi)$ the representation σ of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ defined as

$$\sigma\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right](x) = a^i b^j x, \qquad x \in K^{i,j}$$

and

$$\mathrm{d}\sigma\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right] = N,$$

$$\mathrm{d}\sigma\left[0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = l.$$

It follows from the notations introduced in Lemma 4.5(iii), that one has a Lefschetz decomposition

$$K^{i,j} = \bigoplus_{r,s \geqslant 0} N^r l^s (K^{i-2r,j-2s})_0.$$

The Weyl element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{R}),$$

defines the element $w_2 = (w, w) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ and since the bilinear form ψ defined in (4.6) induces a polarization of $K^{\cdot,\cdot}$ (i.e. ψ satisfies the properties (ii) and (iii) of Lemma 4.6 and Proposition 4.7), it is not difficult to see that the pairing

$$\phi(x,y) := \psi(x, w_2 y),$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

is symmetric and positive definite on $K^{\cdot,\cdot}$ (cf. [11] p. 151 for a proof). It follows from Lemma 4.6 that the differential $d = d' + d''$ admits $w_2^{-1} d w_2$ as transpose ${}^t d$ relative to ϕ. Then, the Laplace operator on $K^{\cdot,\cdot}$

$$\square := \mathrm{d}({}^t d) + ({}^t d)\,\mathrm{d},$$

commutes with the action of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ (cf. *op. cit*. Lemma at p. 153). Using the properties of ϕ mentioned above, one gets

$$H^*(K^*, d) = \mathrm{Ker\ d}/\mathrm{Image\ d} = \mathrm{Ker}\ d \cap \mathrm{Ker}\ {}^t d = \mathrm{Ker}\ \square$$

and $\square$ is invariant for the action of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Then, from the isomorphism of complexes

$$K^* \simeq \mathrm{Ker}\square \oplus \mathrm{Image}\ \square,$$

where $\mathrm{d} = 0$ on (Ker $\square$) and the complex (Image $\square$) is d-acyclic, one deduces an induced action of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ on $H^*(K^*, \mathrm{d})$. Hence, the cohomology $H^*(K^*, \mathrm{d})$ of K^*, equipped with the endomorphisms N and l and the polarization ψ, is a polarized bigraded Lefschetz module. In particular, that means that both N and l satisfy hard Lefschetz theorems on the bigraded module $H^*(K^*, \mathrm{d})$ as in Lemmas 4.2(iii) and 4.5(iii). Therefore, one has the following.

PROPOSITION 4.8. *For all* $q, r \in \mathbb{Z}, q \geqslant 0$ *the operator* N *induces isomorphisms*

$$N^r\colon\ gr^W_{q+r} H^q(\tilde{X}^*) \xrightarrow{\simeq} gr^W_{q-r} H^q(\tilde{X}^*)$$

This result, together with exact sequences as in (3.6) and (3.7) implies (cf. *op. cit.* Theorem (5.3) for the arguments) the following Clemens–Schmidt type exact sequence.

PROPOSITION 4.9. *For* $r \in \mathbb{Z}$ *the following sequence is exact*

$$\cdots \to gr^W_{q+r} H^q_Y(X) \to gr^W_{q+r} H^q(Y) \xrightarrow{sp} gr^W_{q+r} H^q(\tilde{X}^*)$$
$$\xrightarrow{N} gr^W_{q+r-2} H^q(\tilde{X}^*) \xrightarrow{\lambda} gr^W_{q+r} H^{q+2}_Y(X) \to gr^W_{q+r} H^{q+2}(Y) \to \cdots,$$

sp is the composite $gr^W_{q+r} H^q(Y) \to gr^W_{q+r} H^q(X^*) \to gr^W_{q+r} H^q(\tilde{X}^*)$ *and* λ *is the composite* $gr^W_{q+r-2} H^q(\tilde{X}^*) \to gr^W_{q+r} H^{q+1}(X^*) \to gr^W_{q+r} H^{q+2}_Y(X)$ *as described in the sequences* (3.6) *and* (3.7).

Let define, following the notations used in [2] ($p \in \mathbb{Z}$), the groups

$$H^{p,p}_{II} := \mathrm{Coker}(d'd''\colon\ gr^W_{2p} H^{2p}_Y(X) \to gr^W_{2p} H^{2p}(Y))$$

$$H^{p,p}_{I} := \mathrm{Ker}(d'd''\colon\ gr^W_{2p+2} H^{2p+2}_Y(X) \to gr^W_{2p+2} H^{2p+2}(Y)).$$

From Lemma 4.3 and Proposition 4.9 it is easy to deduce

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

COROLLARY 4.10.

(i) $H_{II}^{p,p} = gr_{2p}^W H^{2p}(X^*) = \mathrm{Ker}(N\colon gr_{2p}^W H^{2p}(\tilde{X}^*) \to gr_{2p-2}^W H^{2p}(\tilde{X}^*))$

(ii) $H_I^{p,p} = gr_{2p+2}^W H^{2p+1}(X^*) = \mathrm{Coker}(N\colon gr_{2p+2}^W H^{2p}(\tilde{X}^*) \to gr_{2p}^W H^{2p}(\tilde{X}^*))$.

This Corollary together with Proposition 4.8 implies, arguing as in [2] (6.3).

COROLLARY 4.11.

$$H_{II}^{p,p} \simeq H_I^{p,p}$$

i.e.

$$gr_{2p}^W H^{2p}(X^*) \simeq gr_{2p+2}^W H^{2p+1}(X^*).$$

When $p \geqslant 0$, using the results established in Lemma 4.3(vi) and Corollary 4.4 we deduce an equivalent way to express this isomorphism.

COROLLARY 4.12.

$$H_{\mathcal{D}}^{2p}(X_{\mathbb{C}}, \mathbb{R}(p)) \simeq H_{\mathcal{D}}^{2p+1}(X_{\mathbb{C}}, \mathbb{R}(p+1)),$$

resp. for $X_{\mathbb{R}}$

$$H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \simeq H_{\mathcal{D}}^{2p+1}(X_{\mathbb{R}}, \mathbb{R}(p+1)).$$

Proposition 4.8, together with Lemma 4.3 and Corollary 4.4 permit us to extend the isomorphisms $H_{II}^{p,p} \simeq H_I^{p,p}$ to higher analogs. The symmetry already established in Corollary 4.11 by mean of the isomorphism $N^2\colon gr_{2(p+1)}^W H^{2p}(\tilde{X}^*) \xrightarrow{\simeq} gr_{2(p-1)}^W H^{2p}(\tilde{X}^*)$ represents only the first piece of the following more general result.

Let suppose that $q > 2p$, for $q, p \in \mathbb{Z}$ and q non negative. Then, the isomorphism $(q - 2p + 2 > 2)$

$$N^{q-2p+2}\colon gr_{2(q-p+1)}^W H^q(\tilde{X}^*), \xrightarrow{\simeq} gr_{2(p-1)}^W H^q(\tilde{X}^*),$$

decomposes into the following sequence of maps

$$\begin{aligned} 0 \to gr_{2(q-p+1)}^W H^q(\tilde{X}^*) &\xrightarrow{N} gr_{2(q-p)}^W H^q(\tilde{X}^*) \\ &\xrightarrow{N} \cdots \xrightarrow{N} gr_{2p}^W H^q(\tilde{X}^*) \xrightarrow{N} gr_{2(p-1)}^W H^q(\tilde{X}^*) \to 0. \end{aligned} \tag{4.7}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

According to Corollary 4.4, the first homomorphism N on the left is injective and the last one on the right is surjective. Further,

$$N^{q-2p}\colon\ gr^W_{2(q-p)}H^q(\tilde{X}^*) \xrightarrow{\simeq} gr^W_{2p}H^q(\tilde{X}^*),$$

is also an isomorphism. Therefore putting together these two facts one gets the following decomposition of $gr^W_{2p}H^q(\tilde{X}^*)$ as direct sum

$$\begin{aligned} &gr^W_{2p}H^q(\tilde{X}^*)\\ &\quad = gr^W_{2p}H^q(Y) \oplus \mathrm{Image}(N^{q-2p+1}\colon\ gr^W_{2(q-p+1)}H^q(\tilde{X}^*)\\ &\qquad \to gr^W_{2p}H^q(\tilde{X}^*)).\end{aligned}$$

Finally, noting that

$$\begin{aligned} &\mathrm{Image}(N^{q-2p+1}: gr^W_{2(q-p+1)}H^q(\tilde{X}^*) \to gr^W_{2p}H^q(\tilde{X}^*))\\ &\quad \simeq \mathrm{Image}(N\colon\ gr^W_{2(q-p+1)}H^q(\tilde{X}^*) \to gr^W_{2(q-p)}H^q(\tilde{X}^*)),\end{aligned}$$

we have

$$\begin{aligned} gr^W_{2p}H^q(Y) &= \frac{gr^W_{2p}H^q(\tilde{X}^*)}{\mathrm{Image}(N^{q-2p+1}\colon\ gr^W_{2(q-p+1)}H^q(\tilde{X}^*) \to gr^W_{2p}H^q(\tilde{X}^*))}\\ &\simeq \frac{gr^W_{2(q-p)}H^q(\tilde{X}^*)}{\mathrm{Image}(N\colon\ gr^W_{2(q-p+1)}H^q(\tilde{X}^*) \to gr^W_{2(q-p)}H^q(\tilde{X}^*))}\\ &= gr^W_{2(q-p+1)}H^{q+2}_Y(X).\end{aligned}$$

Summarizing, the theory developed in this section also allows one to recover the following results about Deligne cohomology groups.

PROPOSITION 4.13. *For* $q, p \in \mathbb{Z}, q > 2p, (q \geqslant 0)$

$$gr^W_{2p}H^q(Y) \simeq gr^W_{2(q-p+1)}H^{q+2}_Y(X) \simeq H^{q+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(q+1-p))$$

resp.

$$gr^W_{2p}H^q(Y)^{\bar{F}_\infty = id} \simeq H^{q+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(q+1-p))^{(-1)^q\bar{F}_\infty = id}.$$

In particular, when $p \geqslant 0$ *these isomorphisms show*

$$H^q_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(p)) \simeq H^{q+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(q+1-p))$$

resp.

$$H^q_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) \simeq H^{q+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(q+1-p))^{(-1)^q\bar{F}_\infty = id}.$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

The assertions about Deligne cohomology follow from Proposition 4.1, Lemma 4.3 and Corollary 4.4.

The isomorphisms between Deligne cohomology groups shown in the Proposition can easily be seen a priori by simply working out the definitions of those groups. The above Proposition shows therefore that the theory developed in this chapter behaves correctly.

Finally, our construction shows explicitly a symmetry between homology groups of the right and left hand sides of the 'Deligne complex' $C^{\cdot}_{\mathcal{D}}(p)$ introduced in Proposition 4.1. There, the map in the middle $2\pi\sqrt{-1}d'd'' = -2\pi\sqrt{-1}dd^c$ replaces in the archimedean case the composite $-i^* \cdot i_*$ studied in the non archimedean case: cf. Sect. 3 Proposition 3.2 and Lemma 3.3.

5. Applications: on Deninger infinite determinants

Let X be a smooth projective variety over a global field k and ν be a place of k. Let denote by $H^q(X_\nu)(X_\nu = X \times \mathrm{Spec}(k_\nu))$ the classical q-th Betti cohomology ($q \geqslant 0$) if ν is an archimedean place and the q-th l-adic cohomology ($\nu \nmid l$) otherwise. In [7], [8] and [9] Deninger proved that the local L-factor $L_\nu(H^q(X_\nu), s)$ can be described as it follows, using a suitable definition of an infinite determinant

$$L_\nu(H^q(X_\nu), s) = \begin{cases} \det_\infty(\frac{1}{2\pi}(s-\Theta)|H^q_{\mathrm{ar}}(X_\nu))^{-1}, & \text{if } \nu|\infty \\ \det_\infty(\frac{1}{2\pi}(s-\Theta)|\mathbb{D}(H^q(X_\nu)))^{-1} & \text{if } \nu \text{ is finite.} \end{cases}$$

At an archimedean place, the vector space $H^q_{\mathrm{ar}}(X_\nu)$ (the 'archimedean cohomology') is infinite dimensional over $\mathbb{R}$ and it carries a natural endomorphism Θ. If ν is a non archimedean place of good reduction, $X \to \mathbb{D}(H^q(X_\nu))$ is a cohomology theory on pure good reduction motives over k_ν. In general, $\mathbb{D}(H^q(X_\nu))$ is an object in the category of regular-singular algebraic differential equations on $\mathbb{G}_m/\mathbb{C}$ or equivalently into the category of finite-dimensional complex representations of $\pi_1(\mathbb{C}^*, 1) \simeq \mathbb{Z}$. Finally, N_ν is the number of elements of the residue field of the local field k_ν.

In this paragraph we deal with the 'archimedean cohomology' and we prove that the Euler factor at infinity can be completely recovered using the definition of the infinite determinant introduced in [7] on the infinite dimensional $\mathbb{R}$-vector space $H^q(\tilde{X}^*)^{N=0} = \oplus_p gr^W_{2p} H^q(X^*)$ studied in the last section. On $H^q(\tilde{X}^*)$ we define a linear operator Φ acting on $gr^W_{2p} H^q(X^*)$ as a multiplication by p. The main result proved in this section are the equalities

$$(H^q_{\mathrm{ar}}(X_{\mathbb{C}}), \Theta) = (H^q(\tilde{X}^*)^{N=0}, \Phi)$$

$$(H^q_{\mathrm{ar}}(X_{\mathbb{R}}), \Theta) = (H^q(\tilde{X}^*)^{N=0, \bar{F}_\infty = id}, \Phi).$$

Unfortunately, one cannot expect a similar result at a finite place ν of semistable reduction and recover there the local L-factor $L_\nu(H^q(X_\nu), s)$ using again Deninger

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

formal construction by the infinite determinants applied to the spaces $H^q(\tilde{X}^*)$ we introduced in Sect. 3, the main reason being the finite rank of those vector spaces.

From now on we indicate by X a smooth, projective variety over $K = \mathbb{C}$ or $\mathbb{R}$. Following [7] (cf. p. 253), the 'archimedean cohomology' groups are defined as

$$H^q_{\text{ar}}(X_K) := \mathbb{D}(H^q_{\text{B}}(X_{\mathbb{C}}, \mathbb{C})),$$

where the Betti cohomology groups $H^q_B(X_{\mathbb{C}}, \mathbb{C}) = H^q_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$ are viewed as real Hodge structures over K.$\mathbb{D}(\cdot)$ is a functor from the abelian category of pure Hodge structures over K to the additive category $\mathcal{D}_K$ whose objects are couples (D, Θ_D).D is a free $\mathbb{L}_K$-module of finite rank and Θ is an $\mathbb{R}$-linear endomorphism on D satisfying some module-type compatibility conditions as in *op. cit* (cf. p. 249). $\mathbb{L}_K$ is an $\mathbb{R}$-algebra isomorphic to $\mathbb{R}[T^{-1}]$ if $K = \mathbb{C}$ and to $\mathbb{R}[T^{-2}]$ if $K = \mathbb{R}$, being T an indeterminate. L_K is endowed with a derivation map $\Theta_{\mathbb{L}} = T\frac{d}{dT}$ and the morphisms in the additive category $\mathcal{D}_K$ are $\mathbb{L}$-linear maps commuting with the Θ's. According to the properties of the functor $\mathbb{D}, \mathbb{D}(H^q_{\text{B}}(X_{\mathbb{C}}, \mathbb{C}))$ is a free $(\mathbb{L}, \Theta_L)$-module of rank equal to the q-th Betti number of X.

The main result of *op. cit* (cf. Theorem (4.1)) is the following:

THEOREM 5.1 (Deninger).

$$\det\nolimits_\infty \left(\frac{s}{2\pi} - \frac{\Theta}{2\pi} | H^q_{\text{ar}}(X)\right)^{-1} = L_K(H^q_B(X_{\mathbb{C}}, \mathbb{C}), s).$$

$L_K(H^q_B(X_{\mathbb{C}}, \mathbb{C}), s)$ is the complex (or real) Euler factor (cf. [20]) associated to the real Hodge structure $H^q_B(X_{\mathbb{C}}, \mathbb{C})$. $\det_\infty$ is the infinite determinant defined in *op. cit.* (cf. pp. 246–7). Since it is known that the order of pole of $L_K(H^q_B(X_{\mathbb{C}}, \mathbb{C}), s)$ at any integer $s \leqslant \frac{q}{2}$ is related to the real Deligne cohomology of X_K, a corollary of this theorem is the following (cf. *op. cit.* Proposition (5.1))

PROPOSITION 5.2 (Deninger). *If $q \geqslant 2m$, then*

$$\dim_{\mathbb{R}} H^q_{\text{ar}}(X_K)^{\Theta = m} = \dim_{\mathbb{R}} H^{q+1}_{\mathcal{D}}(X_K, \mathbb{R}(q+1-m)).$$

Let now consider the couple $(H^q(\tilde{X}^*), N)$ as defined in the last section. We recall that $H^q(\tilde{X}^*) = \oplus_p gr^W_{2p} H^q(\tilde{X}^*)$ and by construction $gr^W_{2p} H^q(\tilde{X}^*)$ has weight p. We define on it the following linear operator

$$\Phi\colon gr^W_{2p} H^q(\tilde{X}^*) \to gr^W_{2p} H^q(\tilde{X}^*) \qquad \Phi(a) := p \cdot a.$$

Then, we extend this definition on the whole group $H^q(\tilde{X}^*)$ according to the decomposition of $H^q(\tilde{X}^*)$ as above.

Note that on each graded piece, one might view Φ as the logarithm of a sort of 'frobenius-type' operator

$$\text{Fr}\colon gr^W_{2p} H^q(\tilde{X}^*) \to gr^W_{2p} H^q(\tilde{X}^*) \qquad \text{Fr}(a) := e^p \cdot a. \tag{5.1}$$

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

These definitions might be interpreted as another hint for a classification of the fiber at infinity in the context of the 'bad' fibers.

From now on we will use the following notation

$$H^q(\tilde{X}^*_K)^{N=0} = \begin{cases} H^q(\tilde{X}^*)^{N=0} & \text{if } K = \mathbb{C} \\ H^q(\tilde{X}^*)^{N=0,\bar{F}_\infty=id} & \text{if } K = \mathbb{R}. \end{cases} \tag{5.2}$$

As a first result we have

PROPOSITION 5.3. *For any* $q \geqslant 0$

$$(H^q_{\text{ar}}(X_K), \Theta) = (H^q(\tilde{X}^*_K)^{N=0}, \Phi).$$

Proof. If $q \geqslant 2m$ and $K = \mathbb{C}$, then it follows from Deninger definitions that

$$\begin{aligned} &H^q_{\text{ar}}(X_{\mathbb{C}})^{\langle\Theta=m\rangle} \\ &\quad := \mathbb{D}(H^q_B(X_{\mathbb{C}}, \mathbb{C}))^{\langle\Theta=m\rangle} \\ &\quad = \left[\sum_v F^v H^q_B(X_{\mathbb{C}}, \mathbb{C}) \otimes \sum_{k\leqslant v} \mathbb{R}(k)\right]^{c=id,\Theta=m} \\ &\quad = [F^m H^q(X_{\mathbb{C}}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{R}(m)]^{c=id}, \end{aligned}$$

for c being the $\mathbb{R}$-linear involution induced by the complex cojugation on the coefficients. It is immediate from the description given in Sect. 4 Corollary 4.4 that this group is equal to $gr^W_{2m} H^q(\tilde{X}^*)^{N=0} = gr^W_{2m} H^q(X^*) = H^q(X^*_{\mathbb{C}})^{\Phi=m}$.

When $K = \mathbb{R}$,

$$\begin{aligned} H^q_{\text{ar}}(X_{\mathbb{R}})^{\langle\Theta=m\rangle} &:= \mathbb{D}(H^q(X_{\mathbb{C}}, \mathbb{C}))^{F_\infty=id,\langle\Theta=m\rangle} \\ &= [F^m H^q(X_{\mathbb{C}}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{R}(m)]^{\bar{F}_\infty=id,c=id}, \end{aligned}$$

where F_∞ is the $\mathbb{C}$-linear involution induced by the complex conjugation on the variety $X_{\mathbb{C}}$. Also in this case this group coincides with our $gr^W_{2m} H^q(\tilde{X}^*_{\mathbb{R}})^{N=0} = H^q(X^*_{\mathbb{R}})^{\Phi=m}$: cf. (5.2) above.

If $q < 2m$, $[F^m H^q(X_{\mathbb{C}}, \mathbb{C})]^{c=id} = 0 = gr^W_{2m} H^q(\tilde{X}^*_K)^{N=0}$. Hence, we conclude that

$$H^q_{\text{ar}}(X_K) = \bigoplus_{q\geqslant 2m} H^q_{\text{ar}}(X_K)^{\Theta=m} = \bigoplus_{q\geqslant 2m} gr^W_{2m} H^q(\tilde{X}^*_K)^{N=0} = H^q(\tilde{X}^*_K)^{N=0} \qquad \square$$

As a direct consequence of that and the definition of infinite determinant, we can state our main result

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

THEOREM 5.4.

$$\det_{\infty}\left(\frac{s}{2\pi}-\frac{\Phi}{2\pi}|H^q(\tilde{X}^*_K)^{N=0}\right)^{-1}=L_K(H^q_B(X_{\mathbb{C}},\mathbb{C}),s).$$

We end up by remarking that the isomorphism

$$(H^q(\tilde{X}^*_K)^{N=0})^{\langle\Phi=m\rangle}\simeq H^{q+1}_{\mathcal{D}}(X_K,\mathbb{R}(q+1-m)) \quad (5.3)$$

depends on a choice of $i=\sqrt{-1}$ (i.e. an orientation of $\mathbb{C}$). In our theory this reliance is a clear consequence of the definition of the weighted operator N, since (5.3) is nothing but a way to express the isomorphisms shown in Proposition 4.13, i.e. the identification of $gr^W_{2m}H^q(Y)$ with $gr^W_{2(q-m+1)}H^{q+2}_Y(X)\simeq H^{q+1}_{\mathcal{D}}(X_K,\mathbb{R}(q+1-m))$. In [9] (cf. Proposition (6.10)), the corresponding duality isomorphism between the 'archimedean cohomology' $H^q_{\mathrm{ar}}(X_K)^{\langle\Theta=m\rangle}$ and (a twist of) $H^{q+1}_{\mathcal{D}}(X_K,\mathbb{R}(q+1-m))$ is explained by constructing a natural perfect pairing

$$H^{q+1}_{\mathcal{D}}(X_K,\mathbb{R}(q+1-m))\times H^q_{\mathrm{ar}}(X_K)^{\langle\Theta=m\rangle}\to\mathbb{R}.$$

Because of the Proposition 5.3, that implies the naturality of the corresponding pairing

$$H^{q+1}_{\mathcal{D}}(X_K,\mathbb{R}(q+1-m))\times(H^q(\tilde{X}^*_K)^{N=0})^{\langle\Phi=m\rangle}\to\mathbb{R}.$$

Acknowledgments

This paper is a reduced form of my Ph.D dissertation thesis at the University of Chicago, written under the supervision of Prof. Spencer Bloch. It is my pleasure to thank him for what I learned under his guidance. This paper is the result of his kind and constant support and benefits from several stimulating discussions I had with him in these years.

For many of the arguments developed here I was inspired by Ch. Soulé manuscript 'On Chow groups of semistable models': a first version of [3] and by Steenbrink thesis (cf. [22]). Finally, I would like to thank K. Kato for some suggestions and remarks.

References

1. Bloch, S.: Algebraic cycles and higher K-theory, *Advances in mathematics* 61 (1986) 267–304.
2. Bloch, S., Gillet, H. and Soulé, C.: Non-archimedean Arakelov theory, *Journal of Algebraic Geometry* 4 no. 3 (1995) 427–485.
3. Bloch, S. and Gillet, H.: C. Soulé *Algebraic cycles on degenerate fibers*, Preprint (1995).
4. Burgos Gil, J. I.: *Anillos de Chow aritmeticos*, Ph.D. Thesis Universitat de Barcelona (1994).
5. Consani, C.: *Double complexes and Euler L-factors on degenerations of algebraic varieties*, Ph.D. Thesis The University of Chicago (1996).

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press

6. Deligne, P.: Structures de Hodge mixtes réelles, *Proceedings of Symp. Pure Math.* Vol. 55 I (U. Jannsen et al., eds.), AMS 1994, 509–514.
7. Deninger, C.: On the Γ-factors attached to motives, *Inventiones Math.* 104 (1991) 245–261.
8. Deninger, C.: Local L-factors of motives and regularized determinants, *Inventiones Math.* 107 (1992) 135–150.
9. Deninger, C.: Motivic L-functions and regularized determinants, *Proceedings of Symp. Pure Math.* Vol. 55 I (U. Jannsen et al., eds.), AMS 1994, 707–743.
10. Fulton, W.: *Intersection Theory*, Springer Verlag, Berlin, New York, Tokyo, 1984.
11. Guillén, F. and Navarro Aznar, V.: Sur le théorème local des cycles invariants, *Duke Mathematical Journal* 61 no. 1, (1990) 133–155.
12. Guillén, F., Navarro Aznar, V., Pascual, P. and Puerta, F.: Hyperrésolutions cubiques et descente cohomolologique, *Lect. Notes in Math.* 1335 Springer Verlag, Berlin, Heidelberg (1988).
13. Gillet, H. and Soulé, C.: *Descent, Motives and K-theory*, Preprint (1995).
14. Hanamura, M.: *Homological and cohomological motives of algebraic varieties (Chow cohomology groups)*, Preprint (1995).
15. Manin, Yu. I.: Three-dimensional hyperbolic geometry as ∞-adic Arakelov geometry, *Inventiones Math.* 104 (1991) 223–244.
16. Rapoport, M. and Zink, Th.: On the local zeta function of Shimura varieties. Monodromy filtration and vanishing cycles in unequal characteristic, *Inventiones Math.* 68 (1982) 21–101.
17. Saito, M.: *Modules de Hodge polarisables*, Publ. RIMS 24 (1988) 849–995.
18. Schmid, W.: Variation of Hodge structure: the singularities of the period mapping, *Inventiones Math.* 22 (1973) 211–320.
19. Serre, J. P.: *Corps locaux*, Hermann, Paris, 1968.
20. Serre, J. P.: *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Séminaire Delange–Pisot–Poitou, exp. 19, 1969–70.
21. Soulé, C.: Groupes the Chow et K-théorie de variétés sur un corps fini, *Math. Ann.* 268 (1984) 317–345.
22. Steenbrink, J. H.: Limits of Hodge structures, *Inventiones Math.* 31 (1976) 229–257.
23. Wells, R. O.: *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, Berlin, 1980.

https://doi.org/10.1023/A:1000362027455 Published online by Cambridge University Press