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ON EIGENVALUE PROBLEMS FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES

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Abstract This paper is devoted to the Dirichlet problem for quasilinear elliptic hemivariational inequalities at resonance as well as at non-resonance. Using Clarke's notion of the generalized gradient and the property of the first eigenfunction, we also build a Landesman–Lazer theory in the non-smooth framework of quasilinear elliptic hemivariational inequalities.

Keywords: elliptic hemivariational inequality; generalized Clarke subdifferential; pseudo-monotone operator; existence of solutions

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1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$, let $H_0^1(\Omega)$ be the usual Sobolev space with the norm

$$\|u\| = \left\{ \int_\Omega |\nabla u|^2 \, \mathrm{d}x \right\}^{1/2}.$$

For convenience, we denote $H_0^1(\Omega)$ by V in the following. Let A be a mapping from V to its dual space V^*, which is defined by

$$\langle Au, v\rangle = \sum_{i=1}^n \int_\Omega A_i(x, u, \nabla u) D_i v \, \mathrm{d}x + \int_\Omega A_0(x, u, \nabla u) v \, \mathrm{d}x \quad \text{for all } u, v \in V, \tag{1.1}$$

where $\langle \cdot\,, \cdot\rangle$ denotes the duality pairing between V^* and V, $\nabla = (D_1, \dots, D_n)$, $D_i = \partial/\partial x_i$ and the functions A_i, $i = 0, 1, \dots, n$, satisfy suitable regularity and growth assumptions (see below).

Let K be a non-empty, closed and convex subset of the Hilbert space V. In order to simplify some computations we shall assume that $0 \in K$. The norm convergence in V and V^* is denoted by '$\to$' and the weak convergence is denoted by '$\rightharpoonup$'. We formulate the following variational hemivariational inequality (HVI): find $u \in K$ such that

$$\langle Au, v-u\rangle + \int_{\Omega} j^0(x, u; v-u)\,\mathrm{d}x \geqslant \int_{\Omega} f(x, u)(v-u)\,\mathrm{d}x \quad \text{for all } v \in K. \tag{HVI}$$

Here $j^0(x, u(x); v(x))$ denotes the generalized derivative of $j(x, \cdot)$ at $u(x)$ in the direction $v(x)$ (see [**1**]).

Problem (HVI) has been considered by the author under the coerciveness conditions (see [**7**, **8**]). In the case when A is a linear elliptic operator of second order, Goeleven *et al*. [**3**, **4**] and Mansour and Riahi [**9**] studied the semilinear eigenvalue problems. They provided the existence and stability of solutions for the semilinear problem (HVI) by using the critical-points method, the Leray–Schauder degree and by a version of Ky Fan's minimax inequality, respectively. Beyond the theoretical interest, solution of the problem (HVI) at resonance (as well as at non-resonance) is strongly motivated by applications in mechanics. This kind of inequality expression in (HVI) was introduced by Panagiotopoulos [**11**] in order to deal with problems in mechanics whose variational forms are such inequalities that express the principle of virtual work or power. Such formulations include a wide-ranging class of complicated problems in mechanics and engineering which could not previously be treated correctly by the methods of partial differential equations or variational inequalities, e.g. the case for the study of adhesive joints in structural mechanics, the behaviour of composites, the unilateral contact and non-monotone friction problems in cracks, the adhesive grasping problem in robotics, etc. [**4**, **10**, **11**]. Among the important mechanical phenomena leading to problem (HVI) at resonance, a typical example is offered by a model of loading and unloading presented in [**4**].

The aim of the present paper is the mathematical study of nonlinear eigenvalue problems arising in the stability analysis of mechanical systems subjected to realistic non-monotone boundary conditions [**10**]. Our main results are formulated in Theorems 3.1, 3.2 and 3.4, pointing out a new insight in the general setting of the nonlinear problem (HVI) at resonance as well as non-resonance, for the celebrated Landesman and Lazer conditions [**5**, **6**] as well as for other classical sufficient conditions of solvability such as those due to Chang [**1**], Goeleven *et al*. [**3**, **4**] and Mansour and Riahi [**9**]. We build a Landesman–Lazer theory in the non-smooth framework of nonlinear problem (HVI).

2. Preliminaries and assumptions

Recall that an operator $T : V \to 2^{V^*}$ is pseudo-monotone if and only if the following three conditions are fulfilled:

(i) for each $u \in V$, the set Tu is non-empty, bounded, closed and convex;

(ii) if $\{(u_n^*, u_n,)\}_{n=1}^{\infty} \subset G(T)$ (the graph of the operator T) is such that $u_n \rightharpoonup u$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \leqslant 0$, then for each $v \in V$ there exists $v^*(v) \in T(u)$ with the property that

$$\liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle \geqslant \langle v^*(v), u - v\rangle;$$

(iii) the restriction of T to any finite-dimensional subspace F of V is weakly upper semi-continuous as an operator from F to V^*.

For a locally Lipschitzian functional $h : V \to \mathbb{R}$, we denote by $h^0(u,v)$ the Clarke generalized directional derivative of h at u in the direction v, i.e.

$$h^0(u,v) := \limsup_{\lambda \to 0+,\, w \to u} \frac{h(w+\lambda v) - h(w)}{\lambda}.$$

Recall also at this point that

$$\partial h(u) := \{u^* \in V^* \mid h^0(u,v) \geqslant \langle u^*, v\rangle \text{ for all } v \in V\} \tag{2.1}$$

denotes the generalized Clarke subdifferential.

From [**1**], we have

$$h^0(u,v) = \max\{\langle w, v\rangle \mid w \in \partial h(u)\} \tag{2.2}$$

In the following we assume that the coefficients A_i, $i = 0, 1, \dots, n$, are functions of $x \in \Omega$ and of $\xi = (\eta, \zeta) \in \mathbb{R}^{n+1}$, where $\eta \in \mathbb{R}$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$. We assume that each $A_i(x,\xi)$ is a Carathéodory function, i.e. it is measurable in x for fixed $\xi \in \mathbb{R}^{n+1}$ and continuous in ξ for almost all $x \in \Omega$. We suppose that the $A_i(x,\xi)$, $i = 0, 1, \dots, n$, satisfy the following conditions.

(A_1) There exist $c_1 > 0$ and $a \in L^2(\Omega)$ such that

$$|A_i(x,\xi)| \leqslant c_1|\xi| + a(x)$$

for a.e. $x \in \Omega$, for all $\xi = (\eta, \zeta) \in \mathbb{R}^{n+1}$.

(A_2) $\sum_{i=1}^{n} [A_i(x,\eta,\zeta) - A_i(x,\eta,\zeta')](\zeta_i - \zeta_i') > 0$ for a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}$ and $\zeta, \zeta' \in \mathbb{R}^n$ with $\zeta \neq \zeta'$.

(A_3) There exists a positive c_2 and a non-negative function $Z \in L^1(\Omega)$ such that

$$\sum_{i=1}^{n} A_i(x,\eta,\zeta)\zeta_i \geqslant c_2 \sum_{i=1}^{n} |\zeta_i|^2 - Z(x)$$

for a.e. $x \in \Omega$, for all $\eta \in \mathbb{R}$ and $\zeta, \zeta' \in \mathbb{R}^n$.

We define the first eigenvalue as

$$\lambda_1 = \liminf_{\|u\|_{L^2} \to \infty} \frac{\langle Au, u\rangle}{\|u\|_{L^2}^2}, \quad u \in V. \tag{2.3}$$

Concerning problem (HVI) we deal with the functional $J : V(\subseteq L^2(\Omega)) \to \mathbb{R}$ of the following type:

$$J(u) = \int_{\Omega} j(x, u(x))\, \mathrm{d}x, \quad u \in V. \tag{2.4}$$

We assume that $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following condition.

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(H_1) $j(\cdot\,,s):\Omega\to\mathbb{R}$ is measurable, for all $s\in\mathbb{R}$. $j(x,\cdot):\mathbb{R}\to\mathbb{R}$ is locally Lipschitz for all $x\in\Omega$, $j(\cdot\,,0)\in L^1(\Omega)$ and $|z|\leqslant d(x)+c|s|^{\sigma-1}$ for all $s\in\mathbb{R}$, a.e. $x\in\Omega$, for all $z\in\partial_s j(x,s)$, with constants $c>0$ and $1\leqslant\sigma<2$ and $d\in L^2(\Omega)$.

The above assumptions on j ensure that J is locally Lipschitz on V and that

$$\int_\Omega j^0(x,u(x);v(x))\,\mathrm{d}x\geqslant J^0(u,v)\quad\text{for all } u,v\in V. \tag{2.5}$$

We also make the following assumptions.

(H_2) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (i.e. $f(\cdot\,,s):\Omega\to\mathbb{R}$ is measurable, for all $s\in\mathbb{R}$, and $f(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous, for almost all $x\in\Omega$). There exist $L>0$ and $b(x)\in L^2(\Omega)$ such that $|f(x,t)|\leqslant b(x)+L|t|^{q-1}$ for a.e. $x\in\Omega$, for all $t\in\mathbb{R}$, where $1\leqslant q<2n/(n-2)$ for $n>2$ and $q>2$ for $n=1,2$.

(H_3) There exist $\varepsilon_0>0$ and $c(x)\in L^2$ such that $tf(x,t)\leqslant(\lambda_1-\varepsilon_0)|t|^2+c(x)|t|$ for a.e. $x\in\Omega$, for all $t\in\mathbb{R}$, where $c(x)\geqslant 0$ a.e. in Ω and λ_1 is defined by (2.3).

Let F be the mapping from V to its dual space V^*, which is defined by

$$\langle Fu,v\rangle:=\int_\Omega f(x,u)v\,\mathrm{d}x\quad\text{for all } u,v\in V.$$

The following lemma will be useful (see, for example, [**2**]).

Lemma 2.1. *Let $T:V\to 2^{V^*}$ be a pseudo-monotone operator and let $C\subseteq V$ be non-empty, bounded, closed and convex. Then, for a given $f\in V^*$, there exist $u\in C$ and $u^*\in T(u)$ such that*

$$\langle u^*-f,v-u\rangle\geqslant 0\quad\textit{for all } v\in C.$$

In order to establish the existence results of problem (HVI), we also need the following lemma (see, for example, [**7**, **8**]).

Lemma 2.2. *Suppose that the assumptions (A_1)–(A_3) and (H_1), (H_2) hold. Then the sum operator $A-F+\partial J:V\to 2^{V^*}$ is pseudo-monotone.*

Lemma 2.3 (Mansour and Riahi [9]). *Assume that (H_1), (H_2) hold. Then, for all v in V, the functional*

$$u\to g(u,v):=\int_\Omega j^0(x,u;v-u)\,\mathrm{d}x$$

is weakly upper semi-continuous, while

$$u\to\langle Fu,v-u\rangle:=\int_\Omega f(x,u)(v-u)\,\mathrm{d}x$$

is weakly continuous in V.

Lemma 2.4 (Shapiro [12]). *Let Ω be a bounded open connected set. Then, under the assumptions (A_1)–(A_3), λ_1 defined by (2.3) is finite valued, i.e. $-\infty<\lambda_1<\infty$.*

3. Main results

The first theorem we intend to prove is the following.

Theorem 3.1. *Let K be a non-empty, bounded, closed, convex subset of V. Suppose in addition that assumptions (A_1)–(A_3) and (H_1), (H_2) hold. Then problem (HVI) has at least one solution.*

Proof. By means of Lemmas 2.1 and 2.2, there exist $u \in K$ and $u^* \in \partial J(u)$ such that

$$\langle Au - Fu + u^*, v - u\rangle \geqslant 0 \quad \text{for all } v \in K.$$

By use of (2.2), we have that

$$\langle Au - Fu, v - u\rangle + J^0(u, v - u) \geqslant 0 \quad \text{for all } v \in K.$$

By virtue of (2.5), we obtain

$$\langle Au - Fu, v - u\rangle + \int_\Omega j^0(x, u(x); v(x) - u(x))\,\mathrm{d}x \geqslant 0 \quad \text{for all } v \in K.$$

Therefore, from the definition of operator F, we get

$$\langle Au, v-u\rangle + \int_\Omega j^0(x, u(x); v(x)-u(x))\,\mathrm{d}x \geqslant \int_\Omega f(x, u(x))(v(x)-u(x))\,\mathrm{d}x \quad \text{for all } v \in K.$$

This ends the proof of the theorem. □

Theorem 3.2. *Let K be a non-empty, closed, convex subset of V. Suppose in addition that assumptions (A_1)–(A_3) and (H_1)–(H_3) hold. Then problem (HVI) has at least one solution.*

Proof. Set $K_n := \{v \in K \mid \|v\| \leqslant n\}$. Using Theorem 3.1, we get the existence of $u_n \in K_n$ such that

$$\langle Au_n - Fu_n, v - u_n\rangle + \int_\Omega j^0(x, u_n; v - u_n)\,\mathrm{d}x \geqslant 0 \quad \text{for all } v \in K_n. \tag{3.1}$$

Step 1. There exists $M > 0$ such that

$$\|u_n\|_{L^2} \leqslant M \quad \text{for } n = 1, 2, \dots. \tag{3.2}$$

Suppose to the contrary that (3.2) does not hold. Then, without loss of generality, we may assume that

$$\lim_{n\to\infty} \|u_n\|_{L^2} = \infty. \tag{3.3}$$

By taking $v = 0$ in (3.1), we have

$$\langle Au_n, u_n\rangle \leqslant \int_\Omega j^0(x, u_n; -u_n)\,\mathrm{d}x + \int_\Omega f(x, u_n)u_n\,\mathrm{d}x. \tag{3.4}$$

But it then follows from (2.3) and (3.3) that

$$\lambda_1 \leqslant \liminf_{n\to\infty}\left\{\frac{\int_\Omega j^0(x,u_n;-u_n)\,\mathrm{d}x}{\|u_n\|_{L^2}^2}+\frac{\int_\Omega f(x,u_n(x))u_n(x)\,\mathrm{d}x}{\|u_n\|_{L^2}^2}\right\}. \tag{3.5}$$

By virtue of (2.2) and (H_1), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\frac{\int_\Omega j^0(x,u_n;-u_n)\,\mathrm{d}x}{\|u_n\|_{L^2}^2} &\leqslant \limsup_{n\to\infty}\frac{\int_\Omega \max\{|z_n(x)u_n|,\ z_n(x)\in\partial j(x,u_n)\}\,\mathrm{d}x}{\|u_n\|_{L^2}^2}\\ &\leqslant \limsup_{n\to\infty}\frac{\int_\Omega (d(x)|u_n|+c|u_n|^\sigma)\,\mathrm{d}x}{\|u_n\|_{L^2}^2}\\ &=0.\end{aligned}$$

By use of (H_3), we also have

$$\begin{aligned}\limsup_{n\to\infty}\frac{\int_\Omega f(x,u_n(x))u_n(x)\,\mathrm{d}x}{\|u_n\|_{L^2}^2} &\leqslant \limsup_{n\to\infty}\frac{\int_\Omega [(\lambda_1-\varepsilon_0)|u_n(x)|^2+c(x)|u_n(x)|]\,\mathrm{d}x}{\|u_n\|_{L^2}^2}\\ &=\lambda_1-\varepsilon_0.\end{aligned}$$

From the last two inequalities and (3.5), we obtain a contradiction, which has proved that the inequality (3.2) holds.

Step 2. There exists $M_1>0$ such that

$$\|u_n\|\leqslant M_1 \quad \text{for } n=1,2,\dots. \tag{3.6}$$

Using (3.4) once again, we obtain from (A_3) that

$$\begin{aligned}c_2\int_\Omega |\nabla u_n|^2\,\mathrm{d}x \leqslant \int_\Omega Z(x)\,\mathrm{d}x-\int_\Omega A_0(x,u_n,\nabla u_n)u_n\,\mathrm{d}x\\ +\int_\Omega j^0(x,u_n;-u_n)\,\mathrm{d}x+\int_\Omega f(x,u_n(x))u_n(x)\,\mathrm{d}x.\end{aligned}$$

Using (A_1), (A_3), (H_1) and (H_2), we obtain

$$\begin{aligned}c_2\|u_n\|^2 \leqslant \int_\Omega Z(x)\,\mathrm{d}x+\int_\Omega c_1\{|\nabla u_n|^2+|u_n|^2\}^{1/2}|u_n|\,\mathrm{d}x+\int_\Omega a(x)|u_n|\,\mathrm{d}x\\ +\int_\Omega (d(x)|u_n|+c|u_n|^\sigma)\,\mathrm{d}x+\int_\Omega [(\lambda_1-\varepsilon_0)|u_n(x)|^2+c(x)|u_n(x)|]\,\mathrm{d}x.\end{aligned}$$

Therefore, by Hölder's inequality, there exists a positive constant C such that

$$\|u_n\|^2\leqslant C\left[\int_\Omega Z(x)\,\mathrm{d}x+\|u_n\|\,\|u_n\|_{L^2}+\|u_n\|+\|u_n\|_{L^2}^2\right] \quad \text{for } n=1,2,\dots, \tag{3.7}$$

which implies that the inequalities (3.6) hold due to (3.2).

By virtue of (3.6), there exists a positive integer n_0 such that $\|u_{n_0}\|<n_0$.

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Step 3. u_{n_0} solves problem (HVI).

Since $\|u_{n_0}\| < n_0$, we have, for each $y \in K$, the existence of an $\varepsilon > 0$ such that $u_{n_0} + \varepsilon(y - u_{n_0}) \in K_{n_0}$. It suffices to take

$$\varepsilon \begin{cases} < (n_0 - \|u_{n_0}\|)/\|y - u_{n_0}\| & \text{if } y \neq u_{n_0}, \\ = 1 & \text{if } y = u_{n_0}. \end{cases}$$

We have

$$\langle Au_{n_0} - Fu_{n_0}, v - u_{n_0} \rangle + \int_{\Omega} j^0(x, u_{n_0}; v - u_{n_0})\,\mathrm{d}x \geqslant 0 \quad \text{for all } v \in K_{n_0}. \tag{3.8}$$

If we set $v = u_{n_0} + \varepsilon(y - u_{n_0})$ in (3.8), we obtain

$$\langle Au_{n_0} - Fu_{n_0}, \varepsilon(y - u_{n_0}) \rangle + \int_{\Omega} j^0(x, u_{n_0}; \varepsilon(y - u_{n_0}))\,\mathrm{d}x \geqslant 0. \tag{3.9}$$

Since $j^0(x, u, v)$ is positively homogeneous in v (see [**1**]), we have that

$$\varepsilon\langle Au_{n_0} - Fu_{n_0}, y - u_{n_0} \rangle + \varepsilon \int_{\Omega} j^0(x, u_{n_0}; y - u_{n_0})\,\mathrm{d}x \geqslant 0. \tag{3.10}$$

Dividing (3.10) by $\varepsilon > 0$, we finally obtain

$$\langle Au_{n_0} - Fu_{n_0}, y - u_{n_0} \rangle + \int_{\Omega} j^0(u_{n_0}, y - u_{n_0})\,\mathrm{d}x \geqslant 0 \quad \text{for all } y \in K. \tag{3.11}$$

This completes the proof. □

Now we turn to the solvability of problem (HVI) involving resonance. It is an easy matter in this case to give examples that show that Theorem 3.2 is false if $\varepsilon_0 = 0$ in (H_3) since this is already well known if A given in (1.1) is linear. Consequently, a further condition is necessary to ensure that the conclusion of Theorem 3.2 holds for the situation $\varepsilon_0 = 0$ in (H_3). Results of this nature are referred to in the literature as resonance results (see [**3**–**6**]). We shall present one such result here that will hold for the Hilbert space $V(= H_0^1(\Omega))$. In order to do this, we first recall some facts concerning linear elliptic theory.

Let $a : V \times V \to \mathbb{R}$ be a continuous, symmetric, bilinear form which is coercive:

$$a(u, u) \geqslant \alpha\|u\|^2 \quad \text{for all } u \in V,$$

with a constant $\alpha > 0$. Thus,

$$\|\cdot\|_V := a(\cdot\,,\cdot)^{1/2}$$

is an equivalent norm on V, i.e. there exist two positive constants c_3 and c_4 such that

$$c_3\|u\|^2 \leqslant a(u, u) \leqslant c_4\|u\|^2. \tag{3.12}$$

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Denote by

$$\sigma_1 < \sigma_2 \leqslant \cdots \leqslant \sigma_n \cdots \to +\infty \tag{3.13}$$

the sequence of eigenvalues of the linear problem

$$a(u, v) = \sigma\langle u, v\rangle_{L^2} \quad \text{for all } v \in V. \tag{3.14}$$

We also consider a basis $\{\varphi_n\}_{n=1}^{\infty}$ for V consisting of eigenfunctions, where φ_n corresponds to σ_n, i.e. $u = \varphi_n$ and $\sigma = \sigma_n$ in (3.14), which is normalized in the following sense:

$$a(\varphi_i, \varphi_j) = \delta_{ij}, \tag{3.15}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

In this statement we use essentially the compactness of the embedding $V \subset L^2(\Omega)$. The fact that σ_1 is simple and the corresponding eigenfunction does not change sign in Ω follows from the Krein–Rutman theorem (see, for example, [**13**]).

Remark 3.3. For example, we may assume that the bilinear form $a(\cdot\,,\cdot)$ is defined by the following linear elliptic operator of second order:

$$Lu := -\sum_{ij=1}^{N} D_j(a_{ij}(x)D_i u) + q(x)u$$

if L satisfies some suitable assumptions.

In Theorem 3.4, we shall replace (H_3) by the following.

(H_4) For all $\varepsilon > 0$, there exists $h_\varepsilon \in L^2$ such that $tf(x, t) \leqslant (\lambda_1 + \varepsilon)|t|^2 + h_\varepsilon(x)|t|$ for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, where $h_\varepsilon(x) \geqslant 0$ a.e. in Ω.

We observe that (H_4) is a generalization of the case when $\varepsilon_0 = 0$ in (H_3). We shall also set the following assumption in Theorem 3.4.

(H_5) $\lambda_1 = \sigma_1$, where λ_1 is given by (2.3), and

$$\liminf_{\|u\|\to\infty} \frac{\langle Au, u\rangle - a(u, u)}{\|u\|^2} \geqslant 0, \quad u \in V.$$

Also in Theorem 3.4, we shall set

$$f_\pm = \limsup_{t\to\pm\infty} \frac{f(x, t)}{t}. \tag{3.16}$$

We intend to establish the following result.

Theorem 3.4. *Let K be a non-empty, closed, convex subset of V. Suppose in addition that assumptions (A_1)–(A_3), (H_1), (H_2) and (H_4), (H_5) hold. Suppose furthermore that $f(x, t)$ satisfies*

$$\sigma_1 \int_\Omega \varphi_1^2\,\mathrm{d}x > \max\left\{\int_\Omega f_+\varphi_1^2\,\mathrm{d}x, \int_\Omega f_-\varphi_1^2\,\mathrm{d}x\right\}, \tag{3.17}$$

where φ_1 is the first eigenfunction of the linear problem (3.14). Then problem (HVI) has at least one solution.

Proof. We first recall (see [**13**]) that

$$\sigma_1 = \inf \frac{a(u,u)}{\|u\|_{L^2}^2}, \quad u \in V. \tag{3.18}$$

Next, for n a positive integer we set

$$f_n(x,t) = f(x,t) - \frac{t}{n} \tag{3.19}$$

and choose $\varepsilon = (2n)^{-1}$ in (H_4). It then follows that

$$tf_n(x,t) \leqslant [\lambda_1 - (2n)^{-1}]t^2 + h_{(2n)^{-1}}(x)|t| \tag{3.20}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, where $h_{(2n)^{-1}} \in L^2$.

With (3.20) at our disposal, we see from Theorem 3.2 that there exists a $u_n \in K$ such that

$$\langle Au_n, v - u_n\rangle + \int_\Omega j^0(x, u_n, (v-u_n))\,\mathrm{d}x \geqslant \int_\Omega f(x,u_n)(v-u_n)\,\mathrm{d}x - \int_\Omega \frac{u_n(v-u_n)}{n}\,\mathrm{d}x \tag{3.21}$$

for all $v \in K$.

We claim that there exists an $M_2 > 0$ such that

$$\|u_n\| \leqslant M_2 \quad \text{for all } n. \tag{3.22}$$

Suppose that (3.22) is false. Then we can assume, without loss of generality, that

$$\lim_{n\to\infty} \|u_n\| = \infty. \tag{3.23}$$

We shall show that (3.23) leads to a contradiction.

Let $\varepsilon > 0$ be given. It then follows from (H_5) and (3.23) that there exists an n_0 such that

$$\langle Au_n, u_n\rangle - a(u_n,u_n) \geqslant -\varepsilon\|u_n\|^2 \quad \text{for all } n \geqslant n_0.$$

Taking $v = 0$ in (3.21) and using the above inequality, we see that

$$a(u_n,u_n) + \frac{\|u_n\|_{L^2}^2}{n} \leqslant \int_\Omega j^0(x,u_n,-u_n)\,\mathrm{d}x + \int_\Omega f(x,u_n)u_n\,\mathrm{d}x + \varepsilon\|u_n\|^2 \quad \text{for all } n \geqslant n_0. \tag{3.24}$$

It follows in turn from (H_4), (H_5) and inequality (3.24) that

$$\begin{aligned} a(u_n,u_n) - \sigma_1\|u_n\|_{L^2}^2 \leqslant{}& \int_\Omega j^0(x,u_n,-u_n)\,\mathrm{d}x \\ &+ \int_\Omega h_\varepsilon(x)|u_n|\,\mathrm{d}x + \varepsilon(\|u_n\|^2 + \|u_n\|_{L^2}^2) \quad \text{for all } n \geqslant n_0. \end{aligned} \tag{3.25}$$

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Since $\{\varphi_n\}_{n=1}^{\infty}$ is a normalized basis for V, we may set $u_n = \sum_{k=1}^{\infty} b_n(k)\varphi_k$. Then $b_n(k) = a(u_n, \varphi_k)$ and

$$(u_n, \varphi_k) = \int_{\Omega} u_n \varphi_k \, \mathrm{d}x = \sigma_k^{-1} b_n(k), \qquad (u_n, u_n) = \int_{\Omega} u_n^2 \, \mathrm{d}x = \sum_{k=1}^{\infty} \sigma_k^{-1} b_n^2(k).$$

Consequently, it follows that

$$a(u_n, u_n) - \sigma_1 \|u_n\|_{L^2}^2 = \sum_{k=1}^{\infty} \left(1 - \frac{\sigma_1}{\sigma_k}\right) |b_n(k)|^2. \tag{3.26}$$

Since σ_1 is simple, it then follows from (3.25), (3.26) that

$$\sum_{k=2}^{\infty} \left(1 - \frac{\sigma_1}{\sigma_k}\right) |b_n(k)|^2 \leqslant \int_{\Omega} j^0(x, u_n, -u_n) \, \mathrm{d}x + \int_{\Omega} h_{\varepsilon}(x)|u_n| \, \mathrm{d}x + \varepsilon(\|u_n\|^2 + \|u_n\|_{L^2}^2) \quad \text{for all } n \geqslant n_0. \tag{3.27}$$

Next, we set

$$y_n = b_n(1)\varphi_1, \qquad w_n = \sum_{k=2}^{\infty} b_n(k)\varphi_k. \tag{3.28}$$

Therefore,

$$u_n = y_n + w_n, \qquad (y_n, w_n) = a(y_n, w_n) = 0. \tag{3.29}$$

We also set

$$U_n = \frac{u_n}{\|u_n\|}, \qquad Y_n = \frac{y_n}{\|u_n\|}, \qquad W_n = \frac{w_n}{\|u_n\|}. \tag{3.30}$$

Next, we observe from (3.13) that

$$\text{there exists } \gamma > 0 \text{ such that } \gamma \leqslant \left(1 - \frac{\sigma_1}{\sigma_k}\right) \text{ for all } k \geqslant 2. \tag{3.31}$$

Since $\|w_n\|_V^2 = a(w_n, w_n) = \sum_{k=2}^{\infty} |b_n(k)|^2$, it follows from (3.12), (3.27) and (3.31) that

$$\gamma c_3 \|w_n\|^2 \leqslant \int_{\Omega} j^0(x, u_n; -u_n) \, \mathrm{d}x + \int_{\Omega} h_{\varepsilon}(x)|u_n| \, \mathrm{d}x + \varepsilon(\|u_n\|^2 + \|u_n\|_{L^2}^2) \quad \text{for all } n \geqslant n_0.$$

By virtue of (2.2) and (H_1), we obtain

$$\begin{aligned} \int_{\Omega} j^0(x, u_n; -u_n) \, \mathrm{d}x &\leqslant \int_{\Omega} \max\{|z_n(x)u_n|,\ z_n(x) \in \partial j(x, u_n)\} \, \mathrm{d}x \\ &\leqslant \int_{\Omega} (d(x)|u_n| + c|u_n|^{\sigma}) \, \mathrm{d}x. \end{aligned}$$

Since $h_{\varepsilon} \in L^2$, it follows from Hölder's inequality, and the two inequalities above that there exists an $M_3 > 0$ such that

$$\gamma c_3 \|w_n\|^2 \leqslant M_3(\|u_n\|_{L^2} + \|u_n\|_{L^2}^{\sigma}) + \varepsilon(\|u_n\|^2 + \|u_n\|_{L^2}^2) \quad \text{for all } n \geqslant n_0.$$

By Poincaré's inequality, there exists a positive constant c_5 such that $\|u\|_{L^2} \leqslant c_5\|u\|$ for all $u \in V$. It follows that

$$\gamma c_3 \|w_n\|^2 \leqslant M_3(c_5\|u_n\| + c_5^\sigma \|u_n\|^\sigma) + \varepsilon(1 + c_5^2)\|u_n\|^2.$$

Dividing both sides of this inequality by $\|u_n\|^2$ and using (3.23) and (3.30), we see that

$$\gamma c_3 \limsup_{n\to\infty} \|W_n\|^2 \leqslant \varepsilon(1 + c_5^2).$$

Since γ, c_3 and c_5 are positive constants and $\varepsilon > 0$ is arbitrary, we conclude from the above inequality that

$$\lim_{n\to\infty} \|W_n\|^2 = 0. \tag{3.32}$$

Since $W_n = U_n - Y_n$ and $\|U_n\| = 1$, it follows from (3.32) that $\limsup_{n\to\infty}\|Y_n\| \leqslant 1$. But then we obtain from (3.12) that $\{a(Y_n, Y_n)\}_n^\infty$ is a uniformly bounded sequence. From (3.28), we also see that $Y_n = (b_n(1)/\|u_n\|)\varphi_1$. Then $\{|b_n(1)/\|u_n\||\}_{n=1}^\infty$ is a uniformly bounded sequence. We consequently conclude (where for ease of notation we use the full sequence rather than a subsequence) that

$$\text{there exists } b \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \|Y_n - Y\| = 0, \text{ where } Y = b\varphi_1. \tag{3.33}$$

We next see that

$$\|U_n - Y\| \leqslant \|Y_n - Y\| + \|W_n\|.$$

Hence, it follows from (3.32) and (3.33) that

$$\lim_{n\to\infty} \|U_n - Y\| = 0. \tag{3.34}$$

From this fact (and once again using the entire sequence rather than a subsequence), we see that

$$\lim_{n\to\infty} U_n(x) = Y(x) \quad \text{a.e. in } \Omega. \tag{3.35}$$

Next, we divide both sides of the inequality in (3.24) by $\|u_n\|^2$, and use (3.29) and (3.30) to obtain

$$\begin{aligned} a(Y_n, Y_n) + a(W_n, W_n) + \frac{\|U_n\|_{L^2}^2}{n} \leqslant \int_\Omega \frac{j^0(x, u_n, -u_n)}{\|u_n\|^2}\,\mathrm{d}x \\ + \int_\Omega \frac{f(x, u_n)U_n}{\|u_n\|}\,\mathrm{d}x + \varepsilon \quad \text{for all } n \geqslant n_0. \end{aligned} \tag{3.36}$$

Now from (3.12) and (3.32), we see that

$$\lim_{n\to\infty} a(W_n, W_n) = 0. \tag{3.37}$$

Since $a(\cdot\,,\cdot)$ is an inner product on V and $a(Y_n - Y, Y_n - Y) \to 0$ as $n \to \infty$, by (3.12) and (3.33) we obtain that

$$\lim_{n\to\infty} a(Y_n, Y_n) = a(Y, Y). \tag{3.38}$$

Also, from Poincaré's inequality, we obtain

$$\left.\begin{aligned}\lim_{n\to\infty}\frac{\|U_n\|_{L^2}^2}{n}&\leqslant c_5^2\lim_{n\to\infty}\frac{\|U_n\|^2}{n}=c_5\lim_{n\to\infty}\frac{1}{n}=0,\\ \limsup_{n\to\infty}\frac{\int_\Omega j^0(u_n,-u_n)\,\mathrm{d}x}{\|u_n\|^2}&\leqslant\limsup_{n\to\infty}\frac{\int_\Omega\max\{|z_n(x)u_n|,\ z_n(x)\in\partial j(x,u_n)\,\mathrm{d}x\}}{\|u_n\|^2}\\ &\leqslant\limsup_{n\to\infty}\frac{\int_\Omega\{(d(x)|u_n|+c|u_n|^\sigma)\,\mathrm{d}x\}}{\|u_n\|^2}\\ &=0.\end{aligned}\right\}\tag{3.39}$$

We conclude from this last inequality and (3.36)–(3.39) that

$$a(Y,Y)\leqslant\limsup_{n\to\infty}\int_\Omega f(x,u_n)U_n\|u_n\|^{-1}\,\mathrm{d}x+\varepsilon.$$

But $\varepsilon>0$ is arbitrary. Hence, we obtain from the latter inequality that

$$a(Y,Y)\leqslant\limsup_{n\to\infty}\int_\Omega f(x,u_n)U_n\|u_n\|^{-1}\,\mathrm{d}x.\tag{3.40}$$

Since $Y=b\varphi_1$, we have

$$a(Y,Y)=b^2=\sigma_1\int_\Omega Y^2\,\mathrm{d}x.\tag{3.41}$$

Since $u_n=U_n\|u_n\|$ and $U_n(x)\to Y(x)=b\varphi_1(x)$ a.e. in Ω by (3.35), we have $u_n(x)\to+\infty$, if $b>0$. Therefore, we obtain

$$\limsup_{n\to\infty}\int_\Omega f(x,u_n)U_n\|u_n\|^{-1}\,\mathrm{d}x\leqslant\int_\Omega f_+Y^2\,\mathrm{d}x.\tag{3.42}$$

Similarly, $u_n(x)\to-\infty$, if $b<0$, and we have

$$\limsup_{n\to\infty}\int_\Omega f(x,u_n)U_n\|u_n\|^{-1}\,\mathrm{d}x\leqslant\int_\Omega f_-Y^2\,\mathrm{d}x.\tag{3.43}$$

From (3.40)–(3.43) we consequently have that

$$\sigma_1\int_\Omega\varphi_1^2\,\mathrm{d}x\leqslant\max\biggl\{\int_\Omega f_+\varphi_1^2\,\mathrm{d}x,\ \int_\Omega f_-\varphi_1^2\,\mathrm{d}x\biggr\},$$

which is a direct contradiction to the inequality in (3.17). We conclude that (3.23) is false and therefore that our claim (3.22) is indeed true.

Since V is a Hilbert space and is embedded compactly into $L^2(\Omega)$, by (3.22) there exists $u\in K\subseteq V$ such that (where we have once again used the full sequence)

$$u_n\rightharpoonup u\quad\text{in }V,\qquad u_n\to u\quad\text{in }L^2(\Omega).\tag{3.44}$$

Taking $v=u$ in (3.21), we have

$$\langle Au_n,u-u_n\rangle+\int_\Omega j^0(x,u_n,(u-u_n))\,\mathrm{d}x\geqslant\int_\Omega f(x,u_n)(u-u_n)\,\mathrm{d}x-\int_\Omega\frac{u_n(u-u_n)}{n}\,\mathrm{d}x.$$

By virtue of (3.44) and Lemma 2.3, we obtain

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leqslant 0.$$

Since the operator A is pseudo-monotone, we have

$$\liminf_{n\to\infty}\langle Au_n, u_n - v\rangle \geqslant \langle Au, u - v\rangle \quad \text{for all } v \in V. \tag{3.45}$$

Letting $n \to \infty$ in (3.21), using Lemma 2.3 and (3.45), we finally have

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, u, (v - u))\,\mathrm{d}x \geqslant \int_\Omega f(x, u)(v - u)\,\mathrm{d}x \quad \text{for all } v \in K,$$

which has proved that u is a solution of problem (HVI). □

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