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Abstract

We show that if A and H are Hopf algebras that have equivalent tensor categories of comodules, then one can transport what we call a free Yetter–Drinfeld resolution of the counit of A to the same kind of resolution for the counit of H, exhibiting in this way strong links between the Hochschild homologies of A and H. This enables us to obtain a finite free resolution of the counit of $\mathcal{B}(E)$, the Hopf algebra of the bilinear form associated with an invertible matrix E, generalizing an earlier construction of Collins, Härtel and Thom in the orthogonal case $E = I_n$. It follows that $\mathcal{B}(E)$ is smooth of dimension 3 and satisfies Poincaré duality. Combining this with results of Vergnioux, it also follows that when E is an antisymmetric matrix, the L^2-Betti numbers of the associated discrete quantum group all vanish. We also use our resolution to compute the bialgebra cohomology of $\mathcal{B}(E)$ in the cosemisimple case.

1. Introduction

Let $n \in \mathbb{N}^*$ and let $A_o(n)$ be the algebra (over the field of complex numbers) presented by generators $(u_{ij})_{1\leqslant i,j\leqslant n}$ and relations making the matrix $u = (u_{ij})$ orthogonal. This is a Hopf algebra, introduced by Dubois-Violette and Launer [DL90] and independently by Wang [Wan95] in the compact quantum group setting. The Hopf algebras $A_o(n)$ play an important role in quantum group theory, since any finitely generated Hopf algebra of Kac type (the square of the antipode is the identity), and in particular any semisimple Hopf algebra, is a quotient of one of these. They have been studied from several perspectives, in particular from the (co)representation theory viewpoint [Ban96, Bic03a, BDV06] and the probabilistic and operator algebraic viewpoint [BC07, BCZ09, VV07, VV08, Voi11].

The homological study of $A_o(n)$ begins in [CHT09], where Collins *et al.* define an exact sequence of $A_o(n)$-modules

$$0 \to A_o(n) \longrightarrow M_n(A_o(n)) \longrightarrow M_n(A_o(n)) \longrightarrow A_o(n) \xrightarrow{\varepsilon} \mathbb{C} \to 0 \quad (\star)$$

thus yielding a resolution of the counit of $A_o(n)$ by free $A_o(n)$-modules. From this exact sequence, they deduce some important homological information on $A_o(n)$:

(i) $A_o(n)$ is smooth of dimension 3;

(ii) $A_o(n)$ satisfies Poincaré duality;

(iii) the L^2-Betti numbers of $A_o(n)$ all vanish.

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An inconvenient feature of [CHT09] is that the verification of the exactness of $\star$ is a very long computation involving tedious Gröbner basis computations. It is the aim of the present paper to propose a simpler and more conceptual proof of the exactness of the sequence $\star$, together with a generalization to a larger class of Hopf algebras.

Our starting point is the combination of the following two known facts.

(i) For $q \in \mathbb{C}^*$, there exists a resolution of the counit of $\mathcal{O}(\mathrm{SL}_q(2))$ having the same length as that of the sequence $\star$ (see e.g. [HK05]).

(ii) For q satisfying $q + q^{-1} = -n$, there exists an equivalence of tensor categories of comodules $\mathcal{M}^{\mathcal{O}(\mathrm{SL}_q(2))} \simeq^{\otimes} \mathcal{M}^{A_o(n)}$ (see [Bic03a]).

Therefore, although one cannot expect that a tensor equivalence between categories of comodules induces isomorphisms between Hochschild homologies, it is tempting to believe that it is possible to use the above monoidal equivalence to transport a resolution of the counit of $\mathcal{O}(\mathrm{SL}_q(2))$ by free modules having appropriate additional structures (in particular, a comodule structure) to get a resolution of the counit of $A_o(n)$ having the same length.

The appropriate structure we find is that of a free Yetter–Drinfeld module, see § 3 for the definition. These are Yetter–Drinfeld modules that are, in particular, free as modules. We show that if A and H are Hopf algebras that have equivalent tensor categories of comodules, then one can transport a free Yetter–Drinfeld resolution of the counit of A to the same kind of resolution for the counit of H (with preservation of the length of the resolution).

Now let $E \in \mathrm{GL}_n(\mathbb{C})$ with $n \geqslant 2$ and consider the algebra $\mathcal{B}(E)$ presented by generators $(u_{ij})_{1 \leqslant i,j \leqslant n}$ and relations

$$E^{-1} u^t E u = I_n = u E^{-1} u^t E,$$

where u is the matrix $(u_{ij})_{1 \leqslant i,j \leqslant n}$. The Hopf algebra $\mathcal{B}(E)$ was defined in [DL90], and corresponds to the quantum symmetry group of the bilinear form associated with E. We have $\mathcal{B}(I_n) = A_o(n)$ and $\mathcal{O}(\mathrm{SL}_q(2)) = \mathcal{B}(E_q)$, where

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

We construct, for any $E \in \mathrm{GL}_n(\mathbb{C})$, an exact sequence of $\mathcal{B}(E)$-modules

$$0 \to \mathcal{B}(E) \longrightarrow M_n(\mathcal{B}(E)) \longrightarrow M_n(\mathcal{B}(E)) \longrightarrow \mathcal{B}(E) \xrightarrow{\varepsilon} \mathbb{C} \to 0 \quad (\star_E).$$

See § 5. For $E = I_n$, the sequence is that of Collins–Härtel–Thom in [CHT09]. The verification of exactness goes as follows.

(i) We endow each constituent of the sequence with a free Yetter–Drinfeld module structure.

(ii) We use the previous construction to transport sequences of free Yetter–Drinfeld modules to show that for $E \in \mathrm{GL}_n(\mathbb{C})$, $F \in \mathrm{GL}_m(\mathbb{C})$ with $\mathrm{tr}(E^{-1}E^t) = \mathrm{tr}(F^{-1}F^t)$ and $m, n \geqslant 2$ (so that $\mathcal{M}^{\mathcal{B}(E)} \simeq^{\otimes} \mathcal{M}^{\mathcal{B}(F)}$; see [Bic03a]), the sequence $\star_E$ is exact if and only if the sequence $\star_F$ is exact.

(iii) We check that for any $q \in \mathbb{C}^*$, the sequence $\star_{E_q}$ is exact (this is less than a one-page computation). Now for any $E \in \mathrm{GL}_n(\mathbb{C})$ with $n \geqslant 2$, we pick $q \in \mathbb{C}^*$ such that $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1} = \mathrm{tr}(E_q^{-1}E_q^t)$, and we conclude from the previous item that $\star_E$ is exact.

Similarly as in [CHT09], the exactness of the sequence $\star_E$ has several interesting consequences. The first is that $\mathcal{B}(E)$ is smooth of dimension 3 for any $E \in \mathrm{GL}_n(\mathbb{C})$, $n \geqslant 2$ (recall [VdB98] that an algebra A is said to be smooth of dimension d if the A-bimodule A has a finite resolution of

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length d by finitely generated projective A-bimodules, with d being the smallest possible length for such a resolution).

The second consequence is that $\mathcal{B}(E)$ satisfies a Poincaré duality between its Hochschild homology and cohomology. Since Van den Bergh's seminal paper [VdB98], Poincaré duality for algebras has been the subject of many papers, in which the authors propose axioms that will have Poincaré duality as a corollary, see e.g. [KK10] for a recent general and powerful framework. Let us emphasize that the exact sequence $\star_E$ enables us to establish Poincaré duality in a straightforward manner, without having to check any condition such as those proposed in [BZ08] (where, moreover, noetherianity assumptions were done, while $\mathcal{B}(E)$ is not noetherian if $n \geqslant 3$).

A third consequence concerns bialgebra cohomology (also known as Gerstenhaber–Schack cohomology [GS90, GS92]), for which only very few full computations are known for non-cocommutative and non-commutative Hopf algebras (see [Tai07]). The fact that the exact sequence $\star_E$ consists of free Yetter–Drinfeld modules enables us to compute the bialgebra cohomology of $\mathcal{B}(E)$ (and hence, in particular, of $\mathcal{O}(\mathrm{SL}_q(2))$) in the cosemisimple case.

The last consequence concerns L^2-Betti numbers. Recall that the definition of L^2-Betti numbers for groups [Luc02] can be generalized to discrete quantum groups (compact Hopf algebras of Kac type) [Kye08]. Combining the results in [CHT09] with a result of Vergnioux [Ver12] (vanishing of the first Betti number of $A_o(n)$), Collins *et al.* have shown that the L^2-Betti numbers of $A_o(n)$ all vanish. Using similar arguments we show that the L^2-Betti numbers of $A_o(J_m)$ all vanish, where J_m is any antisymmetric invertible matrix. This completes the computation of the L^2-Betti numbers of all universal orthogonal discrete quantum groups of Kac type.

The paper is organized as follows. Section 2 is devoted to preliminaries. In § 3 we introduce free Yetter–Drinfeld modules and remark that the standard resolution of the counit of a Hopf algebra is a free Yetter–Drinfeld resolution. In § 4 we show how to transport free Yetter–Drinfeld resolutions for Hopf algebras having equivalent tensor categories of comodules. In § 5 we define and prove the exactness of the announced resolution of the counit of $\mathcal{B}(E)$. Section 6 is devoted to the several applications we have already announced, and § 7 consists of concluding remarks.

2. Preliminaries

2.1 Notation and conventions

We assume that the base field is $\mathbb{C}$, the field of complex numbers (although our results, except in § 6.4, do not depend on the base field). All algebras are assumed to be unital. We assume that the reader is familiar with the theory of Hopf algebras and their tensor categories of comodules, as explained, e.g., in [Kas95, KS97, Mon93]. If A is a Hopf algebra, as usual, Δ, ε and S stand for the comultiplication, counit and antipode of A, respectively. We use Sweedler's notation in the standard way. The category of right A-comodules is denoted $\mathcal{M}^A$. If M is an A-bimodule, then $H_*(A, M)$ and $H^*(A, M)$ denote the respective Hochschild homology and cohomology groups of A (with coefficients in M).

2.2 Hochschild homology of Hopf algebras and resolutions of the counit

In this section we recall how the Hochschild homology and cohomology of a Hopf algebra A can be described by using suitable Tor and Ext groups on the category of left or right A-modules

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and resolutions of the counit. This has been discussed under various forms in several papers (see [BZ08, CHT09, FT91, GK93, HK05]) and probably has its origins in [CE56, § 6, ch. X].

Proposition 2.1. *Let A be a Hopf algebra and let M be an A-bimodule. Define a left A-module structure on M and a right A-module structure on M by*

$$a \rightarrow x = a_{(2)} \cdot x \cdot S(a_{(1)}), \quad x \leftarrow a = S(a_{(1)}) \cdot x \cdot a_{(2)}$$

and denote by M' and M'' the respective corresponding left A-module and right A-module. Then for all $n \in \mathbb{N}$ there exist isomorphisms of vector spaces

$$H_n(A, M) \simeq \mathrm{Tor}_n^A(\mathbb{C}_\varepsilon, M'), \quad H^n(A, M) \simeq \mathrm{Ext}_A^n(\mathbb{C}_\varepsilon, M'').$$

The previous Ext groups are those in the category of right A-modules. In lack of a reference that would give exactly the isomorphisms of Proposition 2.1, we provide an explicit proof, which is the object of the rest of the subsection. Writing down the proof also gives us the opportunity to review the material involved in the statement of the proposition [Wei94].

Proof. Let (A, ε) be an augmented algebra, i.e. A is an algebra and $\varepsilon : A \longrightarrow \mathbb{C}$ is an algebra map that we call the counit of A. We view $\mathbb{C}$ as a right A-module via ε and we denote by $\mathbb{C}_\varepsilon$ this right A-module. Recall that the standard resolution of $\mathbb{C}_\varepsilon$ (the standard resolution of the counit) is given by the complex of free right A-modules

$$\cdots \longrightarrow A^{\otimes n+1} \longrightarrow A^{\otimes n} \longrightarrow \cdots \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

where each differential is given by

$$A^{\otimes n+1} \longrightarrow A^{\otimes n}$$

$$a_1 \otimes \cdots \otimes a_{n+1} \longmapsto \varepsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

Given a left A-module M', the vector spaces $\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$ are given by the homology of the complex obtained by tensoring any projective resolution of $\mathbb{C}_\varepsilon$ by $- \otimes_A M'$. Thus, using the standard resolution of the counit, after suitable identifications, we see that the vector spaces $\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$ are given by the homology of the following complex

$$\cdots \longrightarrow A^{\otimes n} \otimes M' \xrightarrow{d} A^{\otimes n-1} \otimes M' \xrightarrow{d} \cdots \xrightarrow{d} A \otimes M' \xrightarrow{d} M' \longrightarrow 0$$

where the differential $d : A^{\otimes n} \otimes M' \longrightarrow A^{\otimes n-1} \otimes M'$ is given by

$$d(a_1 \otimes \cdots \otimes a_n \otimes x) = \varepsilon(a_1)a_2 \otimes \cdots \otimes a_n \otimes x + \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes x$$
$$+ (-1)^n a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n \cdot x.$$

Recall now that if A is an algebra and M is an A-bimodule, the Hochschild homology groups $H_*(A, M)$ are the homology groups of the complex

$$\cdots \longrightarrow M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M \longrightarrow 0$$

where the differential $b : M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ is given by

$$b(x \otimes a_1 \otimes \cdots \otimes a_n) = x \cdot a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^n a_n \cdot x \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

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Assume now that A is a Hopf algebra and let M be an A-bimodule. Consider the linear map

$$\theta : M \otimes A^{\otimes n} \longrightarrow A^{\otimes n} \otimes M'$$
$$x \otimes a_1 \otimes \cdots \otimes a_n \longmapsto a_{1(2)} \otimes \cdots \otimes a_{n(2)} \otimes x \cdot (a_{1(1)} \cdot \cdots \cdot a_{n(1)}).$$

It is straightforward to see that θ is an isomorphism with inverse given by

$$\theta^{-1} : A^{\otimes n} \otimes M' \longrightarrow M \otimes A^{\otimes n}$$
$$a_1 \otimes \cdots \otimes a_n \otimes x \longmapsto x \cdot S(a_{1(1)} \cdots a_{n(1)}) \otimes (a_{1(2)} \otimes \cdots \otimes \cdots a_{n(2)})$$

and that $d \circ \theta = \theta \circ b$. Hence, θ induces an isomorphism between the complexes defining $H_*(A, M)$ and $\mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$ and we get the first isomorphism $H_*(A, M) \simeq \mathrm{Tor}_*^A(\mathbb{C}_\varepsilon, M')$.

For the second isomorphism in Proposition 2.1, let us come back to the situation of an augmented algebra (A, ε). Given a right A-module M'', the vector spaces $\mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$ are given by the cohomology of the complex obtained by applying the functor $\mathrm{Hom}_A(-, M'')$ to any projective resolution of $\mathbb{C}_\varepsilon$. Thus, using the standard resolution of the counit, we see that after suitable identifications, the vector spaces $\mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$ are given by the cohomology of the following complex

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M'') \xrightarrow{\partial} \mathrm{Hom}(A, M'') \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathrm{Hom}(A^{\otimes n}, M'') \xrightarrow{\partial} \mathrm{Hom}(A^{\otimes n+1}, M'') \xrightarrow{\partial} \cdots$$

where the differential $\partial : \mathrm{Hom}(A^{\otimes n}, M'') \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, M'')$ is given by

$$\partial(f)(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}.$$

If A is an algebra and M is an A-bimodule, the Hochschild cohomology groups $H^*(A, M)$ are the cohomology groups of the complex

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M) \xrightarrow{\delta} \mathrm{Hom}(A, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta : \mathrm{Hom}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, M)$ is given by

$$\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \cdot f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}.$$

Assume now that A is a Hopf algebra and let M be an A-bimodule. Consider the linear map

$$\vartheta : \mathrm{Hom}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}(A^{\otimes n}, M'')$$
$$f \longmapsto \hat{f}, \quad \hat{f}(a_1 \otimes \cdots \otimes a_n) = S(a_{1(1)} \cdots a_{n(1)}) f(a_{1(2)} \otimes \cdots \otimes a_{n(2)}).$$

It is easy to see that ϑ is an isomorphism and that $\partial \circ \vartheta = \vartheta \circ \delta$. Hence, ϑ induces an isomorphism between the complexes defining $H^*(A, M)$ and $\mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$ and we get the second isomorphism $H^*(A, M) \simeq \mathrm{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$. □

2.3 The Hopf algebra $\mathcal{B}(E)$

Let $E \in \mathrm{GL}_n(\mathbb{C})$. Recall that the algebra $\mathcal{B}(E)$ (see [DL90]) is presented by generators $(u_{ij})_{1 \leqslant i,j \leqslant n}$ and relations

$$E^{-1} u^t E u = I_n = u E^{-1} u^t E,$$

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where u is the matrix $(u_{ij})_{1\leqslant i,j\leqslant n}$. It has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u) = E^{-1}u^t E.$$

For the matrix $E_q \in \mathrm{GL}_2(\mathbb{C})$ in the introduction, we have $\mathcal{B}(E_q) = \mathcal{O}(\mathrm{SL}_q(2))$, and thus the Hopf algebras $\mathcal{B}(E)$ are generalizations of $\mathcal{O}(\mathrm{SL}_q(2))$. It is shown in [Bic03a] that the isomorphism class of the Hopf algebra $\mathcal{B}(E)$ only depends on the bilinear form associated with the matrix E, and that for $q \in \mathbb{C}^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$, the tensor categories of comodules over $\mathcal{B}(E)$ and $\mathcal{O}(\mathrm{SL}_q(2))$ are equivalent.

The fundamental n-dimensional $\mathcal{B}(E)$-comodule is denoted by V_E: it has a basis $e_1^E, \dots, e_n^E$ and right coaction $\alpha : V_E \to V_E \otimes \mathcal{B}(E)$ defined by $\alpha(e_i^E) = \sum_{k=1}^n e_k^E \otimes u_{ki}$. For future use, we record that the following linear maps

$$\begin{aligned} \delta : \mathbb{C} &\longrightarrow V_E \otimes V_E, & \varphi : V_E &\longrightarrow V_E^* \\ 1 &\longmapsto \sum_{i,j=1}^{n} E_{ij}^{-1} e_i^E \otimes e_j^E, & e_i^E &\longmapsto \sum_{k=1}^{n} E_{ik} e_k^{E*} \end{aligned}$$

are morphisms of $\mathcal{B}(E)$-comodules (where $E^{-1} = (E_{ij}^{-1})$ and $\{e_i^{E*}\}$ is the dual basis of $\{e_i^E\}$).

We now define some maps that will be used in § 6.

(i) The sovereign character of $\mathcal{B}(E)$ is the algebra map $\Phi : \mathcal{B}(E) \to \mathbb{C}$ defined by $\Phi(u) = E^{-1}E^t$. It satisfies $S^2 = \Phi * \mathrm{id} * \Phi^{-1}$.

(ii) The modular automorphism of $\mathcal{B}(E)$ is the algebra automorphism σ of $\mathcal{B}(E)$ defined by $\sigma(u) = E^{-1}E^t u E^{-1}E^t$, i.e. $\sigma = \Phi * \mathrm{id} * \Phi$.

(iii) We denote by θ is the algebra anti-automorphism of $\mathcal{B}(E)$ defined by $\theta = S * \Phi * \Phi$, i.e. $\theta(u) = S(u)E^{-1}E^t E^{-1}E^t$. We have $S \circ \theta = \sigma$.

3. Free Yetter–Drinfeld modules

In this section we introduce the concept of free Yetter–Drinfeld module, which will be essential for our purpose. We begin by recalling the basics on Yetter–Drinfeld modules.

Let A be a Hopf algebra. Recall that a (right–right) Yetter–Drinfeld module over A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftharpoonup a)_{(0)} \otimes (v \leftharpoonup a)_{(1)} = v_{(0)} \leftharpoonup a_{(2)} \otimes S(a_{(1)}) v_{(1)} a_{(3)}.$$

The category of Yetter–Drinfeld modules over A is denoted $\mathcal{YD}_A^A$: the morphisms are the A-linear A-colinear maps. Endowed with the usual tensor product of modules and comodules, it is a tensor category.

An important example of Yetter–Drinfeld module is the right coadjoint Yetter–Drinfeld module A_{coad}: as a right A-module $A_{\mathrm{coad}} = A$ and the right A-comodule structure is defined by

$$\mathrm{ad}_r(a) = a_{(2)} \otimes S(a_{(1)}) a_{(3)}, \quad \forall a \in A.$$

The following result, which will be of vital importance for us, generalizes the construction of the right coadjoint comodule.

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Proposition 3.1. *Let A be a Hopf algebra and let V be a right A-comodule. Endow $V \otimes A$ with the right A-module structure defined by multiplication on the right. Then the linear map*

$$\begin{aligned} V \otimes A &\longrightarrow V \otimes A \otimes A \\ v \otimes a &\longmapsto v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)}) v_{(1)} a_{(3)} \end{aligned}$$

endows $V \otimes A$ with a right A-comodule structure, and with a Yetter–Drinfeld module structure. We denote by $V \boxtimes A$ the resulting Yetter–Drinfeld module, and this construction produces a functor

$$\begin{aligned} L : \mathcal{M}^A &\longrightarrow \mathcal{YD}_A^A \\ V &\longmapsto V \boxtimes A. \end{aligned}$$

Proof. This is a direct verification. □

Note that when $V = \mathbb{C}$ is the trivial comodule, then $\mathbb{C} \boxtimes A = A_{\mathrm{coad}}$.

Definition 3.2. Let A be a Hopf algebra. A Yetter–Drinfeld module over A is said to be free if it is isomorphic to $V \boxtimes A$ for some right A-comodule V.

Of course a free Yetter–Drinfeld is free as a right A-module. The terminology is further justified by the following result.

Proposition 3.3. *Let A be a Hopf algebra. The functor $L = - \boxtimes A : \mathcal{M}^A \longrightarrow \mathcal{YD}_A^A$ is left adjoint to the forgetful functor $R : \mathcal{YD}_A^A \longrightarrow \mathcal{M}^A$. In particular, if P is a projective object in $\mathcal{M}^A$, then $L(P)$ is a projective object in $\mathcal{YD}_A^A$.*

Proof. Let $V \in \mathcal{M}^A$ and $X \in \mathcal{YD}_A^A$. It is a direct verification to check that we have a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}^A}(V, R(X)) &\longrightarrow \mathrm{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, X) \\ f &\longmapsto \tilde{f}, \quad \tilde{f}(v \otimes a) = f(v) \leftarrow a \end{aligned}$$

and thus $L = - \boxtimes A$ is left adjoint to the forgetful functor R. The last assertion is a standard fact, see e.g. [Wei94, Proposition 2.3.10]. □

A left–right version of Proposition 3.3 was given in [CMZ97, Corollary 2.8]. More generally, it is worth noting that the existence of a left adjoint functor to the forgetful functor $R : \mathcal{YD}_A^A \longrightarrow \mathcal{M}^A$ follows from the general situations studied in [CMZ02].

Recall [Lin77] that the category $\mathcal{M}^A$ of right A-comodules has enough projectives if and only if A is co-Frobenius (A is said to be co-Frobenius is there exists a non-zero right A-colinear map $A \to \mathbb{C}$).

Corollary 3.4. *Let A be a co-Frobenius Hopf algebra. Then the category $\mathcal{YD}_A^A$ has enough projective objects.*

Proof. Let $V \in \mathcal{YD}_A^A$ and let P be a projective object in $\mathcal{M}^A$ with an epimorphism $f : P \twoheadrightarrow R(V)$. We have a surjective morphism of Yetter–Drinfeld modules

$$\begin{aligned} L(P) = P \boxtimes A &\twoheadrightarrow V \\ x \otimes a &\mapsto (f(x) \leftarrow a) \end{aligned}$$

with $L(P)$ projective, and we are done. □

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Note that when A is a projective generator in $\mathcal{M}^A$ (A is then said to be quasi-co-Frobenius) then $A \boxtimes A$ is a projective generator of $\mathcal{YD}_A^A$, by [CMZ97, Corollary 2.10].

Definition 3.5. Let A be a Hopf algebra and let $M \in \mathcal{YD}_A^A$. A free Yetter–Drinfeld resolution of M consists of a complex of free Yetter–Drinfeld modules

$$\mathbf{P}_\cdot = \cdots P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0$$

for which there exists a Yetter–Drinfeld module map $\epsilon : P_0 \to M$ such that

$$\cdots P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is an exact sequence.

In particular, a free Yetter–Drinfeld resolution of M is a resolution of M (as a right A-module) by free A-modules. A basic motivation for considering this special kind of resolutions comes from the fact that the standard resolution of the counit is in fact a free Yetter–Drinfeld resolution. Before making this statement precise, we need the following construction.

For any $n \in \mathbb{N}$, we define the comodule $A^{\boxtimes n}$ as follows:

$$A^{\boxtimes 0} = \mathbb{C},\ A^{\boxtimes 1} = \mathbb{C} \boxtimes A = A_{\mathrm{coad}},\ A^{\boxtimes 2} = A^{\boxtimes 1} \boxtimes A, \ldots,\ A^{\boxtimes (n+1)} = A^{\boxtimes n} \boxtimes A, \ldots.$$

It is straightforward to check that after the obvious vector space identification of $A^{\boxtimes n}$ with $A^{\otimes n}$, the right A-module structure of $A^{\boxtimes n}$ is given by right multiplication and its comodule structure is given by

$$\begin{aligned} \mathrm{ad}_r^{(n)} : A^{\boxtimes n} &\longrightarrow A^{\boxtimes n} \otimes A \\ a_1 \otimes \cdots \otimes a_n &\longmapsto a_{1(2)} \otimes \cdots \otimes a_{n(2)} \otimes S(a_{1(1)} \cdots a_{n(1)}) a_{1(3)} \cdots a_{n(3)}. \end{aligned}$$

Proposition 3.6. *Let A be Hopf algebra. The standard resolution of the counit of A is a resolution of $\mathbb{C}$ by free Yetter–Drinfeld modules.*

Proof. It is a direct verification to check that for any $n \geqslant 0$, the map

$$\begin{aligned} A^{\boxtimes (n+1)} &\longrightarrow A^{\boxtimes n} \\ a_1 \otimes \cdots \otimes a_{n+1} &\longmapsto \varepsilon(a_1) a_2 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \end{aligned}$$

is a morphism of Yetter–Drinfeld modules. This gives the result since the Yetter–Drinfeld modules $A^{\boxtimes (n+1)}$ are free by construction. □

We close the section by recording the following elementary result, to be used in § 5. The proof is left to the reader.

Lemma 3.7. *Let A be a Hopf algebra, let V be a finite-dimensional A-comodule with coaction $\alpha : V \to V \otimes A$, with basis $e_1, \ldots, e_n$, and let $(u_{ij}) \in M_n(A)$ be such that $\alpha(e_i) = \sum_k e_k \otimes u_{ki}$. The linear maps*

$$\begin{aligned} \Phi_V^1 : V^* \otimes V &\longrightarrow \mathbb{C} \boxtimes A = A_{\mathrm{coad}} \\ e_i^* \otimes e_j &\longmapsto u_{ij} \end{aligned}$$

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$$\Phi_V^2 : V^* \otimes V^* \otimes V \otimes V \longrightarrow (V^* \otimes V) \boxtimes A$$
$$e_i^* \otimes e_j^* \otimes e_k \otimes e_l \longmapsto e_j^* \otimes e_k \otimes u_{il}$$

are A-colinear.

4. Equivalences between tensor categories of comodules

In this section we present the technical core of the paper: the fact that if A and H are Hopf algebras that have equivalent tensor categories of comodules, then one can transport a free Yetter–Drinfeld resolution of the counit of A to the same kind of resolution for the counit of H (with preservation of the length of the resolution). The precise result is as follows.

Theorem 4.1. *Let A and H be some Hopf algebras. Assume that there exists an equivalence of linear tensor categories $\Theta : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^H$. Then Θ induces an equivalence of linear tensor categories $\widehat{\Theta} : \mathcal{YD}_A^A \simeq^{\otimes} \mathcal{YD}_H^H$ together with, for all $V \in \mathcal{M}^A$, natural isomorphisms*

$$\widehat{\Theta}(V \boxtimes A) \simeq \Theta(V) \boxtimes H.$$

The functor $\widehat{\Theta}$ associates to any free Yetter–Drinfeld resolution of the counit of A

$$V_\cdot \boxtimes A : \cdots V_{n+1} \boxtimes A \to V_n \boxtimes A \to \cdots \to V_0 \boxtimes A \to 0$$

a free Yetter–Drinfeld resolution of the counit of H

$$\Theta(V_\cdot) \boxtimes H : \cdots \Theta(V_{n+1}) \boxtimes H \to \Theta(V_n) \boxtimes H \to \cdots \to \Theta(V_0) \boxtimes H \to 0.$$

Proof. Let $R_A : \mathcal{YD}_A^A \longrightarrow \mathcal{M}^A$ and $R_H : \mathcal{YD}_H^H \longrightarrow \mathcal{M}^H$ be the respective forgetful functors with their respective left adjoint $L_A : \mathcal{M}^A \longrightarrow \mathcal{YD}_A^A$ and $L_H : \mathcal{M}^H \longrightarrow \mathcal{YD}_H^H$. The description of $\mathcal{YD}_A^A$ as the weak center of the monoidal category $\mathcal{M}_A$ (this is stated in [Sch04, Appendix], the proof can be done along similar lines as the proof for modules over a finite-dimensional Hopf algebra, given in [Kas95, Theorem XIII.5.1]) ensures the existence of an equivalence of linear tensor categories $\widehat{\Theta} : \mathcal{YD}_A^A \simeq \mathcal{YD}_H^H$ such that $R_H\widehat{\Theta} \simeq \Theta R_A$ as functors. Denote by Θ^{-1} a quasi-inverse of Θ. Then we have, for any $U \in \mathcal{M}^H$ and $X \in \mathcal{YD}_H^H$, natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{YD}_H^H}(\widehat{\Theta}L_A\Theta^{-1}(U), X) &\simeq \mathrm{Hom}_{\mathcal{YD}_A^A}(L_A\Theta^{-1}(U), \widehat{\Theta}^{-1}(X))\\ &\simeq \mathrm{Hom}_{\mathcal{M}^A}(\Theta^{-1}(U), R_A\widehat{\Theta}^{-1}(X))\\ &\simeq \mathrm{Hom}_{\mathcal{M}^A}(\Theta^{-1}(U), \Theta^{-1}R_H(X))\\ &\simeq \mathrm{Hom}_{\mathcal{M}^H}(U, R_H(X)).\end{aligned}$$

The uniqueness of adjoint functors ensures that $\widehat{\Theta}L_A\Theta^{-1} \simeq L_H$, so that $\widehat{\Theta}L_A \simeq L_H\Theta$, as required. The last assertion is then immediate. □

In the next section, in order to transport an explicit resolution, we will need to know the explicit form of a tensor equivalence $\Theta : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^H$ and of the associated tensor equivalence $\widehat{\Theta} : \mathcal{YD}_A^A \simeq \mathcal{YD}_H^H$ (whose description will be independent of the description of the category of Yetter–Drinfeld modules as a monoidal center).

It was shown by Schauenburg [Sch96] that equivalences of linear tensor categories $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^H$ always arise from Hopf A–H bi-Galois objects. The axioms of Hopf bi-Galois objects were symmetrized [Bic03b, Gru04], leading to the use of the language of cogroupoids [Bic10], that we now recall.

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First recall that a cocategory (of algebras over $\mathbb{C}$) $\mathcal{C}$ consists of:

- a set of objects $\mathrm{ob}(\mathcal{C})$;
- for any $X, Y \in \mathrm{ob}(\mathcal{C})$, an algebra $\mathcal{C}(X, Y)$;
- for any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, algebra morphisms
$$\Delta^Z_{X,Y} : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \quad \text{and} \quad \varepsilon_X : \mathcal{C}(X, X) \longrightarrow \mathbb{C}$$
such that some natural coassociativity and counit diagrams, dual to the usual associativity and unit diagrams in a category, commute. In particular, $\mathcal{C}(X, X)$ is a bialgebra for any object X.

A cogroupoid $\mathcal{C}$ consists of a cocategory $\mathcal{C}$ together with, for any $X, Y \in \mathrm{ob}(\mathcal{C})$, linear maps
$$S_{X,Y} : \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(Y, X)$$
such that natural diagrams (dual to the invertibility diagrams in a groupoid) commute. In particular, $\mathcal{C}(X, X)$ is a Hopf algebra for any object X. A cogroupoid is said to be connected if for any $X, Y \in \mathrm{ob}(\mathcal{C})$, the algebra $\mathcal{C}(X, Y)$ is non-zero.

The following theorem is the cogroupoid reformulation of Schauenburg's results in [Sch96], see [Bic10].

THEOREM 4.2. *Let $\mathcal{C}$ be a connected cogroupoid. Then for any $X, Y \in \mathrm{ob}(\mathcal{C})$ we have linear equivalences of tensor categories that are inverse of each other*
$$\mathcal{M}^{\mathcal{C}(X,X)} \simeq^{\otimes} \mathcal{M}^{\mathcal{C}(Y,Y)} \qquad \mathcal{M}^{\mathcal{C}(Y,Y)} \simeq^{\otimes} \mathcal{M}^{\mathcal{C}(X,X)}$$
$$V \longmapsto V \,\square_{\mathcal{C}(X,X)}\, \mathcal{C}(X, Y) \qquad V \longmapsto V \,\square_{\mathcal{C}(Y,Y)}\, \mathcal{C}(Y, X).$$
Conversely, if A and H are Hopf algebras such that $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^H$, then there exists a connected cogroupoid with two objects X, Y such that $A = \mathcal{C}(X, X)$ and $B = \mathcal{C}(Y, Y)$.

Here the symbol $\square$ stands for the cotensor product of a right comodule by a left comodule, see e.g. [Mon93].

In order to extend the previous monoidal equivalences to categories of Yetter–Drinfeld modules, let us now recall Sweedler's notation for cocategories and cogroupoids. Let $\mathcal{C}$ be a cocategory. For $a^{XY} \in \mathcal{C}(X, Y)$, we write
$$\Delta^Z_{X,Y}(a^{XY}) = a^{XZ}_{(1)} \otimes a^{ZY}_{(2)}.$$
The cocategory axioms are
$$(\Delta^T_{X,Z} \otimes 1) \circ \Delta^Z_{X,Y}(a^{XY}) = a^{XT}_{(1)} \otimes a^{TZ}_{(2)} \otimes a^{ZY}_{(3)} = (1 \otimes \Delta^Z_{T,Y}) \circ \Delta^T_{X,Y}(a^{XY})$$
$$\varepsilon_X(a^{XX}_{(1)}) a^{XY}_{(2)} = a^{XY} = \varepsilon_Y(a^{YY}_{(2)}) a^{XY}_{(1)}$$
and the additional cogroupoid axioms are
$$S_{X,Y}(a^{XY}_{(1)}) a^{YX}_{(2)} = \varepsilon_X(a^{XX}) 1 = a^{XY}_{(1)} S_{Y,X}(a^{YX}_{(2)}).$$
The following result is [Bic10, Proposition 6.2].

PROPOSITION 4.3. *Let $\mathcal{C}$ be cogroupoid, let $X, Y \in \mathrm{ob}(\mathcal{C})$ and let V be a right $\mathcal{C}(X, X)$-module.*

(i) *$V \otimes \mathcal{C}(X, Y)$ has a right $\mathcal{C}(Y, Y)$-module structure defined by*
$$(v \otimes a^{XY}) \leftarrow b^{YY} = v \leftarrow b^{XX}_{(2)} \otimes S_{YX}(b^{YX}_{(1)}) a^{XY} b^{XY}_{(3)}.$$

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Endowed with the right $\mathcal{C}(Y,Y)$-comodule defined by $1\otimes\Delta^Y_{X,Y}$, $V\otimes\mathcal{C}(X,Y)$ is a Yetter–Drinfeld module over $\mathcal{C}(Y,Y)$.

(ii) *If moreover V is Yetter–Drinfeld module, then $V\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y)$ is Yetter–Drinfeld submodule of $V\otimes\mathcal{C}(X,Y)$.*

We now can write down the explicit form of the tensor equivalence between categories of Yetter–Drinfeld modules induced by a tensor equivalence between categories of comodules.

THEOREM 4.4. *Let $\mathcal{C}$ be connected cogroupoid. Then for any $X,Y\in\mathrm{ob}(\mathcal{C})$, the functor*

$$\begin{aligned}\mathcal{YD}^{\mathcal{C}(X,X)}_{\mathcal{C}(X,X)}&\longrightarrow\mathcal{YD}^{\mathcal{C}(Y,Y)}_{\mathcal{C}(Y,Y)}\\ V&\longmapsto V\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y)\end{aligned}$$

is an equivalence of linear tensor categories. Moreover, we have natural isomorphisms

$$\begin{aligned}(V\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y))\boxtimes\mathcal{C}(Y,Y)&\longrightarrow(V\boxtimes\mathcal{C}(X,X))\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y)\\ v\otimes a^{XY}\otimes b^{YY}&\longmapsto v\otimes b^{XX}_{(2)}\otimes S_{Y,X}(b^{YX}_{(1)})a^{XY}b^{XY}_{(3)}.\end{aligned}$$

Proof. The fact that this indeed defines an equivalence of tensor categories is proved in [Bic10, Theorem 6.3]. The announced natural isomorphism is that induced by the uniqueness of adjoint functors as in the proof of Theorem 4.1. For the reader's convenience, let us write down explicitly the inverse isomorphism. Let

$$\begin{aligned}\xi_V:&(V\boxtimes\mathcal{C}(X,X))\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y)\longrightarrow\\ &((V\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y))\boxtimes\mathcal{C}(Y,Y))\,\square_{\mathcal{C}(Y,Y)}\,(\mathcal{C}(Y,X)\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y))\\ &v\otimes a^{XX}\otimes b^{XY}\longmapsto v_{(0)}\otimes v^{XY}_{(1)}\otimes a^{YY}_{(2)}\otimes S_{XY}(a^{XY}_{(1)})v^{YX}_{(2)}a^{YX}_{(3)}\otimes b^{XY}.\end{aligned}$$

The explicit inverse of the morphism in the statement is then $(\mathrm{id}\otimes(\varepsilon_Y f))\xi_V$, where f is the inverse isomorphism to $\Delta^X_{YY}:\mathcal{C}(Y,Y)\to\mathcal{C}(Y,X)\,\square_{\mathcal{C}(X,X)}\,\mathcal{C}(X,Y)$, see [Bic10, proof of Lemma 2.14]. □

We end the section by recalling that $\mathcal{B}(E)$ is part of a cogroupoid. Let $E\in\mathrm{GL}_m(\mathbb{C})$ and let $F\in\mathrm{GL}_n(\mathbb{C})$. Recall [Bic03a] that the algebra $\mathcal{B}(E,F)$ is the universal algebra with generators u_{ij}, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant n$, satisfying the relations

$$F^{-1}u^tEu=I_n;\quad uF^{-1}u^tE=I_m.$$

Of course the generator u_{ij} in $\mathcal{B}(E,F)$ is denoted u^{EF}_{ij} to express the dependence on E and F, when needed. It is clear that $\mathcal{B}(E,E)=\mathcal{B}(E)$.

We get a cogroupoid $\mathcal{B}$ whose objects are the invertible matrices $E\in\mathrm{GL}_n(\mathbb{C})$, where the algebras $\mathcal{B}(E,F)$ are those just defined and where the structural morphisms are the algebra maps defined as follows

$$\begin{aligned}\Delta^G_{E,F}:\mathcal{B}(E,F)&\longrightarrow\mathcal{B}(E,G)\otimes\mathcal{B}(G,F)\\ u^{EF}_{ij}&\longmapsto\sum_k u^{EG}_{ik}\otimes u^{GF}_{kj}\end{aligned}$$

$$\begin{aligned}S_{E,F}:\mathcal{B}(E,F)&\longrightarrow\mathcal{B}(F,E)^{\mathrm{op}}\\ u&\longmapsto E^{-1}u^tF\end{aligned}$$

and where ε_E is the counit of $\mathcal{B}(E)$.

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When $\mathcal{B}(E,F)\neq 0$ (i.e. when $\mathrm{tr}(E^{-1}E^t)=\mathrm{tr}(F^{-1}F^t)$ and the matrices E, F have size $\geqslant 2$, see [Bic03a]), we know, by Theorems 4.2 and 4.4, that the cotensor product by $\mathcal{B}(E,F)$ induces equivalences of tensor categories

$$\mathcal{M}^{\mathcal{B}(E)}\simeq^{\otimes}\mathcal{M}^{\mathcal{B}(F)},\quad \mathcal{YD}^{\mathcal{B}(E)}_{\mathcal{B}(E)}\simeq^{\otimes}\mathcal{YD}^{\mathcal{B}(F)}_{\mathcal{B}(F)}.$$

The following $\mathcal{B}(F)$-comodule isomorphisms will be used in the next section:

$$\begin{aligned} &V_F\longrightarrow V_E\,\square_{\mathcal{B}(E)}\,\mathcal{B}(E,F), && V_F^*\otimes V_F\longrightarrow (V_E^*\otimes V_E)\,\square_{\mathcal{B}(E)}\,\mathcal{B}(E,F)\\ &e_i^F\longmapsto\sum_k e_k\otimes u_{ki}^{EF}, && e_i^{F*}\otimes e_j^F\longmapsto\sum_{k,l} e_k^{E*}\otimes e_l^E\otimes S_{FE}(u_{ik}^{FE})u_{lj}^{EF}. \end{aligned}$$

5. A resolution of the counit for $\mathcal{B}(E)$

In this section we write down the announced resolution $\star_E$ for the counit of $\mathcal{B}(E)$.

5.1 The resolution

Theorem 5.1. *Let $E\in\mathrm{GL}_n(\mathbb{C})$, $n\geqslant 2$, and let V_E be the fundamental n-dimensional $\mathcal{B}(E)$-comodule. There exists an exact sequence of Yetter–Drinfeld modules over $\mathcal{B}(E)$*

$$0\to\mathbb{C}\boxtimes\mathcal{B}(E)\xrightarrow{\phi_1}(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)\xrightarrow{\phi_2}(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)\xrightarrow{\phi_3}\mathbb{C}\boxtimes\mathcal{B}(E)\xrightarrow{\varepsilon}\mathbb{C}\to 0$$

which thus yields a free Yetter–Drinfeld resolution of the counit of $\mathcal{B}(E)$.

Of course the first thing to do is to define the maps ϕ_1, ϕ_2, ϕ_3 in Theorem 5.1.

Definition 5.2. Let $e_1^E,\dots,e_n^E$ be the canonical basis of V_E. The linear maps ϕ_1, ϕ_2, ϕ_3 in Theorem 5.1 are defined as follows:

$$\begin{aligned} \phi_1:\mathbb{C}\boxtimes\mathcal{B}(E)&\longrightarrow(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)\\ x&\longmapsto\sum_{i,j}e_i^{E*}\otimes e_j^E\otimes((E^tE^{-1})_{ij}-(Eu(E^t)^{-1})_{ij})x \end{aligned}$$

$$\begin{aligned} \phi_2:(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)&\longrightarrow(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)\\ e_i^{E*}\otimes e_j^E\otimes x&\longmapsto e_i^{E*}\otimes e_j^E\otimes x+\sum_{k,l}e_k^{E*}\otimes e_l^E\otimes(u(E^t)^{-1})_{il}E_{jk}x \end{aligned}$$

$$\begin{aligned} \phi_3:(V_E^*\otimes V_E)\boxtimes\mathcal{B}(E)&\longrightarrow\mathbb{C}\boxtimes B(E)\\ e_i^{E*}\otimes e_j^E\otimes x&\longmapsto(u_{ij}-\delta_{ij})x. \end{aligned}$$

When $E=I_n$, the maps ϕ_1, ϕ_2, ϕ_3 are those defined in [CHT09].

5.2 Proof of Theorem 5.1

Lemma 5.3. *The maps ϕ_1, ϕ_2, ϕ_3 in Theorem 5.1 are morphisms of Yetter–Drinfeld modules.*

Proof. This can be checked directly, but for future use we describe ϕ_1, ϕ_2, ϕ_3 as linear combination of maps that are known to be morphisms of Yetter–Drinfeld modules. Let ϕ_1', ϕ_1''

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be defined by the following compositions of $\mathcal{B}(E)$-colinear maps

$$\phi_1' : \mathbb{C} \xrightarrow{\delta} V_E \otimes V_E \xrightarrow{\varphi\otimes\mathrm{id}} V_E^* \otimes V_E \xrightarrow{\mathrm{id}\otimes u} (V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)$$
$$\phi_1'' : \mathbb{C} \xrightarrow{\delta\otimes\delta} V_E \otimes V_E \otimes V_E \otimes V_E \xrightarrow{\varphi\otimes\varphi\otimes\mathrm{id}} V_E^* \otimes V_E^* \otimes V_E \otimes V_E \xrightarrow{\Phi_V^2} (V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)$$

where u is the unit map $\mathbb{C} \hookrightarrow \mathcal{B}(E)$, the maps φ, δ were defined in § 2 and Φ_V^2 was defined in Lemma 3.7. We have $\phi_1 = \tilde{\phi_1'} - \tilde{\phi_1''}$ (where the notation $\tilde{}$ has the same meaning as in the proof of Proposition 3.3), hence ϕ_1 is a Yetter–Drinfeld map. Define now ϕ_2' by the composition of the following colinear maps

$$\phi_2' : V_E^* \otimes V_E \xrightarrow{\mathrm{id}\otimes\delta} V_E^* \otimes V_E \otimes V_E \otimes V_E \xrightarrow{\mathrm{id}\otimes\varphi\otimes\mathrm{id}} V_E^* \otimes V_E^* \otimes V_E \otimes V_E \xrightarrow{\Phi_V^2} (V_E^* \otimes V_E) \boxtimes \mathcal{B}(E).$$

We have $\phi_2 = \mathrm{id} + \tilde{\phi_2'}$, hence ϕ_2 is a Yetter–Drinfeld map. Finally, we have $\phi_3 = \tilde{\Phi_V^1} - \tilde{\phi_3'}$, where ϕ_3' is the evaluation map $V_E^* \otimes V_E \to \mathbb{C} \hookrightarrow \mathbb{C} \boxtimes \mathcal{B}(E)$, and hence ϕ_3 is also a morphism of Yetter–Drinfeld modules. □

Lemma 5.4. *The sequence in Theorem 5.1 is a complex.*

Proof. It is straightforward to check that $\varepsilon \circ \phi_3 = 0$ and that $\phi_3 \circ \phi_2 = 0$. The identity $\phi_2 \circ \phi_1 = 0$ follows from the observation that $\tilde{\phi_2'} \circ \tilde{\phi_1'} = \tilde{\phi_1''}$ and $\tilde{\phi_2'} \circ \tilde{\phi_1''} = \tilde{\phi_1'}$. □

Lemma 5.5. *Let $E \in \mathrm{GL}_n(\mathbb{C})$, $F \in \mathrm{GL}_m(\mathbb{C})$ with $m, n \geqslant 2$ and $\mathrm{tr}(E^{-1}E^t) = \mathrm{tr}(F^{-1}F^t)$. Then the sequence in Theorem 5.1 is exact for $\mathcal{B}(E)$ if and only if it is exact for $\mathcal{B}(F)$.*

Proof. We know from [Bic03a] and Theorems 4.1 and 4.4 that we have an equivalence of tensor categories ${}^{\mathcal{B}(E)}_{\mathcal{B}(E)}\mathcal{YD} \simeq^{\otimes} {}^{\mathcal{B}(F)}_{\mathcal{B}(F)}\mathcal{YD}$ that preserves freeness of Yetter–Drinfeld modules. Let us check that this tensor equivalence transforms the complex of Yetter–Drinfeld modules of Theorem 5.1 for $\mathcal{B}(E)$ into the complex of Yetter–Drinfeld modules of Theorem 5.1 for $\mathcal{B}(F)$. First consider the following diagram.

$$\begin{array}{ccc}
\mathbb{C} \boxtimes \mathcal{B}(F) & \xrightarrow{\phi_1^F} & (V_F^* \otimes V_F) \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
(\mathbb{C} \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) & & ((V_E^* \otimes V_E) \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
(\mathbb{C} \boxtimes \mathcal{B}(E)) \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F) & \xrightarrow{\phi_1^E\otimes\mathrm{id}} & ((V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)) \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F)
\end{array}$$

The first vertical arrow on the left is given by the identification $\mathbb{C} \simeq \mathbb{C} \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F)$ (this identification follows from the fact that $\mathcal{B}(E,F)$ is a left $\mathcal{B}(E)$-Galois object) and the second vertical arrow is that of Theorem 4.4 for the trivial comodule $\mathbb{C}$. So the composition of the vertical arrows on the left is

$$\begin{aligned}
\mathbb{C} \boxtimes \mathcal{B}(F) &\longrightarrow (\mathbb{C} \boxtimes \mathcal{B}(E)) \,\square_{\mathcal{B}(E)}\, \mathcal{B}(E,F) \\
1 \otimes x^{FF} &\longmapsto x^{EE}_{(2)} \otimes S_{FE}(x^{FE}_{(1)}) x^{EF}_{(3)}\,.
\end{aligned}$$

The first vertical arrow on the right is that given at the end of § 4, while the second vertical arrow is that of Theorem 4.4 for the comodule $V_E^* \otimes V_E$. So the composition of the vertical arrows on

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the right is

$$\begin{aligned}(V_F^* \otimes V_F) \boxtimes \mathcal{B}(F) &\longrightarrow ((V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)\\ e_i^{F*} \otimes e_j^F \otimes x^{FF} &\longmapsto \sum_{k,l} e_k^{E*} \otimes e_l^E \otimes x_{(2)}^{EE} \otimes S_{FE}(x_{(1)}^{FE}) S_{FE}(u_{ik}^{FE}) u_{lj}^{EF} x_{(3)}^{EF}.\end{aligned}$$

The vertical arrows are compositions of isomorphisms so are isomorphisms. It is a direct verification to check that the previous diagram is commutative. Similarly one checks the commutativity of the following diagrams.

$$\begin{array}{ccc}
(V_F^* \otimes V_F) \boxtimes \mathcal{B}(F) & \xrightarrow{\phi_2^F} & (V_F^* \otimes V_F) \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
((V_E^* \otimes V_E) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) & & ((V_E^* \otimes V_E) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
((V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F) & \xrightarrow{\phi_2^E \otimes \mathrm{id}} & ((V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)
\end{array}$$

$$\begin{array}{ccc}
(V_F^* \otimes V_F) \boxtimes \mathcal{B}(F) & \xrightarrow{\phi_3^F} & \mathbb{C} \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
((V_E^* \otimes V_E) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) & & (\mathbb{C} \square_{\mathcal{B}(E)} \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) \\
\downarrow & & \downarrow \\
((V_E^* \otimes V_E) \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F) & \xrightarrow{\phi_3^E \otimes \mathrm{id}} & (\mathbb{C} \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F)
\end{array}$$

$$\begin{array}{ccc}
\mathbb{C} \boxtimes \mathcal{B}(F) & \xrightarrow{\varepsilon} & \mathbb{C} \\
\downarrow & & \downarrow \\
(\mathbb{C} \square_{\mathcal{B}(E)} \mathcal{B}(E,F)) \boxtimes \mathcal{B}(F) & & \\
\downarrow & & \\
(\mathbb{C} \boxtimes \mathcal{B}(E)) \square_{\mathcal{B}(E)} \mathcal{B}(E,F) & \xrightarrow{\varepsilon \otimes \mathrm{id}} & \mathbb{C} \square_{\mathcal{B}(E)} \mathcal{B}(E,F)
\end{array}$$

The vertical isomorphisms are those defined previously. Thus, we conclude that the complex in Theorem 5.1 is exact for $\mathcal{B}(E)$ if and only if it is exact for $\mathcal{B}(F)$. □

LEMMA 5.6. *Let $q \in \mathbb{C}^*$. The sequence of Theorem 5.1 is exact when $E = E_q$.*

Proof. Put $A = \mathcal{B}(E_q) = \mathcal{O}(\mathrm{SL}_q(2))$. As usual we put $a = u_{11}$, $b = u_{12}$, $c = u_{21}$ and $d = u_{22}$. We will frequently use the well-known fact that A and its quotients $A/(b)$, $A/(c)$ and $A/(b,c)$ are integral domains, see e.g. [BG02, § I.1]. For $x \in A$, we have

$$\begin{aligned}\phi_1(x) = e_1^* \otimes e_1 \otimes ((-q^{-1} + qd)x) + e_1^* \otimes e_2 \otimes (-cx) + e_2^* \otimes e_1 \otimes (-bx)\\ + \ e_2^* \otimes e_2 \otimes ((-q + q^{-1}a)x)\end{aligned}$$

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$$\phi_2(e_1^*\otimes e_1\otimes x)=e_1^*\otimes e_1\otimes x+e_2^*\otimes e_1\otimes(-qbx)+e_2^*\otimes e_2\otimes ax$$
$$\phi_2(e_1^*\otimes e_2\otimes x)=e_1^*\otimes e_1\otimes bx+e_1^*\otimes e_2\otimes(1-q^{-1}a)x$$
$$\phi_2(e_2^*\otimes e_1\otimes x)=e_2^*\otimes e_1\otimes(1-qd)x+e_2^*\otimes e_2\otimes cx$$
$$\phi_2(e_2^*\otimes e_2\otimes x)=e_1^*\otimes e_1\otimes dx+e_1^*\otimes e_2\otimes(-q^{-1}cx)+e_2^*\otimes e_2\otimes x$$
$$\phi_3(e_1^*\otimes e_1\otimes x)=(a-1)x,\quad \phi_3(e_1^*\otimes e_2\otimes x)=bx,$$
$$\phi_3(e_2^*\otimes e_1\otimes x)=cx,\quad \phi_3(e_2^*\otimes e_2\otimes x)=(d-1)x.$$

The injectivity of ϕ_1 follows from the fact that A is an integral domain and the surjectivity of ϕ_3 is easy (and well known). Let $X=\sum_{i,j}e_i^*\otimes e_j\otimes x_{ij}\in\mathrm{Ker}(\phi_3)$. We have

$$X+\phi_2(-e_1^*\otimes e_1\otimes x_{11})=e_1^*\otimes e_2\otimes x_{12}+e_2^*\otimes e_1\otimes(qbx_{11}+x_{21})+e_2^*\otimes e_2\otimes(-ax_{11}+x_{22})$$

and, hence, to show that $X\in\mathrm{Im}(\phi_2)$, we can assume that $x_{11}=0$. We have

$$bx_{12}+cx_{21}+(d-1)x_{22}=0$$

which gives $(d-1)x=0$ in the integral domain $A/(b,c)$ and, thus, $x_{22}=b\alpha+c\beta$ for some $\alpha,\beta\in A$. Then we have

$$X+\phi_2(e_1^*\otimes e_2\otimes qd\alpha-e_2^*\otimes e_1\otimes\beta-e_2^*\otimes e_2\otimes b\alpha)=e_1^*\otimes e_2\otimes x+e_2^*\otimes e_1\otimes y$$

for some $x,y\in A$, and hence we also can assume that $x_{22}=0$. Then we have $bx_{12}+cx_{21}=0$, which gives $bx_{12}=0$ in the integral domain $A/(c)$, hence $x_{12}=c\alpha$ for some $\alpha\in A$, and moreover $x_{21}=-b\alpha$. Then we have

$$\phi_2(q^{-1}e_1^*\otimes e_1\otimes\alpha+e_1^*\otimes e_2\otimes c\alpha-q^{-1}e_2^*\otimes e_2\otimes a\alpha)=X$$

and we conclude that $\mathrm{Ker}(\phi_3)=\mathrm{Im}(\phi_2)$.

Let $X=\sum_{i,j}e_i^*\otimes e_j\otimes x_{ij}\in\mathrm{Ker}(\phi_2)$. Then $-qbx_{11}+(1-qd)x_{21}=0$, hence $(1-qd)x_{21}=0$ in the integral domain $A/(b)$ and, hence, $x_{21}=b\alpha$ for some $\alpha\in A$. Hence,

$$\begin{aligned}X+\phi_1(\alpha)=e_1^*\otimes e_1\otimes(x_{11}+(-q^{-1}+qd)\alpha)+e_1^*\otimes e_2\otimes(x_{12}-c\alpha)\\+e_2^*\otimes e_2\otimes(x_{22}+(-q+q^{-1}a)\alpha)\end{aligned}$$

and we can assume that $x_{21}=0$. But then, using the fact that A is an integral domain, we see that $X=0$ since $X\in\mathrm{Ker}(\phi_2)$. We conclude that $\mathrm{Ker}(\phi_2)=\mathrm{Im}(\phi_1)$. □

We are now ready to prove Theorem 5.1. Let $E\in\mathrm{GL}_n(\mathbb{C})$, $n\geqslant 2$, and let $q\in\mathbb{C}^*$ be such that $\mathrm{tr}(E^{-1}E^t)=-q-q^{-1}=\mathrm{tr}(E_q^{-1}E_q^t)$. Lemmas 5.5 and 5.6 combined together yield that the sequence in Theorem 5.1 is exact.

6. Applications

In this section we present several applications of the resolution built in the previous section.

6.1 Smoothness and Poincaré duality

Theorem 6.1. *Let $E\in\mathrm{GL}_n(\mathbb{C})$ with $n\geqslant 2$. The algebra $\mathcal{B}(E)$ is smooth of dimension 3. In particular, $H_n(\mathcal{B}(E),M)=(0)=H^n(\mathcal{B}(E),M)$ for any $\mathcal{B}(E)$-bimodule M and any $n\geqslant 4$.*

Proof. It is a standard fact (explained e.g. in [CHT09, § 3]) that the resolution of Theorem 5.1 will induce a length 3 resolution of the trivial bimodule $\mathcal{B}(E)$, and hence $\mathcal{B}(E)$ is smooth of dimension

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$d \leqslant 3$. Moreover using Proposition 2.1 and Theorem 5.1, we see that $H_3(\mathcal{B}(E), {}_\Phi\mathbb{C}_{\Phi^{-1}}) \simeq \mathbb{C}$, so the resolution in Theorem 5.1 has minimal length and we conclude that $\mathcal{B}(E)$ is smooth of dimension 3. □

We now show that Poincaré duality holds for the algebras $\mathcal{B}(E)$.

PROPOSITION 6.2. *Let M be a right $\mathcal{B}(E)$-module. Then for any $n \in \{0,1,2,3\}$ we have isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{B}(E)}(\mathbb{C}_\varepsilon, M) \simeq \mathrm{Tor}^{\mathcal{B}(E)}_{3-n}(\mathbb{C}_\varepsilon, {}_\theta M)$$

where θ is the algebra anti-automorphism of $\mathcal{B}(E)$ defined by $\theta(u) = S(u)E^{-1}E^tE^{-1}E^t$ and where ${}_\theta M$ has the left $\mathcal{B}(E)$-module structure given by $a \cdot x := x \cdot \theta(a)$.

Proof. After applying the functor $\mathrm{Hom}_A(-, M)$ to the resolution of Theorem 5.1 and standard identifications, the complex to compute the Ext-groups on the left becomes

$$0 \to M \xrightarrow{\phi_3^t} V^* \otimes V \otimes M \xrightarrow{\phi_2^t} V^* \otimes V \otimes M \xrightarrow{\phi_1^t} M \to 0$$

where $V = V_E$ and with

$$\phi_3^t(x) = \sum_{i,j} e_i^* \otimes e_j \otimes x \cdot (u_{ji} - \delta_{ji})$$

$$\phi_2^t(e_i^* \otimes e_j \otimes x) = e_i^* \otimes e_j \otimes x + \sum_{k,l} e_k^* \otimes e_l \otimes x \cdot ((u(E^t)^{-1})_{li} E_{kj})$$

$$\phi_1^t(e_i^* \otimes e_j \otimes x) = x \cdot ((E^tE^{-1})_{ji} - (Eu(E^t)^{-1})_{ji}).$$

Consider now the isomorphisms $\iota : M \to A \otimes_A ({}_\theta M)$, $x \mapsto 1 \otimes_A x$, and

$$f : V^* \otimes V \otimes M \to V^* \otimes V \otimes A \otimes_A ({}_\theta M), \quad \phi \otimes v \otimes x \mapsto \phi \otimes f_0(v) \otimes 1 \otimes_A x$$

where f_0 is the automorphism of V whose matrix in the canonical basis is $(E^t)^{-1}E$. The following diagrams commute

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{\phi_3^t} & V^* \otimes V \otimes M & \xrightarrow{\phi_2^t} & V^* \otimes V \otimes M & \xrightarrow{\phi_1^t} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & A \otimes_A ({}_\theta M) & \xrightarrow{\phi_1 \otimes_A \mathrm{id}} & V^* \otimes V \otimes A \otimes_A ({}_\theta M) & \xrightarrow{\phi_2 \otimes_A \mathrm{id}} & V^* \otimes V \otimes A \otimes_A ({}_\theta M) & \xrightarrow{\phi_3 \otimes_A \mathrm{id}} & A \otimes_A ({}_\theta M) & \longrightarrow & 0
\end{array}$$

and hence since the homology of the lower complex gives the Tor-groups in the proposition, we obtain the result. □

Using Propositions 2.1, 6.2 and the fact that $S \circ \theta = \sigma$, the proof of the following result is a direct verification that we leave to the reader, who might also consult [BZ08, § 3.3].

COROLLARY 6.3. *Let M be a $\mathcal{B}(E)$-bimodule. Then for any $n \in \{0,1,2,3\}$ we have isomorphisms*

$$H^n(\mathcal{B}(E), M) \simeq H_{3-n}(\mathcal{B}(E), {}_\sigma M)$$

where σ is the modular automorphism of $\mathcal{B}(E)$ given by $\sigma(u) = E^{-1}E^tuE^{-1}E^t$.

See [BZ08] for more examples of (noetherian) Hopf algebras satisfying Poincaré duality.

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6.2 Some homology computations

In this short subsection we record the computation of the Hochschild homology of $\mathcal{B}(E)$ when the bimodule of coefficients has dimension 1 as a vector space.

Proposition 6.4. *Let $\alpha, \beta \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{B}(E), \mathbb{C})$. Put $\gamma = \beta^{-1} * \alpha$, where $\beta^{-1} = \beta \circ S$ is the convolution inverse of β. Then*

$$H_0(\mathcal{B}(E), {}_\alpha\mathbb{C}_\beta) \simeq \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \mathbb{C} & \text{if } \alpha = \beta \end{cases} \qquad H_3(\mathcal{B}(E), {}_\alpha\mathbb{C}_\beta) \simeq \begin{cases} 0 & \text{if } \alpha \neq \beta * \Phi^2 \\ \mathbb{C} & \text{if } \alpha = \beta * \Phi^2 \end{cases}$$

$$H_1(\mathcal{B}(E), {}_\alpha\mathbb{C}_\beta) \simeq \frac{\{M \in M_n(\mathbb{C}) \mid \mathrm{tr}(M\gamma(u)^t) = \mathrm{tr}(M)\}}{\{M + E^t M^t \gamma(u)(E^t)^{-1}, M \in M_n(k)\}}$$

$$H_2(\mathcal{B}(E), {}_\alpha\mathbb{C}_\beta) \simeq \frac{\{M \in M_n(\mathbb{C}), M + E^t M^t \gamma(u)(E^t)^{-1} = 0\}}{\{\lambda(\sum_{i,j} e_i^* \otimes e_j \otimes ((E^t E^{-1})_{ij} - (E\gamma(u)(E^t)^{-1})_{ij})), \lambda \in \mathbb{C}\}}.$$

The proof is a direct computation by using Proposition 2.1 and Theorem 5.1. The computation of the cohomology groups follows by using Poincaré duality.

6.3 Bialgebra cohomology of $\mathcal{B}(E)$

The cohomology of a bialgebra was introduced by Gerstenhaber and Schack [GS90, GS92]: it is defined by means of an explicit bicomplex whose arrows are modelled on the Hochschild complex of the underlying algebra and columns are modelled on the Cartier complex of the underlying coalgebra. If A is a Hopf algebra, let us denote by $H_b^*(A)$ the resulting cohomology. Taillefer [Tai04a] proved that

$$H_b^*(A) \cong \mathrm{Ext}^*_{\mathcal{M}(A)}(A, A)$$

where $\mathcal{M}(A)$ is the category of Hopf bimodules over A. Combined with the monoidal equivalence between Hopf bimodules and Yetter–Drinfeld modules [Sch94], this yields an isomorphism

$$H_b^*(A) \cong \mathrm{Ext}^*_{{}^A_A\mathcal{YD}}(\mathbb{C}, \mathbb{C}).$$

Bialgebra cohomology of algebras of polynomial functions on linear algebraic groups was studied by Parshall and Wang [PW90], with a complete description in the connected reductive case. It seems that only very few full computations of $H_b^*(A)$ are known in the non-cocommutative case, see [Tai07] and the references therein. The following result gives in particular the description of the bialgebra cohomology of $\mathcal{O}(\mathrm{SL}_q(2))$ for q generic, i.e. $q = \pm 1$ or q not a root of unity.

Theorem 6.5. *Assume that $\mathcal{B}(E)$ is cosemisimple, i.e. that the solutions of the equation $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$ are generic. Then we have*

$$H_b^n(\mathcal{B}(E)) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3. \end{cases}$$

Proof. The assumption that $\mathcal{B}(E)$ is cosemisimple ensures that every object in $\mathcal{M}^{\mathcal{B}(E)}$ is projective, so by Proposition 3.3 every free Yetter–Drinfeld module is a projective object in ${}^{\mathcal{B}(E)}_{\mathcal{B}(E)}\mathcal{YD}$ and the category ${}^{\mathcal{B}(E)}_{\mathcal{B}(E)}\mathcal{YD}$ has enough projective objects by Corollary 3.4. The description of bialgebra cohomology as an Ext-functor [Tai04a, Tai04b] now shows that the bialgebra cohomology of $\mathcal{B}(E)$ is given by the cohomology of the complex obtained by applying the functor $\mathrm{Hom}_{{}^{\mathcal{B}(E)}_{\mathcal{B}(E)}\mathcal{YD}}(-, \mathbb{C})$ to any projective resolution of the trivial Yetter–Drinfeld module $\mathbb{C}$. Thus,

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using the resolution of Theorem 5.1, the description of $H_b^*(\mathcal{B}(E))$ is a direct computation, that we leave to the reader. □

6.4 L^2-Betti numbers

Let A be compact Hopf algebra, i.e. A is the Hopf $*$-algebra of polynomial functions on a compact quantum group [Wor98], and assume that A is of Kac type (the Haar state of A is tracial, or equivalently the square of the antipode is the identity). The L^2-Betti numbers of A have been defined in [Kye08]. There are several possible equivalent definitions [CS05, Kye08, Tho08] and the one we shall use is

$$\beta_k^{(2)}(A) = \dim_{\mathcal{M}^{\mathrm{op}}} \mathrm{Tor}_k^A(\mathbb{C}_\varepsilon, {}_A\mathcal{M})$$

where $\mathcal{M}$ is the (finite) von Neumann algebra of A, the left A-module structure on ${}_A\mathcal{M} = \mathcal{M}$ is given by left multiplication through the natural inclusion $A \subset \mathcal{M}$, and $\dim_{\mathcal{M}^{\mathrm{op}}}$ is Lück's dimension function for modules over finite von Neumann algebras [Luc98].

Now let $F \in \mathrm{GL}_n(\mathbb{C})$ with $F\overline{F} \in \mathbb{R}I_n$. Recall [Ban96] that $A_o(F)$ is the universal $*$-algebra with generators (u_{ij}), $1 \leqslant i, j \leqslant n$ and relations making the matrix $u = (u_{ij})$ unitary and $u = F\overline{u}F^{-1}$. The $*$-algebra $A_o(F)$ has a natural Hopf $*$-algebra structure (it is isomorphic, as a Hopf algebra, to the previous $\mathcal{B}((F^t)^{-1})$) and is a compact Hopf algebra. Moreover $A_o(F)$ is of Kac type if and only if $F^t = \pm F$. Thus, in this case

$$A_o(F) \simeq A_o(I_n) = A_o(n) \quad \text{or } A_o(F) \simeq A_o(J_{2m})$$

where for $2m = n$, $J_{2m} \in \mathrm{GL}_{2m}(\mathbb{C})$ is the antisymmetric matrix

$$J_{2m} = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}.$$

Collins *et al.* have shown, combining the results in [CHT09] with the vanishing of $\beta_1^{(2)}(A_o(n))$ by Vergnioux [Ver12], that all of the L^2-Betti numbers $\beta_k^{(2)}(A_o(n))$ vanish. Similar arguments lead to the following result, which completes the computation of the L^2-Betti numbers of the $A_o(F)$'s of Kac type.

Theorem 6.6. *For any $m \geqslant 1$ and $k \geqslant 0$, we have $\beta_k^{(2)}(A_o(J_{2m})) = 0$.*

Proof. For $m = 1$ it is already known [Kye08] that the L^2-Betti numbers of $A_o(J_2)$ all vanish, since $A_o(J_2)$ is commutative. So we assume that $m \geqslant 2$. First, by Theorem 5.1, we have $\beta_k^{(2)}(A_o(J_{2m})) = 0$ for $k > 3$. We have $\beta_0^{(2)}(A_o(J_{2m})) = 0$ by [Kye 11] and hence by Poincaré duality and the fact that the L^2-Betti numbers can be defined in terms of L^2-cohomology [Tho08] we have $\beta_3^{(2)}(A_o(J_{2m})) = 0 = \beta_0^{(2)}(A_o(J_{2m}))$. Similarly by Poincaré duality we have $\beta_2^{(2)}(A_o(J_{2m})) = \beta_1^{(2)}(A_o(J_{2m}))$. By [Ver12, Theorem 4.4, proof of Corollary 5.2] ([Ver12, proof of Corollary 5.2] is valid for $A_o(J_{2m})$ since it has the property of rapid decay [Ver07]), we have $\beta_1^{(2)}(A_o(J_{2m})) = 0$, and we are done. □

7. Conclusion

We have shown that there might exist strong links between the Hochschild (co)homologies of Hopf algebras that have equivalent tensor categories of comodules, although the ring-theoretical properties of the underlying algebras might be very different. We cannot expect to have functoriality at the level of the computation of Hochschild (co)homology groups, the situation

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is rather that if one of the Hopf algebras has a very special homological feature (a free Yetter–Drinfeld resolution of the counit), then so has the other.

A similar situation had been observed in the work of Voigt [Voi11] on the K-theory of free orthogonal quantum groups: the existence of a tensor category equivalence does not seem to imply functoriality at the level of K-theory groups, but is enough to ensure that one can transport a special homological situation, namely the validity of the Baum–Connes conjecture.

We hope that the present paper will bring further evidence to convince the reader that tensor category methods can be useful tools in the homological study of Hopf (C^*-)algebras.

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